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Geometric Structure of Chemistry-relevant Graphs: zigzags and central circuits

Structure Theory

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This book is a companion to our book [DeDu08], which considered the notions of polycycles, face-regularity and weak face-regularity of plane graphs and toroidal maps. The central actors in the present monograph are the *zigzags* and *central-circuits* of 3- or 4-regular plane graphs, which allow to obtain a double covering or covering of the edge-set.

This study is mainly focused on specific classes of bifaced plane graphs, i.e. those without faces of negative curvature. This contains, as a particular case, the *fullerenes*, i.e., 3-regular plane graphs with faces of size 5 or 6, which are prominent Chemistry-relevant graphs. The class also contains the *octahedrites*, i.e., 4-regular plane graphs with faces of size 3 or 4. We also consider three classes of graphs which are self-dual. For those graphs we consider how to enumerate them, their possible symmetry groups, their connectivity and other structural properties. We also study the *icosahedrites*, i.e. 5-regular plane graphs with faces of size 3 or 4; those have faces of negative curvature and so, their number grows exponentially. Finally, we consider *disk-fullerenes*, i.e. 3-regular partitions of a disk by 5- and 6-gons.

For all those classes of graphs, we treat the notion of zigzags and central-circuits; sometimes, at the same time. We consider simplicity of circuits, possible configuration, tightness and enumeration of the tight graphs with simple circuits. We also address extremal questions, such as the maximum number of circuits of tight graphs.

For the classes of graphs with maximal symmetry, such as the fullerenes of icosahedral symmetry, a special construction, named *Goldberg-Coxeter construction*, allows to describe them explicitly in term of two integer parameters k and l. This construction is studied systematically for 3-, 4- and 6-valent graphs and allow us to describe many classes in a simple way. We study the zigzags and central-circuits of the obtained graphs and build a new (k,l)-*product* algebraic formalism that allow us to describe the zigzags and central-circuits of the obtained graphs explicitly. For classes of graphs with non-maximal symmetry more complex description are needed. We explain how this can be done in practice by presenting the formalism of hyperbolic complex geometry derived by William Thurston in [Thur98].

For dimensions higher than 2, the possible similar structures are more complicated. In that case, we limit ourselves to zigzags and compute them for several infinite families of complexes and the regular, semiregular and regular-faced polytopes.

Contents

Chapter 1
Introduction: main *ZC*-notions

In this Chapter we summarize the main notions considered in this book and, briefly, the results that we obtain. Specifically, we define the pure graph theoretic and plane graph theoretic notions needed for this work with emphasis on symmetries. Then we define the main object of this book, i.e., zigzags and central circuits for 3- and 4-valent graphs and the corresponding notions of intersections. We explain how perfect matchings can be obtained from zigzags and knots from central circuits.

The notion of curvature of faces is also considered and the 10 classes of plane bifaced graphs without faces of negative curvature, that we consider, are described. We summarize in Tables the main obtained results on growth rates, symmetry groups, connectivity, zigzags/central circuits, etc. Finally, we give the smallest examples of such graphs occurring on projective plane, torus and Klein bottle.

1.1 Graphs

A *graph* $G = (V, E)$ consists of a vertex-set $V = V(G)$ and an edge-set $E = E(G)$, such that either one or two vertices are assigned to each edge as its ends. A graph is said to be *simple* it it has no loops or multiple edges.

A *k-connected graph* is the one which remains connected after removing any set of $k-1$ vertices. So, a plane graph is not 3-connected if it has a 2-gon (doubled edge) and not 2-connected if it has an 1-gon (loop). We mainly deal with 3- or 2-connected plane graphs, whose vertex- and edge-sets are finite.

A *plane graph* is a particular embedding (i.e., drawing) of a graph in the Euclidean plane $\mathbb{E}^2$ using smooth curves that cross each other only at the vertices of the graph. A graph, which has at least one such drawing, is called *planar*. In general, given a surface $\mathbb{F}^2$, an $\mathbb{F}^2$-*graph* is a graph embedded in $\mathbb{F}^2$.

Main structural vectors of an $\mathbb{F}^2$-graph G are the *v-vector* $\mathbf{v}(G) = (\dots, v_i, \dots)$ and *p-vector* $\mathbf{p}(G) = (\dots, p_i, \dots)$, which enumerates the numbers v_i of vertices of degree i and, respectively, the numbers p_i of faces of *gonality* i, i.e., having i sides.

The problem of existence of plane graphs with a fixed p-vector is an active subject of research; see, say, [YHZQ10]. A *k-regular graph* is the one with $v_i = 0$ for $i \neq k$.

For a *connected* (i.e., 1-connected) $\mathbb{F}^2$-graph G, its *dual graph* is the graph G^* on the set of faces of G with two faces being adjacent if they share an edge. Clearly, $\mathbf{v}(G^*) = \mathbf{p}(G)$ and $\mathbf{p}(G^*) = \mathbf{v}(G)$. A graph G is *self-dual* if $G = G^*$.

A *truncation* (or $\frac{1}{3}$*-truncation*) $Tr(G)$ of a plane graph G is an operation replacing, for any i with $v_i > 0$, its i-valent vertices by "small" i-gons. The *leapfrog graph* of G is $Tr(G^*)$. The *medial* (or $\frac{1}{2}$*-truncation*) will be defined in Sect. 1.3.

A *bipartite graph* is the one with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that every edge connects a vertex in V_1 to one in V_2. A plane graph is bipartite if and only if its faces have even gonality, i.e., its dual is an *Eulerian graph*. A connected graph is *Eulerian* if its vertices have even degrees or, equivalently, it has an *Eulerian circuit*, i.e., a circuit that passes through every vertex and uses every edge exactly once.

The *skeleton* of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices being adjacent if they generate a face. The famous *Steinitz Theorem* ([Ste16]) is that a graph is the skeleton of a *polyhedron* (3-dimensional polytope) if and only if it is planar and 3-connected. A polyhedron is usually represented by the *Schlegel diagram* (projection from $\mathbb{R}^3$ into $\mathbb{R}^2$ through a point beyond one of its facets) of its skeleton; the program used is *CaGe*. Such drawing is unique up to diffeomorphisms $\mathbb{R}^2$, but there exist many distinct polyhedra having the same graph as their skeleton; all such polyhedra are of the same *combinatorial type*. A *simple polyhedron* is the one with 3-regular skeleton.

1.2 Symmetries

A *point group* is a finite subgroup of the group $O(3)$ of isometries of $\mathbb{R}^3$, fixing the origin (0). The connection with graphs come from representing them on the sphere centered at (0). The point group of isometries of a polyhedron is a subgroup of the algebraic *symmetry group* $Aut(G)$ of its skeleton G, consisting of automorphisms of G. The *rotation group* $Rot(G)$ consists of all rotations preserving G; it is a subgroup of index 1 or 2 of $Aut(G)$. We use Schoenflies nomenclature for point groups.

By the *Mani's Theorem* ([Ma71], a refinement of Steinitz's theorem) valid for 3-connected plane graphs, there is, for each combinatorial type, at least one polyhedron, for which this inclusion becomes equality, i.e., $Aut(G)$ can be realized as the point group of a convex polyhedron with the skeleton G. So, one can identify the polyhedron and its graph, as well as $Aut(G)$ and the point group; the maximal symmetry groups of plane graphs are identified with the corresponding point groups.

Theorem 1.1. *For any plane graph G, the group $Aut(G)$ of symmetries preserving the set of vertices, edges and faces and their incidence can be identified with a group of isometries of* 3*-space.*

Proof. For a plane graph G, we define $order(G)$ to be the planar graph of the *Hasse diagram* of G, i.e., the vertex-set of $order(G)$ is formed by the vertices, edges and faces of G. Two vertices of $order(G)$ are adjacent if their corresponding vertex, edge or face in G are incident. Clearly, $order(G)$ is a plane graph as well. The faces of $order(G)$ correspond to the triples (v,e,f) with v, e and f being vertex, edge and face of G satisfying $v \in e \subset f$.

If G is a general plane graph, i.e., possibly, with loops, vertices of degree 1 and disconnecting vertices, then $order(G)$ is 2-connected. If G is 2-connected, then $order(G)$ is 3-connected. Any symmetry of G preserving incidence between vertices, edges and faces induces a symmetry of $order(G)$ as well. So, if G a plane graph, then $G_2 = order(order(G))$ is a 3-connected plane graph. So, Mani's Theorem implies that the symmetry group of G_2 is isomorphic to a group of symmetries of 3-space. Since $Aut(G)$ is a subgroup of $Aut(G_2)$, it can also be realized in 3-space. □

In the presence of 2-gons, i.e., multiple edges, one cannot speak of convex polyhedra and so, Mani's Theorem has no equivalent in that setting.

The list of point groups is split into two classes: the infinite families and the sporadic cases. Every point group has a normal subgroup formed by the rotations.

The group, denoted C_m, is the cyclic group of rotations by angle $\frac{2\pi}{m}k$ with $0 \leq k \leq m-1$ around a fixed axis Δ. Both groups C_{mv} and C_{mh} contains C_m as normal subgroups of rotations. The group C_{mh} (respectively, C_{mv}) is the group, generated by C_m and a symmetry of plane P, with P being orthogonal to Δ (respectively, containing Δ). The group D_m is the group, generated by C_m and a rotation by angle π, whose axis is orthogonal to Δ. The point group D_{mh} (respectively, D_{md}) is generated by C_{mv} and a rotation by angle π, whose axis is orthogonal to Δ and contained in a plane of symmetry (respectively, going between two planes of symmetry). Both, D_{mh} and D_{md}, contain D_m as a normal subgroup. If N is even, one defines the point group S_N to be the cyclic group generated by the composition of a rotation by angle $\frac{2\pi}{N}$ with axis Δ and a symmetry of plane P with P being orthogonal to Δ. The particular cases C_1, $C_s = C_{1v} = C_{1h}$, $C_i = S_2$ correspond to the trivial group, plane symmetry group and central symmetry inversion group.

