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★**Geometry of cuts and metrics. (English summary)**
Algorithms and Combinatorics, 15.
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The central, unifying object of this book is a convex polytope $P_n \subset \mathbf{R}^{\binom{n}{2}}$, called the cut polytope, which is defined as follows. Let us interpret $\mathbf{R}^{\binom{n}{2}}$ as the vector space of all real-valued functions on the set of unordered pairs $\{i, j\} \subset \{1, \cdots, n\}$. For a subset $S \subset \{1, \cdots, n\}$, let $\delta_S(i, j)$ be the indicator function of the cut associated with S: $\delta_S(i, j) = 1$ if $i \in S$ and $j \notin S$ or $i \notin S$ and $j \in S$, and $\delta_S(i, j) = 0$ if $i \in S$ and $j \in S$ or $i \notin S$ and $j \notin S$. Finally, let us define P_n as the convex hull of all functions δ_S. The cut polytope P_n provides (for large n) a counterexample, found by J. Kahn and G. Kalai, to Borsuk's conjecture, which claimed that every convex body in $\mathbf{R}^d$ can be partitioned into $d + 1$ pieces whose diameter is strictly smaller than the diameter of the body. A point from P_n can be interpreted as a semimetric on $\{1, \cdots, n\}$, that is, a symmetric non-negative function which satisfies a triangle inequality. It turns out that the points from P_n are precisely those semimetrics ρ for which there exist a probability space (Ω, μ) and n events $A_1, \cdots, A_n \subset \Omega$ such that $\rho(i, j) = \mu(A_i \triangle A_j)$. Let K_n be the convex cone spanned by P_n, that is, the set of all non-negative combinations of the indicator functions of cuts. It turns out that the points from K_n are precisely those semimetrics ρ for which there is an isometric embedding of $\{1, \cdots, n\}$ into $\mathbf{R}^N$ with L^1 metrics. Moreover, non-negative integer combinations of δ_S are precisely those metrics that arise from an embedding into the Boolean cube $\{0, 1\}^n$.

These properties give rise to many interesting connections, described in the book, among polyhedral combinatorics, local Banach geometry, optimization, graph theory, geometry of numbers, and probability. For example, testing whether a given finite metric space is L^1 isometrically embeddable amounts to testing whether a given point of $\mathbf{R}^{\binom{n}{2}}$ belongs to the cut cone K_n. This problem is NP-complete and so it is reasonable to try to describe some linear inequalities satisfied by K_n, as necessary conditions for L^1 embeddability. A large class of valid inequalities for K_n is provided by the hypermetric inequalities $\sum_{1\le i<j\le n} b_i b_j x_{ij} \le 0$, where $b_1, \cdots, b_n$ are integer numbers summing up to 1. Of course, these are not all the inequalities satisfied by K_n (for $n \ge 6$) and, for a fixed n, all but finitely many of these inequalities are redundant. Finding out which inequalities are essential leads to questions in the geometry of numbers, to Delaunay polytopes and to Voronoĭ "finiteness" results for point lattices. On the other hand, testing whether a given point belongs to P_n is dual (both in the sense of convex geometry and algorithmically) to optimizing a given linear function on P_n. The problem of finding the maximum value of a given linear function on P_n is a well-known NP-hard problem of finding the maximum weight of a cut in a given weighted graph. Semidefinite programming (linear programming in the space of quadratic forms) was used by M. Goemans and D. Williamson to obtain good approximation algorithms and hence we get a connection to the structure of the cone of positive semidefinite forms and L^2 embeddability questions.

Applying the duality again, we come to the question of testing the membership $\rho \in K_n$ approximately. The book describes J. Bourgain's result on L^1 embeddability with a "distortion" and an application found by N. Linial, E. London, and Y. Rabinovich to multicommodity flows. Questions regarding embeddings into a Boolean cube lead to some interesting connections with graph theory and designs. The interpretation of the coordinates of a point $\rho \in P_n$ as the probabilities of the symmetric differences of some n events in a probability space leads to an interesting question in "computational probability".

In short, this is a very interesting book which is nice to have.

Reviewed by *Alexander I. Barvinok*