The point groups T_d, O_h, I_h are the symmetry groups of Tetrahedron, Cube, Icosahedron; the point groups T, O, I are their normal subgroups of rotations. The point group T_h is formed by all f and $-f$ with $f \in T$.

More detailed description of point groups are available, for example, from [FoMa95] and [Dut04]. Since those groups have been classified long ago and are much used in Chemistry, we will use the chemical nomenclature for them.

On the pictures of this book, in order to express better the (maximal) symmetry of an i-hedrite, we put a double arrow, in order to represent an edge passing at infinity, and a quadruple arrow, in order to represent a vertex at infinity.

1.3 Zigzag and central circuits

Let G be a 3-regular plane graph. Then all edges, incident to a vertex x, can be labeled in counter-clockwise order as $e_1, e_2, \ldots, e_k$, where k is the degree of x in

G. For any edge e_i, $1 \le i \le k$, the edges e_{i+1}, and e_{i-1} (with $i+1$ and $i-1$ being addition modulo k) are called, respectively, the *left* and the *right*. A circuit of edges of G is called a *zigzag* (or a *Petrie path* [Cox73], *geodesic* [GrünMo63], *left-right path* [Sh75]), if, in tracing the circuit, we alternately select, as the next edge, the left neighbour and the right neighbor; see Fig. 1.1. In a 3-regular plane graph, any pair of edges sharing a vertex define a zigzag.

Given an edge, there are two possible directions for extending it to a zigzag. Hence, each edge is covered exactly twice by zigzags.

Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge e of intersection $Z \cap Z'$ is called of *type I* or *type II*, if Z and Z' traverse e in opposite or same direction, respectively; see Fig. 1.1. The types of edge depends on orientation chosen on zigzags except if $Z = Z'$. So, the *edge of self-intersection* of a zigzag Z is called of *type I* or *type II* ([GrünMo63]), if Z traverses it twice in the opposite or in the same direction, respectively. Edges of type I and type II correspond to edges of *cocycle* and *cycle character* in [Sh75]. The *signature of a zigzag* is the pair (α_1, α_2), where α_1 and α_2 are the numbers of its edges of type I and type II, respectively.

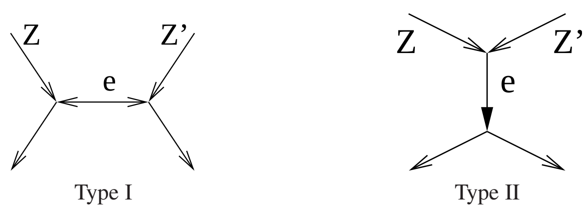


Fig. 1.1 Two types of zigzags intersection in an edge

A zigzag on a non-orientable surface can have either even, or odd number of edges. However, on a orientable surface (so, including a sphere or a disk), the length of each zigzag is even, because we consecutively take left and right turns. The number of intersections of two simple zigzags in a graph on a surface of genus $g \ge 1$ can be odd; see (in the Table 1.1 and on Fig. 1.8) Klein and Dyck maps of the hyperbolic plane $\mathbb{H}^2$. In our Euclidean plane $\mathbb{E}^2$ case, the number of intersections can be only even.

While for dual graphs v- and p-vectors are interchanged, the z-vector $\mathbf{z}$ remains the same, except that type I and type II in the signature are interchanged. Other kinds of dualities - interchanging z- and p-vector $\mathbf{p}$ (or z- and v-vector), but preserving v-vector $\mathbf{v}$ (or, respectively, p-vector) are considered in Sect. 8.4.

Let us give an example of application of zigzags. Given a plane graph G with vertices, $V_1, \dots V_v$, to every zigzag Z of G corresponds ([Sh75]) an element $x(Z) \in \{0,1\}^v$ which is a basic element of the *kernel* (equation $Lx = 0$ over $\{0,1\}$) of the *Laplacian* $L(G) = D(G) - A(G)$ of G. Here $D(G)$ is a diagonal matrix with d_{ii} being the degree of vertex V_i and $A(G)$ is the adjacency matrix $A(G)$ of G. So, the number of zigzags $Z(G)$ is equal to corank of $L(G)$ over $\mathbb{Z}^2$ and corank of any minor of $L(G)$

is $Z(G)-1$. Using this and the *Kirchhoff's theorem* (number of spanning trees of G is equal to the determinant of a minor of $L(G)$), it was proved in [GoRo01] that the number of spanning trees of G is odd if and only if G is *z-knotted*, i.e., has unique zigzag.

A *central circuit* of a plane Eulerian graph (i.e., see Sect. 1.1, having only vertices of even degree) is a circuit, which is obtained by starting with an edge and continuing at each vertex by the edge opposite to the entering one. Such circuit is called also *traverse* ([GaKe94]), *straight ahead* ([Harb97]), *straight Eulerian* (Chap. 17 of [GoRo01]), *cut-through* ([Jeo95]), *intersecting*, etc.

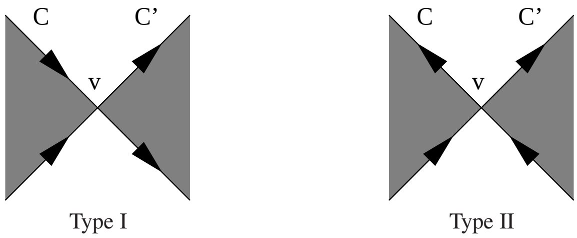


Fig. 1.2 Two types of central circuits intersection in a vertex

Clearly, the edge-set of G is partitioned by all its central circuits. The length of a central circuit is even, since it is twice the number of its points of self-intersection plus the sum of its intersections with other circuits. Two central circuits intersect in an even number number of vertices. However, it is harder to define the intersection type of two central circuits. Like in the zigzag case, the intersection type of two central circuits C and C' at a vertex V will depend on the orientation of the central circuits except if $C=C'$. For a 4-regular plane graph, the dual graph is bipartite, therefore we can choose a *chess coloring* of the faces in white or black such that for any two face of a given color, the adjacent faces are of the opposite color. If an orientation on C, C' and a chess-like coloring are chosen, then the vertex can be called of *type I* or *type II* according to Fig. 1.2. The *signature of a central circuit* is the pair (α_1, α_2), where α_1 and α_2 are the numbers of its vertices of type I and type II, respectively.

Some examples of applications of plane 4-regular graphs are *projections of links*, *rectilinear embedding* in VLSI and *Gauss crossing problem* for plane graphs (see, for example, [Liu98]). A *link* is one or more circles (components of a link) embedded into the space $\mathbb{R}^3$; a link with one component is called *knot*. A *projection of a link* is a drawing of it on the plane with gaps representing underpass and solid line representing overpass. Knot Theory is concerned with characterizing different plane presentations of links (see [Lic97] for a pleasant introduction). An *alternating link* is the one admitting a plane representation in which overlappings and underlappings alternate. (Apropos, there is no known topological characterization of such links.)

So, such projection can be seen just as a 4-regular plane graph. We will consider only *minimal projections*, i.e., those without 1-gons (loops). Clearly, each edge belongs to exactly one central circuit and any 4-regular graph without 1-gons can be seen as a minimal projection of an alternating link with components corresponding to its central circuits going alternately under and over at consecutive intersections; see an example on Fig. 1.3.

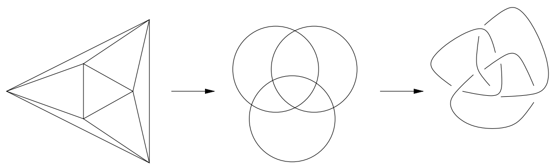


Fig. 1.3 The link 6_2^3 (*Borromean rings*) corresponding to Octahedron

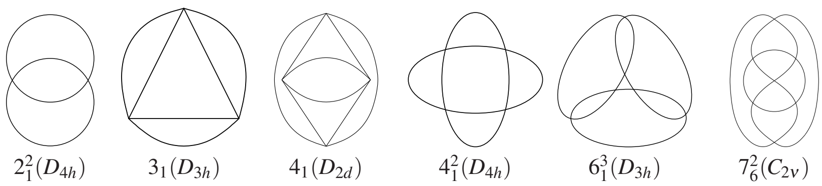


Fig. 1.4 Minimal plane projections of some alternating links

Proposition 1.1. *(i) For any plane bipartite graph G there exist an orientation of zigzags, with respect to which each edge has type I.*

(ii) For any 6*-regular plane graph, there exist an orientation on zigzags and central circuits such that any edge, respectively, vertex of intersection is of type II.*

Proof. We represent the graph on the sphere. The list of vertices, adjacent to a given vertex, can be organized in counter-clockwise order. Let the vertex-set V be partitioned into the two subsets V_1 and V_2 of the bipartition.

(i). Fix a zigzag Z; it will turn left at vertices, say, $v \in V_1$ and right at vertices $v \in V_2$. It is easy to see that the edges of self-intersection of Z can be only of type I. Take another zigzag Z', having a common edge e with Z. We choose an orientation on Z', such that e is an edge of type I. Then Z' will turn left at vertices $v \in V_1$ and right at vertices $v \in V_2$. Iterating this construction, all zigzags will be oriented and all edges will have type I with respect to this orientation of zigzags; (i) is proved.

(ii). For a 6-regular plane graph G, the dual graph G^* is bipartite. Let us take one color c of the faces of G and orient the edges of the face of color c in such a way that they turn clockwise around the face (see Fig. 1.5). It is apparent that such orientation carries over to the zigzags and central circuits of G and that with this orientation all the intersection are of type II. □

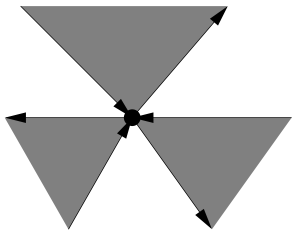


Fig. 1.5 The orientation of the edges of the face

In the case of one zigzag, this proposition was already known ([Sh75]). It is also valid for any bipartite graph, which is embedded in an oriented surface, in view of the well-known topological fact that any two-dimensional orientable manifold admits coherent orientation of its faces.

If all edges of plane graph are of same type for some orientation of its zigzags, then this orientation is unique. In fact, the orientation of one zigzag defines orientation of any zigzag, with which it have a common edge, and so, by a connectivity argument, orientation of any other zigzag. In a graph G, a *perfect matching* is a family PM of edges such that any vertex belongs to exactly one edge in PM.

Proposition 1.2. *([DDF04]) Let G be a plane graph; for any orientation of all zigzags of G, we have:*

(i) The number of edges of type II, which are incident to any fixed vertex*, is even.*

(ii) The number of edges of type I, which are incident to any fixed face*, is even.*

(iii) If all v vertices of G have odd degree, then the number of edges of type I is at least $\frac{v}{2}$*, and in the case of equality, these edges form a perfect matching of G.*

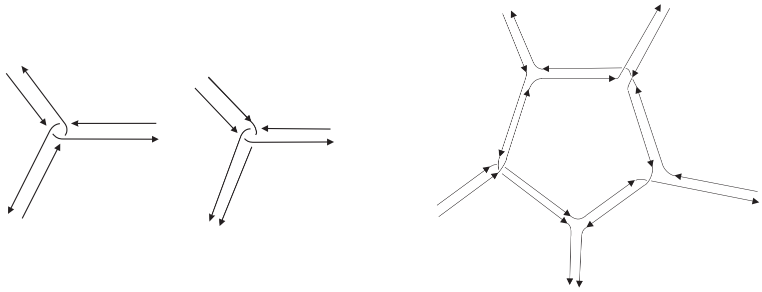
Proof. (i) For each vertex the number of times, that a zigzag enters it, should be equal to the number of times that a zigzag leaves it. This forces the number of edges of type II leaving to be equal to the number of edge II entering to be equal. So, the number of edge of type II incident to a vertex is even. See an example on Fig. 1.6.

(ii) Passing to the dual graph, the type of edges are interchanged.

(iii) follows from (i), since in this case any vertex is incident to odd (and so, positive) number of edges of type I. If the number is exactly $\frac{v}{2}$ then this number is 1 and we have a perfect matching. □

In Table 1.1, zigzag notions are illustrated by the skeletons of Platonic and *semiregular polyhedra*, i.e., such that their symmetry group is transitive on vertices.

Proposition 1.3. *(i) If G is a k-valent, k odd, plane v-vertex graph having only simple zigzags, then v is divisible by* 4.



(a) (b)

Fig. 1.6 Pictorial proof of (i) and (ii) in Proposition 1.2

Nr. edges	Polyhedron	z-vector **z**	Int. vector **Int**
6	*Tetrahedron*3; 3}	4^3	2^2
12	*Cube* {4; 3}	6^4	2^3
12	*Octahedron* {3; 4}	6^4	2^3
30	*Dodecahedron* {5; 3}	10^6	2^5
30	*Icosahedron* {3; 5}	10^6	2^5
24	*Cuboctahedron*	8^6	$2^4, 0$
60	*Icosidodecahedron*	10^{12}	$2^5, 0^6$
48	*Rhombicuboctahedron*	12^8	$2^6, 0$
120	*Rhombicosidodecahedron*	20^{12}	$2^{10}, 0$
72	*Truncated Cuboctahedron*	18^8	$6, 2^6$
180	*Truncated Icosidodecahedron*	30^{12}	$10, 2^{10}$
18	*Truncated Tetrahedron*	12^3	6^2
36	*Truncated Octahedron*	12^6	$4, 2^4$
36	*Truncated Cube*	18^4	6^3
90	*Truncated Icosahedron*	18^{10}	2^9
90	*Truncated Dodecahedron*	30^6	6^5
60	*Snub Cube*	$30^4_{3,0}$	8^3
150	*Snub Dodecahedron*	$50^6_{5,0}$	8^5
3m	$Prism_m,\ m \equiv 0 \pmod 4$	$(\frac{3m}{2})^4$	$(\frac{m}{2})^3$
3m	$Prism_m,\ m \equiv 2 \pmod 4$	$(3m_{\frac{m}{2},0})^2$	$2m$
3m	$Prism_m,\ m \equiv 1,3 \pmod 4$	$6m_{m,2m}$	
4m	$APrism_m, m \equiv 0 \pmod 3$	$(2m)^4$	$(\frac{2m}{3})^3$
4m	$APrism_m, m \equiv 1,2 \pmod 3$	$2m; 6m_{0,2m}$	
84	*Klein map* {3; 7}	8^{21}	$1^8, 0^{12}$
48	*Dyck map* {3; 8}	6^{16}	$1^6, 0^9$

Table 1.1 Zigzag structure of Platonic and semiregular polyhedra; also, of Klein and Dyck maps

(ii) If G is a 4*-valent, k even, v-vertex plane graph having only simple central circuits, then vt is even.*

Proof. (i) Any edge of $G = (V,E)$ is the intersection of two zigzags Z and Z'. However, Z and Z' intersect in an even number of edges. Hence, the edge-set E of G is union of parts of even size and so, has even cardinality itself. By double counting we get easily $kv = 2|E|$, which proves that kv is divisible by 4 and so, v as well.

(ii) Any vertex of G is the intersection of two central circuits C and C'. The circuits C and C' intersect in an even number of vertices. Hence, the vertex-set of G is union of parts of even size and so, has even cardinality itself. □

A *road* in a 3- or 4-regular plane graph is a non-extensible sequence (possibly, with self-intersections) of either 6-gonal faces or of 4-gonal faces, such that any non-end face is adjacent to its neighbors on opposite edges. If the sequence stops on a non-hexagon or, respectively, a non-quadrilateral face, then it is called a *pseudo-road*; otherwise, it is called a *railroad* and it is a circuit by finiteness of the graph.

We associate to each railroad a *representing plane curve* in the following way: in each of its faces one connects, by an arc, the midpoints of opposite edges, on which it is adjacent to its two neighbors. The sequence of those arcs can be seen as a closed curve in the plane and self-intersections of railroad correspond to self-intersections of the curve. Those self-intersections can be only double or, when a railroad consists of hexagons, triple. If the railroad has no triple self-intersection points, then this curve can be seen as a projection (not minimal, if there are 1-gons) of an alternating knot (see, for example, [Kaw96] and [Rol76]) with v crossings, where v is the number of self-intersection points. If there are several railroads without triple intersections and triple self-intersections, then the set of plane curves, representing them, can be seen as a projection of an alternating link.

Many results for 3- and 4-regular graphs will be similar; in such case we will use general notion of "either zigzag, or central circuit" and call it *ZC-circuit*. A zigzag is called *simple* if no edge occurs twice and a central circuit is called *simple* if no vertex occurs twice; so, we defined a *simple ZC-circuit*. A simple railroad is called a *belt*; it is bounded by two simple *ZC*-circuits.

A simple *ZC*-circuit can be seen as a *Jordan curve*, i.e., a simple and closed plane curve, which is a homeomorphic image of the unit circle. A (k,t)*-arrangements of pseudocircles* (or (k,t)*-AP*) is a set of k Jordan curves where any two intersect (triple or tangent points excluded) exactly in t points; so, there are $t(k\text{-}1)$ points. It is a tight pure *cc*-uniform 4-regular graph with k central circuits of length $t(k\text{-}1)$, pairwise intersecting in t points, i.e., with (defined below) $\mathbf{cc} = (tk-t)^k$, $\mathbf{Int} = t^{k-1}$. An *Grünbaum arrangements* of plane curves is an $(k,2)$-AP.

For example, the central circuits of the *medials* (defined in this section) of four truncated polyhedra - Tetrahedron, Cube, Icosahedron, Dodecahedron - form $(3,6)$-, $(4,6)$-, $(10,2)$-, $(6,6)$-APs; see Table 1.1. The medials of seven z-uniform fullerenes from Fig. 2.8 give $(k,2)$-APs with $k = 6,7,9,10,10,12,15$. The central circuits of five *cc*-uniform 8-hedrites from Fig. 4.11 form $(k,2)$-APs with $k = 3,4,4,5,6$.

The *z-vector* (or *cc-vector*) of a graph G is the vector $\mathbf{z}$ (or $\mathbf{cc}$) enumerating *lengths*, i.e., the numbers of edges, of all its zigzags (or, respectively, central circuits) with their signature as subscript. In general, we will use the term *ZC-vector*. The simple *ZC*-circuits are put in the beginning, in non-decreasing order of length, without their signature $(0,0)$, and separated by a semicolon from others. The self-intersecting ones are also ordered by non-decreasing lengths. If there are $m > 1$ *ZC*-circuits of the same length l and the same signature (α_1, α_2), then we write l^m if $\alpha_1 = \alpha_2 = 0$ and $l^m_{\alpha_1,\alpha_2}$, otherwise. For a *ZC*-circuit *ZC*, its *intersection vector* $(\alpha_1, \alpha_2); \dots, c_k^{m_k}, \dots$ is such that $\dots, c_k, \dots$ is an increasing sequence of sizes of its intersection with all other *ZC*-circuits, while m_k denote respective multiplicity.

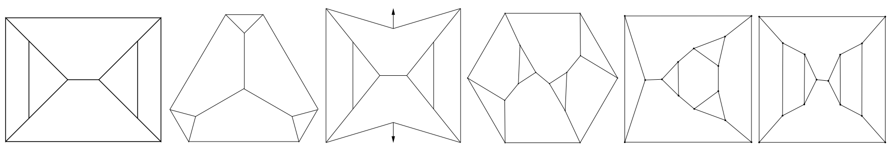
A 3- or 4-regular graph G is called:

- *pure graph* if all its *ZC*-circuits are simple;
- *tight graph* if it has no railroads;
- *ZC-knotted graph* if it has only one ZC-circuit.

If G has more than one *ZC*-circuit, it is called:

- *ZC-transitive graph* if $Aut(G)$ acts transitively on ZC-circuits;
- *ZC-uniform graph* if all its *ZC*-circuits have the same length and signature;
- *ZC-balanced graph* if all its *ZC*-circuits of the same length and signature have identical intersection vectors **Int**.

Clearly, *ZC*-transitivity implies *ZC*-uniformity and *ZC*-balanceness. In *ZC*-uniform case, the length of each of the r central circuits (respectively, zigzags) is $\frac{2v}{r}$ (respectively, $\frac{3v}{r}$). We do not know example of a *ZC*-uniform, but not *ZC*-balanced, graph.



10-2 (C_{2v}) 10-3 (C_{3v}) 12-6 (C_{2v}) 14-21 (C_2) 14-23 (C_s) 14-24 (C_{2v})

Fig. 1.7 Some *z*-uniform 3-regular graphs with their symmetry group; the numbers of graphs are those given by the program *plantri*

Zigzags and central circuits, being local notions, are defined for maps on any surface, even on non-orientable one; see, for example, on Fig. 1.8 a zigzag for the *Klein map* $\{3;7\}$ and the *Dyck map* $\{3;8\}$, which are dual triangulations for such 3-regular maps. The notion of zigzag (respectively, central circuit) is used here (except 6- and 5-regular case in Chap. 5) in 3-regular (respectively, 4-regular) case, but they can be defined on any plane graph (respectively, Eulerian plane graph). [Harb97] consider central circuits for any drawing on the plane of any Eulerian graph, so that edges are mapped into simple curves with at most one crossing point.

The notion of a zigzag was generalized also to locally-finite infinite plane 3-connected graphs. All such *edge-transitive* graphs without self-intersecting zigzags

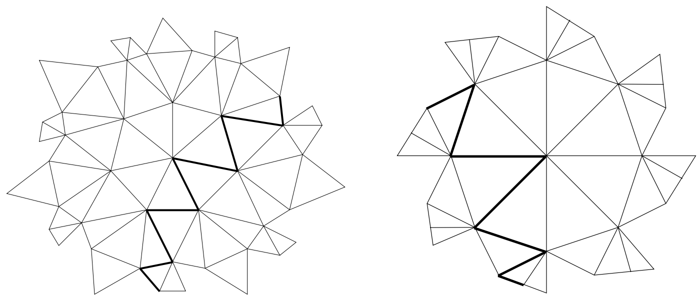


Fig. 1.8 A zigzag of length 8 in Klein map $\{3;7\}$ and of length 6 in Dyck map $\{3;8\}$

(i.e., simple circuits and doubly infinite rays are the only zigzags) were classified in [GrSh87]: these are the three regular partitions $\{6;3\}$, $\{3;6\}$, $\{4;4\}$ of the Euclidean plane $\mathbb{E}^2$, the Archimedean partition $(3.6.3.6)$, its dual and several infinities of partitions of the hyperbolic plane $\mathbb{H}^2$. Also the notion of zigzag (Petrie polygon) was extended in [Cox73] p. 223 for n-dimensional polytopes and honeycombs.

The *medial graph* of a plane graph G, denoted by $Med(G)$, is defined by taking, as vertex-set, the set of edges of G with two edges being adjacent if they share a vertex and belong to the same face of G. $Med(G)$ is 4-regular and its central circuits $C_1,\dots,C_r$ correspond to zigzags $Z_1,\dots,Z_r$ of G (see Fig. 1.9) and the same, up to change of types I and II in the signatures, zigzags of G^*. It holds $Med(G) = Med(G^*)$. Moreover, an orientation of a zigzag Z_i induces an orientation of a central circuit C_i. The set of faces of $Med(G)$ corresponds to the set of vertices and faces of G. If we keep the same orientation, the intersection numbers of C_i and C_j are the same as the intersection numbers of Z_i and Z_j.

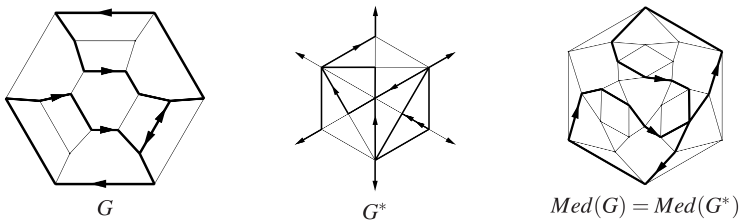


Fig. 1.9 Example of a zigzag $18_{1,0}$ in a plane graph G, the corresponding zigzag $18_{0,1}$ in G^* and the corresponding to them central circuit in $Med(G)$

The graph $(Med(G))^*$ is bipartite, that is the face-set of $Med(G)$ is split into two sets, which correspond to the vertices and faces of the graph G. So, any 4-regular plane graph H is the medial graph for a pair of following mutually dual plane graphs: one can assign two colors to the faces of H in the "chess way", such that no two adjacent faces of H have the same color. Two faces of the same color are said to be *adjacent* if they share a vertex. See example on Fig. 1.10.

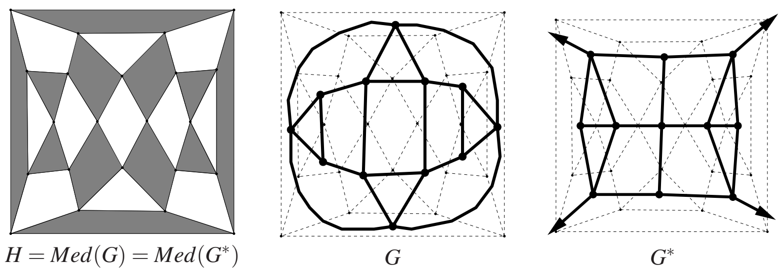


Fig. 1.10 Inverse medial graphs G and G^* of a chess-colored bipartite graph H are $G = H_{black}$ and $G^* = H_{white}$ of the black and white faces of H

Removing of a central circuit is explained by Fig. 1.11. Removing a zigzag in a graph G consists of removing corresponding central circuit in $Med(G)$ and taking one of two inverse medial graphs. A central circuit is called *reducible* if on one of its sides there are only 4-gons. This sequence of 4-gons can be completely eliminated to get a *reduced graph*. For a 3-regular plane graph, a zigzag is called *reducible* if on one side there is only 6-gons. We can reduce the graph by eliminating those 6-gons only if the zigzag is simple. Moreover, there are several possibilities for this reduced graph, while in the 4-regular case, the reduction is uniquely defined. By removing one of two boundary central-circuits for each railroad, any 4-regular graph can be reduced to a tight one. There is no analogous reduction for the 3-regular case.

Proposition 1.4. *Given a tight k-regular map G,* $k \in \{3,4\}$ *with p-vector* $\mathbf{p} = (\dots, p_i, \dots)$*; denote* $\frac{2k}{k-2}$ *by* k'*. The number of ZC-circuits (zigzags or central circuits for* 3- *or* 4-*regular case, respectively) of G is at most*

$$\frac{1}{2}\sum_{i \neq k'} ip_i.$$

Proof. In fact, each $ZC-circuit$ has a non-k'-gonal face on each of its sides, since, otherwise, it would have a railroad on this side; so, the number of incidences between *ZC*-circuits and non-k'-gons is at least twice the number of *ZC*-circuits. On the other hand, the number of those incidences is exactly $\sum_{i \neq k'} ip_i$. □

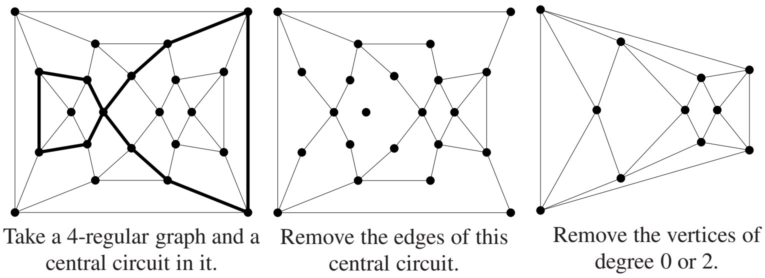


Fig. 1.11 Removing a central circuit

1.4 Curvature of faces

Given $R \subset \mathbb{N}$ and a connected closed (compact and without boundary) surface $\mathbb{F}^2$, let R-$\mathbb{F}^2$ denote a map M on $\mathbb{F}^2$ whose faces have gonalities $i \in R$ and let (R,k)-$\mathbb{F}^2$ denote such a k-regular map. We suppose also $\max\{i \in R\} \geq 3 \leq k$.

The *Euler characteristic* $\chi = \chi(M) = \chi(\mathbb{F}^2)$ is $v - e + f$, where v, $e = \frac{1}{2}\sum_{i \geq 1} ip_i$ and $f = \sum_i p_i$ are the numbers of vertices, edges and faces of M. So, $v = \chi + \frac{1}{2}\sum_{i \geq 1} p_i(i-2)$ and in the k-regular case, $v = \frac{1}{k}\sum_{i \geq 1} ip_i$.

Since $kv = 2e = \sum_i ip_i$, the Euler formula becomes also $\chi = \sum_{i \geq 1} p_i \kappa_i$, where

$$\kappa_i = 1 + \frac{i}{k} - \frac{i}{2}$$

denote the *curvature of a face* of gonality i. Euler formula is a discrete analog of the *Gauss–Bonnet formula*, $2\pi\chi(\mathbb{F}) = \int_{\mathbb{F}} K(x)dx$, for the Gaussian curvature K.

A map (R,k)-$\mathbb{F}^2$ with non-negatively curved faces has $\chi(\mathbb{F}^2) \geq 0$. So, such irreducible surface $\mathbb{F}^2$ is only the sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ (two orientable ones), real projective plane $\mathbb{P}^2$ or Klein bottle $\mathbb{K}^2$ with $\chi = 2, 0, 1, 0$, respectively. Clearly, the set of all maps (R,k)-$\mathbb{F}^2$ with only positively curved faces is finite and consists of

(i) $(\{1,2,3,4,5\},3)$-, $(\{1,2,3\},4)$-, $(\{1,2,3\},5)$-maps on $\mathbb{S}^2$ and

(ii) their antipodal quotients, when they are centrally symmetric, on $\mathbb{P}^2$.

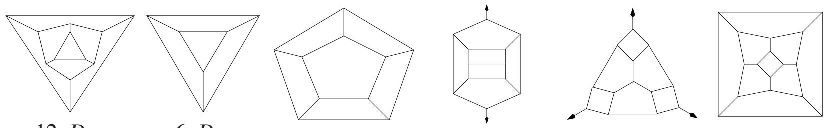
For example, all maps (i) with $\min\{i \in R\} \geq 3$, are only Octahedron, Icosahedron and nine spheres $(\{3,4,5\},3)$-$\mathbb{S}^2$: all eight dual *convex deltahedra* (all faces are equilateral triangles) and *Dürer's octahedron* (truncation of Cube on two opposite vertices). All such non-Platonic spheres are given in Fig. 1.12.

All maps (ii) with $\min\{i \in R\} \geq 3$, are the antipodal quotients of Octahedron, Icosahedron, Cube, Dodecahedron and Dürer's octahedron.

Also, all maps (i) with $\min\{i \in R\} \geq 2$ and $|R| = 2$, are $8, 2, 4$ maps $(\{a,b\},3)$-, $(\{2,3\},4)$-, $(\{2,3\},5)$-$\mathbb{S}^2$, given on Fig. 2.1 [DeDu08].

For fixed R and k, the set of maps (R,k)-$\mathbb{F}^2$ with non-negatively curved faces is infinite if and only if the faces of maximal gonality (say, r) are *flat faces* (i.e., $\kappa_r = 0$), implying $r = \frac{2k}{k-2}$ and so, $(r,k) = (6,3), (4,4)$ or $(3,6)$.

Hence, all such infinite families are:



12, D_{3d}, $\mathbf{z} = 6; 30_{6,6}$ | 6, D_{3h}, $\mathbf{z} = 18_{3,6}$ | 10, D_{5h}, $\mathbf{z} = 30_{5,10}$ | 12, D_{2d}, $\mathbf{z} = 8^2; 20_{0,4}$ | 14, D_{3h}, $\mathbf{z} = 8^3; 18_{0,3}$ | 16, D_{4d}, $\mathbf{z} = 8; 40_{8,8}$

Fig. 1.12 Dürer's octahedron and all non-Platonic dual deltahedra

(i) $(\{1,2,3,4,5,6\},3)$-, $(\{1,2,3,4\},4)$-, $(\{1,2,3\},6)$-maps on $\mathbb{S}^2$,
(ii) their antipodal quotients, when they are centrally symmetric, on $\mathbb{P}^2$,
(iii) $(\{6\},3)$-, $(\{4\},4)$-, $(\{3\},6)$-maps on the torus $\mathbb{T}^2$ and their antipodal quotients, when they are centrally symmetric, on $\mathbb{K}^2$.

Exclusion of negatively curved faces simplifies enumeration, while number p_r of flat faces not being restricted, there is an infinity of such (R,k)-spheres. The number of such v-vertex (R,k)-spheres with $|R| = 2$ increases polynomially with v. Such spheres admit parametrization and description in terms of rings of (*Gaussian* if $k = 4$ and *Eisenstein* if $k = 3,6$) *integers*.

There are 16 (see Sect. 6.4) infinite families of *non-bifaced*, i.e., $|\{3 \le i \le 5 : p_i > 0\}| \ge 2$, $(\{3,4,5,6\},3)$-spheres. Their symmetries are listed explicitly in [DDF09]. It is also done by Proposition 1.5; "iff" there means "if and only if".

Proposition 1.5. *Let G be a non-bifaced* $(\{3,4,5,6\},3)$*-sphere. Then its possible symmetries are given by the criterion below.*

(i) G admit groups C_1, C_s.

(ii) G admits C_i *iff* $gcd(p_3, p_4, p_5)$ *is even, and then it admits* C_2, C_{2v}, C_{2h} *also. Moreover, G admits* S_4, D_2, D_{2h}, D_{2d} *iff* $\frac{p_5}{2}$ *is even and* D_m, D_{mh}, D_{md} *iff* $m = \frac{p_5}{2} > 2$.

(iii) If $gcd(p_3, p_4, p_5)$ *is odd, then G admits* C_2, C_{2v} *iff* p_5 *is even and it admits* C_3, C_{3v} *iff* p_5 *is divided by* 3. *Moreover, G admits* C_{3d} *iff* $gcd(p_3, p_4) = 3$ *and it admits* D_m, D_{mh} *iff* $m = p_4 > 2$ *and* p_3 *is even.*

There are three families of non-bifaced $(\{2,3,4\},4)$-spheres (with $(p_2, p_3) = (3,2), (2,4), (1,6)$) and one family of non-bifaced self-dual $\{2,3,4\}$-spheres (with $(p_2, p_3) = (1,2)$. All of them admit groups C_1, C_s, C_2, C_{2v}. Moreover, first family admits D_3, D_{3h} and second one admits $C_i, C_{2h}, D_2, D_{2h}, D_{2d}$.

1.5 Bifaced maps

In particular, any *bifaced map* (i.e., $(\{a,b\},k)$-$\mathbb{F}^2$, $a < b$) have, by Euler formula,

$$p_a = \frac{b(k-2) - 2k}{2k - a(k-2)} + \chi(\mathbb{F}^2)2k.$$

Besides 16 maps with only positively curved faces (11 on $\mathbb{S}^2$ and 5 on $\mathbb{P}^2$, see Sect. 1.4), each such map has $a < \frac{2k}{k-2} \leq b$ and negatively curved faces are excluded exactly if $b = \frac{2k}{k-2}$, i.e., if b-gons are flat. In this case, $(b,k) = (6,3), (4,4)$ or $(3,6)$, Euler formula $\chi = \kappa_a p_a$ becomes $\chi = (1 - \frac{a}{b}) p_a$, the number p_a is a constant $\frac{\chi b}{b-a}$ and all possible (a, p_a) for a $(\{a,b\},k)$-$\mathbb{S}^2$ are:

- $(5,12),(4,6),(3,4),(2,3)$ for $(b,k) = (6,3)$,
- $(3,8),(2,4)$ for $(b,k) = (4,4)$ and
- $(2,6),(1,3)$ for $(b,k) = (3,6)$.

Those eight families can be seen as spherical analogs of the regular plane partitions $\{6;3\}$, $\{4;4\}$, $\{3;6\}$ with p_a *disclinations* (”defects”) of the curvature κ_a, added to get the curvature 2 of the sphere.

Denote by a_v any v-vertex map $(\{a,6\},3)$-$\mathbb{S}^2$. The smallest (with $p_6 = 0$) 3_v, 4_v and 5_v are Tetrahedron, Cube and Dodecahedron. In fact, all maps $(\{a,b\},3)$-$\mathbb{S}^2$ with $3 \leq a < b < 6$, besides them, are those given on Fig. 1.12.

The 5_v are, actually, the (geometric) *fullerenes*, well known in Organic Chemistry (they will be considered in Chap. 2), while 4_v represent *boron nitrides*. See Fig. 1.13 for a recently discovered by Wang *et al.* (see https://news.brown.edu/articles/2014/07/buckyball and [Borospherene2014]) *Borosphere*.

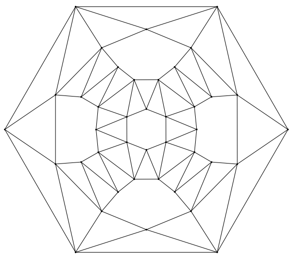


Fig. 1.13 Borosphere has symmetry D_{2d}, $\mathbf{v} = (v_4 = 16, v_5 = 24)$, $\mathbf{p} = (p_3 = 48, p_6 = 2, p_7 = 4)$, $\mathbf{z} = (16^2, 36; 116_{8,24})$ and $286,224$ perfect matchings in $36,090$ orbits

The Chap. 4 treats the maps $(\{2,3,4\},4)$-$\mathbb{S}^2$ and self-dual maps $(\{2,3,4\})$-$\mathbb{S}^2$. The Chap. 5 treats the maps $(\{1,2,3\},6)$-$\mathbb{S}^2$ and $(\{3,4\},5)$-$\mathbb{S}^2$.

Note that all $(\{3,6\},3)$-, $(\{4,6\},3)$-, $(\{2,4\},4)$- and $(\{1,3\},6)$-$\mathbb{S}^2$ are Hamiltonian, but $(\{2,6\},3)$-$\mathbb{S}^2$ with $v \equiv 0 \pmod 4$ are not; see [Good77] and [GrZa74].

Denote by $i \times G$ the graph G with each edge replaced by i edges. The $Bundle_m$ is defined as m-regular graph $m \times K_2$; its z-vector is $2m_{m,0}$ for odd m and $2m_{0,m}$ for even m. The $Foil_m$ is defined as 4-regular graph $2 \times C_m$; its cc-vector is $2m$ for odd m and m^2 for even m. Clearly, $Foil_m$ with $m = 2$, 4 and 3 are (projections of) links 2_1^2, 4_1^2 and knot Trefoil 3_1 (see Fig. 1.4). The *m-rose* is the $2m$-regular 1-vertex plane graph $m \times K_1$ with one m-gonal and m 1-gonal faces; *Trifolium* is the 3-rose.

The maps (R,k)-$\mathbb{S}^2$ with negatively curved faces are much more complicate to study. We will consider in detail only the simplest cases: *icosahedrites* $((\{3,4\},5)$-spheres), *c-disk-fullerenes* $((\{5,6,c\},3)$-spheres with $c>6$ and $p_c=1)$ and any $(\{a,b\},k)$-$\mathbb{S}^2$ with $p_b \le 3$. See Table 1.2 below. Last two lines there give the families of bifaced self-dual plane maps with non-negatively curved faces.

The criterions of existence there were obtained in [GrünMo63] for the $(\{a,6\},3)$-spheres with $3 \le a \le 5$, in [Grün67] for $(\{3,4\},4)$- and in [GrZa74] for $(\{2,6\},3)$- and $(\{1,3\},6)$-spheres. For an infinite family $\mathscr{M}$ of spheres, let $Nr_v(\mathscr{M})$ be the number of such v-vertex maps. In the column "Order" we give the order of magnitude of $Nr_v(\mathscr{M})$. In the column "NrGr", the number of symmetry groups of maps from $\mathscr{M}$ is given. In the Tables 1.2 and 1.4, '?' means "conjectured" or "unknown".

k	(a,b)	smallest one	exists if and only if	connect.	p_a	v	Order	$NrGr$
3	$(5,6)$	Dodecahedron	$p_6 \neq 1$	3-conn.	12	$20+2p_6$	v^9	28
3	$(4,6)$	Cube	$p_6 \neq 1$	3-conn.	6	$8+2p_6$	v^3	16
4	$(3,4)$	Octahedron	$p_4 \neq 1$	3-conn.	8	$6+p_4$	v^5	18
6	$(2,3)$	Bundle$_6$=$6\times K_2$	p_3 is even	2-conn.	6	$2+\frac{p_3}{2}$	v^4	22
3	$(3,6)$	Tetrahedron	p_6 is even	Prop. 3.1	4	$4+2p_6$	v	5
4	$(2,4)$	Bundle$_4$=$4\times K_2$	p_4 is even	2-conn.	4	$2+p_4$	v	5
3	$(2,6)$	Bundle$_3$=$3\times K_2$	$p_6=(k^2+kl+l^2)$-1	2-conn.	3	$2+2p_6$	v	2
6	$(1,3)$	Trifolium	$p_3=2(k^2+kl+l^2)$-1	1-conn.	3	$\frac{1+p_3}{2}$	v	3
5	$(3,4)$	Icosahedron	$p_4 \neq 1$	3-conn.?	$20+2p_4$	$12+2p_4$	exp.	38
self-dual	$(3,4)$	Tetrahedron	$p_4 \ge 0$	2-conn.	4	$4+p_4$	v^3	16
self-dual	$(2,4)$	Bundle$_2$=$2\times K_2$	$p_4 \ge 0$	1-conn.	2	$2+p_4$	v	5

Table 1.2 Main families of considered maps $(\{a,b\},k)$-$\mathbb{S}^2$ and two self-dual $\{a,b\}$-$\mathbb{S}^2$

In Table 1.4, we give information on the ZC-structure of considered graphs. Among 11 families, given in the Table, the $(\{2,6\},3)$- and $(\{1,3\},6)$-spheres come by the Goldberg–Coxeter construction from Bundle$_3$ and Trifolium, i.e., admit a description by 1 complex parameter. All their ZC-circuits self-intersect.

The $(\{3,6\},3)$- ([GrünMo63]) and $(\{2,4\},4)$-spheres ([DeSt03]) admit a simple description by 2 complex parameters, or, moreover, by 3 natural ones: pseudo-road length, number of circumscribing railroads and shift. All their ZC-circuits are simple. Such pure sphere is tight if and only if it has the minimal number (3 or, respectively, 2) ZC-circuits. There is a tight v-vertex $(\{2,4\},4)$-sphere for any possible (i.e., even) v. A tight v-vertex $(\{3,6\},3)$-sphere exists if and only if $\frac{v}{4}$ is odd; all such spheres are tight if and only if $\frac{v}{4}$ is prime >2. See Chaps. 3 and 4 for details.

Aggregating groups as $\mathbf{C_m}$=$\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}$=$\{D_m, D_{mh}, D_{md}\}$, $\mathbf{C'_m}$=$\mathbf{C_m} \setminus \{S_{2m}\}$, $\mathbf{D'_m}$=$\mathbf{D_m} \setminus \{D_{md}\}$ and $\mathbf{T}$=$\{T, T_d, T_h\}$, $\mathbf{O}$=$\{O, O_h\}$, $\mathbf{I}$=$\{I, I_h\}$, $\mathbf{T'} = \mathbf{T} \setminus \{T_h\}$, the symmetries of families of bifaced plane maps become simpler to see:

1. 28 for $(\{5,6\},3)$- ([FoMa95]): $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
2. 22 for $(\{2,3\},6)$- ([DeDu12]): $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D'_6}$, $\mathbf{T}$

k	(a,b)	Groups of symmetry and number of complex parameters	
3	$(5,6)$	(C_1,C_s,C_i), 10 (C_3,C_{3v},C_{3h},S_6), 4 (D_3,D_{3h},D_{3d}), 3 (D_6,D_{6h},D_{6d}), 2 (I,I_h), 1, $GC_{k,l}(Dodecahedron)$	(C_2,C_{2v},C_{2h},S_4), 6 (D_2,D_{2h},D_{2d}), 4 (D_5,D_{5h},D_{5d}), 2 (T,T_h,T_d), 2
3	$(4,6)$	(C_1,C_s,C_i), 4 (D_2,D_{2h},D_{2d}), 2 (D_6,D_{6h}), 1, $GC_{k,l}(Prism_6)$	(C_2,C_{2v},C_{2h}), 3 (D_3,D_{3h},D_{3d}), 2 (O,O_h), 1, $GC_{k,l}(Cube)$
4	$(3,4)$	(C_1,C_s,C_i), 6 (D_2,D_{2h},D_{2d}), 3 (D_4,D_{4h},D_{4d}), 2	(C_2,C_{2v},C_{2h},S_4), 4 (D_3,D_{3h},D_{3d}), 2 (O,O_h), 1, $GC_{k,l}(Octahedron)$
6	$(2,3)$	(C_1,C_s,C_i), 4 (C_3,C_{3v},C_{3h},S_6), 2 (D_3,D_{3h},D_{3d}), 2 (T,T_d,T_h), 1, $GC_{k,l}(K_2 \times Tetrahedron)$	(C_2,C_{2v},C_{2h},S_4), 3 (D_2,D_{2h},D_{2d}), 2 (D_6,D_{6h}), 1, $GC_{k,l}(6 \times K_2)$
3	$(3,6)$	(D_2,D_{2h},D_{2d}), 2	(T,T_d), 1, $GC_{k,l}(Tetrahedron)$
4	$(2,4)$	(D_2,D_{2h},D_{2d}), 2	(D_4,D_{4h}), 1, $GC_{k,l}(Bundle_4)$
3	$(2,6)$	(D_3,D_{3h}), 1, $GC_{k,l}(Bundle_3)$	
6	$(1,3)$	(C_3,C_{3h},C_{3v}), 1, $GC_{k,l}(Trifolium)$	
5	$(3,4)$	(C_1,C_i,C_s), ∞ (C_3,C_{3h},C_{3v},S_6), ∞ $(C_5,C_{5h},C_{5v},S_{10})$, ∞ (D_3,D_{3h},D_{3d}), ∞ (D_5,D_{5h},D_{5d}), ∞ (T,T_d,T_h), ∞	(C_2,C_{2h},C_{2v},S_4), ∞ (C_4,C_{4h},C_{4v},S_8), ∞ (D_2,D_{2h},D_{2d}), ∞ (D_4,D_{4h},D_{4d}), ∞ (O,O_h), ∞ (I,I_h), ∞
self-dual	$(3,4)$	(C_1,C_i,C_s) (C_3,C_{3v}) (D_2,D_{2d},D_{2h})	(C_2,C_{2h},C_{2v},S_4) (C_4,C_{4v}) (T,T_d), 1, $GC_{k,l}(Tetrahedron)$
self-dual	$(2,4)$	(C_2,C_{2v},C_{2h})	(D_2,D_{2h}), 1, $GC_{k,l}(Bundle_2)$

Table 1.3 Number of complex parameters needed for describing spheres from Table 1.2 (see Chap. 6); when this number is 1, a description with Goldberg–Coxeter construction is given; for the self-dual case, we restrict to odd $k+l$, see Theorem 4.10

3. 16 for $(\{4,6\},3)$- ([DeDu05]): $\mathbf{C_1}$, $\mathbf{C'_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D'_6}$, $\mathbf{O}$
4. 18 for $(\{3,4\},4)$- ([DeDuSh03]): $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
5. 5 for $(\{3,6\},3)$- ([FoCr97]): $\mathbf{D_2}$, $\mathbf{T'}$
6. 5 for $(\{2,4\},4)$- ([DeDuSh03]): $\mathbf{D_2}$, $\mathbf{D'_4}$
7. 2 for $(\{2,6\},3)$- ([GrZa74]): $\mathbf{D'_3}$
8. 3 for $(\{1,3\},6)$-([DeDu12]): $\mathbf{C'_3}$
9. 38 for $(\{3,4\},5)$- ([DDS13a]): $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.
10. 16 for self-dual $(\{3,4\})$- ([DuDe11]): $\mathbf{C_1}$, $\mathbf{C_2}$, $\{C_3, C_{3v}\}$, $\{C_4, C_{4v}\}$ $\mathbf{D_2}$, $\mathbf{T'}$.
11. 5 for self-dual $(\{2,4\})$- ([DuDe11]): $\mathbf{C'_2}$, $\mathbf{D'_2}$.

The (R,k)-maps on the projective plane $\mathbb{P}^2$ are the antipodal quotients of centrosymmetric maps (R,k)-$\mathbb{S}^2$; so, with halving of their v- and p-vector. The point groups with inversion are: T_h, O_h, I_h, C_{mv}, D_{mh} with even m and D_{md}, S_{2m} with odd m. So, the symmetries of above 11 families on $\mathbb{P}^2$ are:

k	(a,b)	circuits	zcMax	First zcMax	zcPT
3	$(5,6)$	zigzags	15	$GC_{2,1}(Dodecahedron)$	9?
3	$(4,6)$	zigzags	8?	$4_{88}(D_{2h})$?	2
4	$(3,4)$	central circuits	6	$GC_{2,1}(Cube)$	8
6	$(2,3)$	zigzags	6	$GC_{2,1}(Bundle_6)$	3
		central circuits	6	$GC_{2,1}(Bundle_6)$	1
3	$(3,6)$	zigzags	3	$Tetrahedron$	∞
4	$(2,4)$	central circuits	2	Bundle_4	∞
3	$(2,6)$	zigzags	3	$GC_{1,1}(Bundle_3)$	0
6	$(1,3)$	zigzags	0	0	0
		central circuits	0	0	0
5	$(3,4)$	zigzags	∞?	N/A	3?
		weak zigzags	∞?	N/A	∞?
self-dual	$(3,4)$	zigzags	6	$GC_{2,1}(Tetrahedron)$	6
self-dual	$(2,4)$	zigzags	2	Bundle_2	∞

Table 1.4 *ZC*-structure of maps from Table 1.2; zc*Max* is the maximum number of zc-circuits of tight graphs; First *zc*Max is the first graph attaining the bound; *zc*PT is the number of *z*- or *cc*-tight pure graphs. For the case $k = 5$, or 6, distinct notions of tightness are used, see Chap. 5 for details.

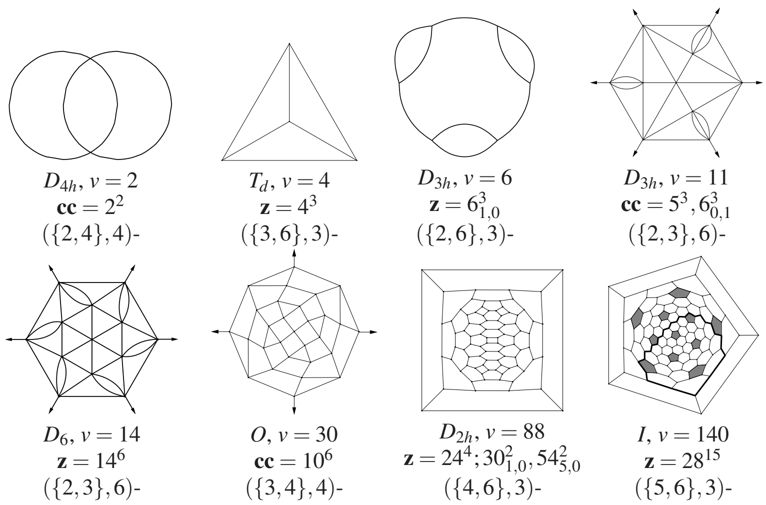


Fig. 1.14 Smallest tight $(\{a,b\},k)$-spheres with maximal number of *ZC*-circuits

1. 9 for $(\{5,6\},3)$-: C_i, C_{2h}, S_6, D_{2h}, D_{3d}, D_{5d}, D_{6h}, T_h, $\mathbf{I_h}$
2. 7 for $(\{2,3\},6)$-: C_i, C_{2h}, S_6, D_{2h}, D_{3d}, $\mathbf{D_{6h}}$, $\mathbf{T_h}$
3. 6 for $(\{4,6\},3)$-: C_i, C_{2h}, D_{2h}, D_{3d}, $\mathbf{D_{6h}}$, $\mathbf{O_h}$
4. 6 for $(\{3,4\},4)$-: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, $\mathbf{O_h}$
5. 2 for $(\{2,4\},4)$-: D_{2h}, $\mathbf{D_{4h}}$
6. 1 for $(\{3,6\},3)$-: D_{2h}
7. 0 for $(\{2,6\},3)$- and $(\{1,3\},6)$-
8. 12 for $(\{3,4\},5)$-: C_i, C_{2h}, C_{4h}, S_6, S_{10}, D_{2h}, D_{3d}, D_{4h}, D_{5d}, T_h, O_h, I_h
9. 2 for self-dual $(\{3,4\})$- and for self-dual $(\{2,4\})$-: C_{2h}, D_{2h}.

Each of 7 above families with bold-faced symmetry is described by one natural parameter and contains $O(\sqrt{v})$ spheres with at most v vertices.

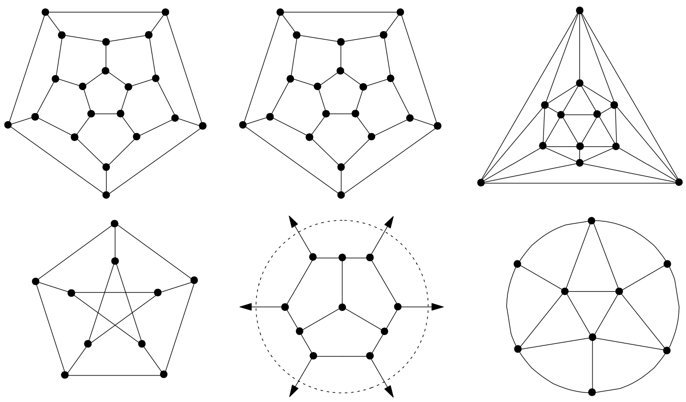
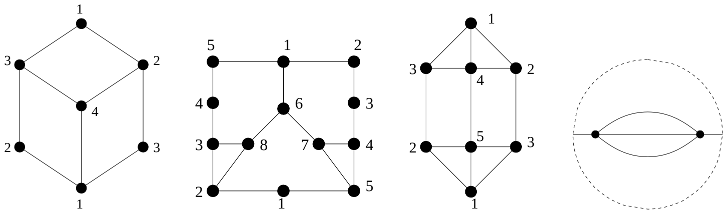


Fig. 1.15 The Petersen graph is the smallest $\mathbb{P}^2$-fullerene. Its $\mathbb{P}^2$-dual, K_6, is the smallest $\mathbb{P}^2$-icosahedrite (*half-Icosahedron*) and smallest $\mathbb{P}^2$-triangulation.

The smallest $(\{a,6\},3)$-$\mathbb{P}^2$ for $a = 5,4,3$ and $(\{3,4\},5)$-$\mathbb{P}^2$ are K_6^* (*Petersen graph*), K_4 (smallest $\mathbb{P}^2$-quadrangulation), K_4 truncated on 2 vertices and K_6, respectively. The smallest $(\{a,4\},4)$-$\mathbb{P}^2$ for $a = 3,2$ and $(\{2,3\},6)$-$\mathbb{P}^2$ are K_5 and points with $2,3$ loops; smallest ones without loops are Bundle$_4$, Bundle$_6$ but on $\mathbb{P}^2$. The smallest self-hedrite maps on $\mathbb{P}^2$ without 1- and 2-gons are given in Table 1.5; here $K_{2,2,2} - P_2$ is the antipodal quotient of the 12-th one in Fig. 4.21.

Clearly, for an $(\{a,b\},k)$-sphere with $a,b,k \geq 3$, all possible (a,k) are $(3,3)$, $(4,3)$, $(3,4)$, $(5,3)$ and $(3,5)$. Permitting $a = 2$, one get a b-vertex $(\{2,b\},k)$-sphere with $p_b = 2$ for any $b,k \geq 3$, not both odd: repeat every of disjoint $\lfloor \frac{b}{2} \rfloor$ edges of the cycle C_b i times, $1 \leq i \leq k-1$, and and repeat every other edge $k-i$ times (it should be $i = k-i$ for odd b). Permitting $a = 1$, one get, say, a $\frac{2b}{3}$-vertex $(\{1,b\},3)$-sphere with $p_b = 2$ if $b \equiv 0 \pmod 3$: put 1-gons, $\frac{b}{3}$ inside and $\frac{b}{3}$ outside, on the $\frac{2b}{3}$-cycle.

Now we give full (except only conjectured completeness for the case $(a,k) = (3,5), p_b = 3$) listing of $(\{a,b\},k)$-spheres with $p_b \leq 3 \leq a,b,k$, obtained explicitly



$(\{4,6\},3)$-$\mathbb{P}^2$ $(\{3,6\},3)$-$\mathbb{P}^2$ $(\{3,4\},4)$-$\mathbb{P}^2$ $(\{2,4\},4)$-$\mathbb{P}^2$

Fig. 1.16 Some bifaced maps on projective plane

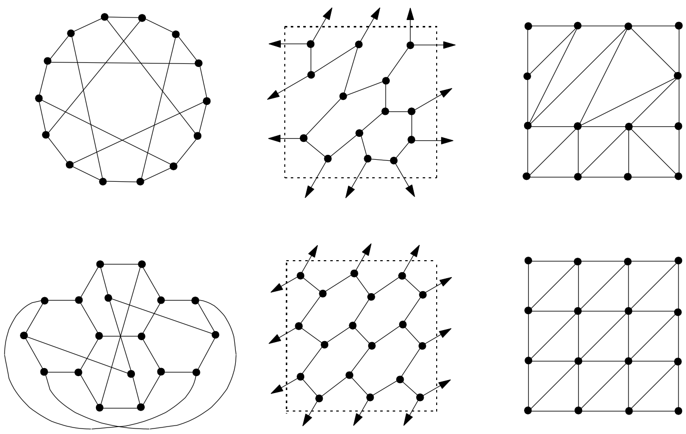


Fig. 1.17 The *Heawood graph* is the smallest $(\{6\},3)$-$\mathbb{T}^2$; its $\mathbb{T}^2$-dual K_7, is the smallest $(\{3\},6)$-$\mathbb{T}^2$. The $K_{3,3,3}$ is the smallest $(\{3\},6)$-$\mathbb{K}^2$; its $\mathbb{K}^2$-dual is the smallest $(\{6\},3)$-$\mathbb{K}^2$. The K_5 and $K_{2,2,2}$ are the smallest $(\{4\},4)$-$\mathbb{T}^2$ and $(\{4\},4)$-$\mathbb{K}^2$.

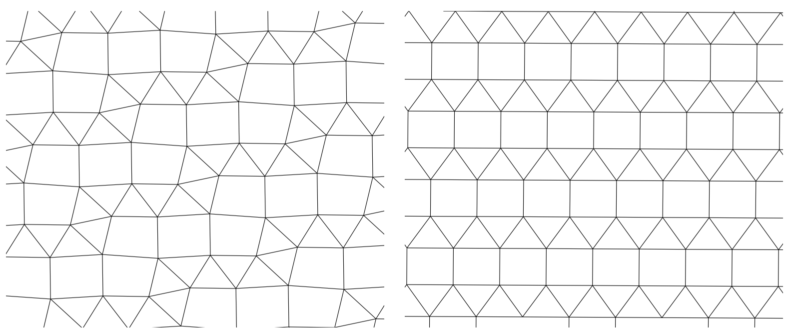


Fig. 1.18 The plane icosahedrites $(\{3,4\},5)$-$\mathbb{E}^2$, the quotient of which are the smallest $(\{3,4\},5)$-$\mathbb{T}^2$; their graphs are isomorphic to K_6

k	(a,b)	smallest on $\mathbb{S}^2$	on $\mathbb{P}^2$	on $\mathbb{T}^2$	on $\mathbb{K}^2$
3	$(5,6)$	Dodecahedron	K_6^*	K_7^*	$K_{3,3,3}^*$
3	$(4,6)$	Cube $=K_2^3$	K_4	K_7^*	$K_{3,3,3}^*$
3	$(3,6)$	Tetrahedron $=K_4$	2-trunc. K_4	K_7^*	$K_{3,3,3}^*$
3	$(2,6)$	Bundle$_3$=$3\times K_2$	-	K_7^*	$K_{3,3,3}^*$
4	$(3,4)$	Octahedron $=K_{2,2,2}$	K_5	K_5	$K_{2,2,2}$
4	$(2,4)$	Bundle$_4$=$4\times K_2$	$4\times K_2$	K_5	$K_{2,2,2}$
6	$(2,3)$	Bundle$_6$=$6\times K_2$	$6\times K_2$	K_7	$K_{3,3,3}$
6	$(1,3)$	Trifolium	-	K_7	$K_{3,3,3}$
5	$(3,4)$	Icosahedron	K_6	K_6	?
self-dual	$(3,4)$	Tetrahedron $=K_4$	$K_{2,2,2}-P_2$	K_5	$K_{2,2,2}$
self-dual	$(2,4)$	Bundle$_2$=$2\times K_2$	K_5-P_3	K_5	$K_{2,2,2}$

Table 1.5 Smallest bifaced maps without loops on irreducible surfaces

in [DDS13a]. The $(\{a,b\},k)$-spheres with $p_b \le 1$ are five Platonic solids $(a;k)$: Tetrahedron, Cube (*Prism*$_4$), Octahedron (*APrism*$_3$), Dodecahedron (snub *Prism*$_5$), Icosahedron (snub *APrism*$_3$). Given two circuits $u_1,\dots,u_m$ and $v_1,\dots,v_m$, an *m-sided prism Prism$_m$* is formed when every u_i is joined to v_i by an edge. An *m-sided antiprism Prism$_m$* is formed by adding the cycle u_1, v_2, u_2, v_3,…, v_m,u_m,v_1,u_1.

There is one *trivial* 3-connected $(\{a,b\},k)$-sphere with $p_b = 2$ for each of $(a,k) = (4,3),(3,4),(5,3)$ and $(3,5)$ – *Prism$_b$*, *APrism$_b$*, *snub Prism$_b$* and *snub APrism$_b$* – defined as two b-gons separated by b-ring of 4-gons, $2b$-ring of 3-gons, two b-rings of 5-gons and two $3b$-rings of 3-gons, respectively.

The all remaining such spheres are 10, for any $t \ge 2$, *non-trivial* $(\{a,at\},k)$-spheres with $p_{at} = 2$: 1,1,2,2 and 4 for $(a,k) = (3,3)$, $(4,3)$, $(5,3)$, $(3,4)$ and $(3,5)$, respectively. All have symmetry D_{th}, except two $(\{3,at\},5)$-spheres, one of C_{th} and one of D_t symmetry. All non-trivial $(\{a,ta\},k)$-spheres with $p_{ta} = 2$ and $k = 3,4,5$, are given, for the case $t = 2$, on Fig. 1.19.

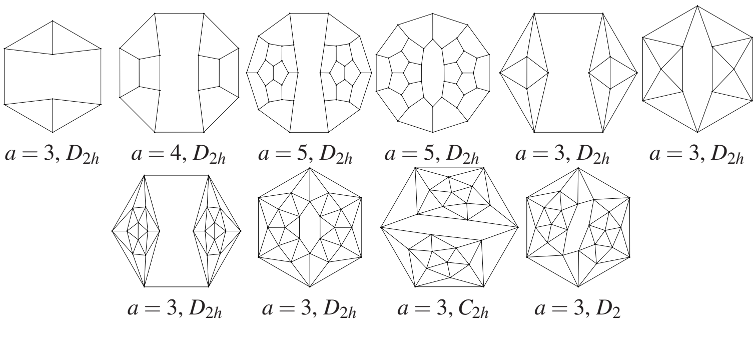


Fig. 1.19 All non-trivial $(\{a,2a\},k)$-spheres with $p_{2a} = 2$ and $k = 3,4,5$

Let $k \geq 3 \leq a < b$. An $(\{a,b\},k)$-sphere with $p_b = 3$ exists if and only if $b \equiv 2, a, 2(a-1) \pmod{2a}$ and, in addition for $a = 5$, $b \equiv 4, 6 \pmod{10}$. Such sphere are unique if b is not $\equiv a \pmod{2a}$ and then their symmetry is D_{3h}, except when $(a,k) = (3,5)$ when the symmetry is D_3. (There is also unique such $(\{a, a-1\},k)$-sphere for $(a,k) = (5,3)$, $(3,4)$, $(3,5)$.) For $b \equiv a \pmod{2a}$, there are $1, 2, 5, 3$ and ≥ 15 of such spheres for $(a,k) = (3,3)$, $(4,3)$, $(5,3)$, $(3,4)$ and $(3,5)$, respectively.

1.6 Computer generation of the families

Many results below are based on extensive computer computations. Main technique: exhaustive search. Sometimes, speedup by proving that a group of faces cannot be completed to the desired graph. The `GAP` computer algebra system ([GAP]), Plantri ([BrMK01]), for general graphs, and `CaGe` ([BFDH97]), for plane graph drawings, were used. The program `CPF` ([BrHaHe03]) generates 3-regular plane graphs with specified p-vector. `ENU` ([Heid98, BrHaHe03]) does the same for 4-regular plane graphs. `CGF` ([Har00]) generates 3-regular orientable maps with specified genus and p-vector. The second author adapted `ENU` to deal with 2-gonal faces also. His package `PlanGraph` ([Dut02]) was used for handling planar graphs in general.