Figurate Numbers: presentation of a book

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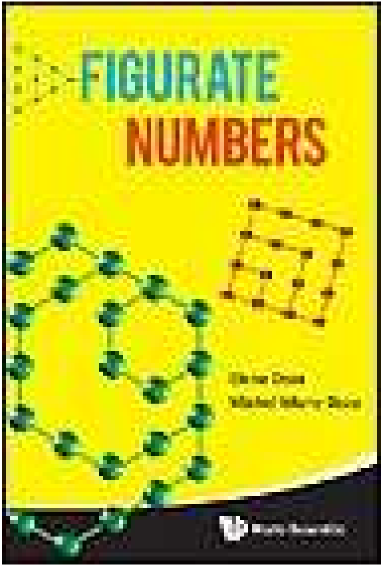
Overview

0. Overview


FIGURATE
NUMBERS

Overview

Figurate numbers, as well as a majority of classes of special numbers, have long and rich history.

They were introduced in Pythagorean school (VI-th century BC) as an attempt to connect Geometry and Arithmetic. Pythagoreans, following their credo "all is number", considered any positive integer as a set of points on the plane.

Overview

In general, a figurate number is a number that can be represented by regular and discrete geometric pattern of equally spaced points. It may be, say, a polygonal, polyhedral or polytopic number if the arrangement form a regular polygon, a regular polyhedron or a reqular polytope, respectively.

Figurate numbers can also form other shapes such as centered polygons, L-shapes, three-dimensional (and multidimensional) solids, etc.

Overview

The theory of figurate numbers does not belong to the central domains of Mathematics, but the beauty of these numbers attracted the attention of many scientists during thousands years.

Overview

Pythagoras of Samos (circa 582 BC - circa 507 BC),
Nicomachus (circa 60 – circa 120),
Hypsicles of Alexandria (190 BC – 120BC),
Diophantus of Alexandria (circa 200/214 - circa 284/298),
Theon of Smyrna (70 – 135),
Leonardo of Pisa, also known as Leonardo Pisano or Leonardo Fibonacci (circa 1170 - circa 1250),
Stifel (1487 – 1557),
Gerolamo Cardano (1501 – 1576),

Overview

Claude Gaspard Bachet de Méziriac (1581 - 1638),
John Pell (1611 - 1685),
Pierre de Fermat (1601 - 1665),
René Descartes, also known as Cartesius (1596 - 1650),
Barnes Wallis (1887 - 1979),
Leonhard Euler (1707 - 1783),
Wacław Franciszek Sierpiński (1882 - 1969)

- that is (not full) list of famous mathematicians, who worked in this domain.

Overview

However, if everyone knows about figurate numbers, there was no book on this topic, and one needed, for full description of it, find information in many different sources. Moreover, the information, which could be found, was very fragmented.

The main purpose of this book is to give thoroughful and complete presentation of the theory of figurate numbers, giving much of their properties, facts and theorems with full proofs.
In particular:

- to find and to organize much of scattered material such as, for example, the proof of famous Cauchy theorem on representation of positive integers as sums of at most m m-gonal numbers.
- to present updated material with all details, in clear and unified way.

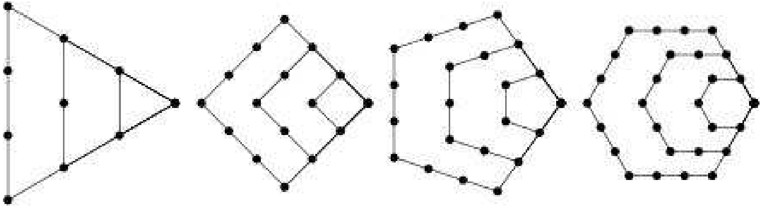
Contents

I. Chapter 1.
Plane figurate numbers

Chapter 1

Polygonal numbers generalize numbers which can be arranged as a triangle (triangular numbers), or a square (square numbers) to an m-gon for any integer $m \geq 3$.

Chapter 1



The above diagrams graphically illustrate the process by which the polygonal numbers are built up.

Chapter 1

The series of polygonal numbers consist of
triangular numbers 1, 3, 6, 10, 15, 21,... (Sloane's A000217),
square numbers 1, 4, 9, 16, 25, 36, ... (Sloane's A000290),
pentagonal numbers 1, 5, 12, 22, 35, 51,... (Sloane's A000326),
hexagonal numbers 1, 6, 15, 28, 45, 66, ... (Sloane's A000384),
etc.

Chapter 1

We consider in the Chapter 1 many interesting properties of polygonal numbers, including:

Theon formula:

$$S_3(n) + S_3(n-1) = S_4(n).$$

It can be demonstrated diagrammatically:

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & \cdot \\ * & * & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \end{array}$$

In the above example, constructed for $n = 4$, the square is formed by two interlocking triangles.

Chapter 1

The Diophantus' formula (or the Plutarch formula):

$$S_4(2n+1) = 8S_3(n) + 1.$$

The geometrical illustration for $n = 2$ is given below.

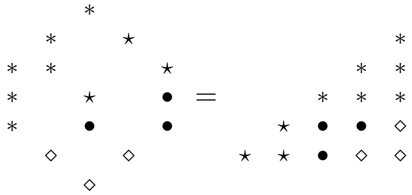
$$\begin{array}{ccccc}
\ast & \ast & \cdot & \cdot & \star \\
\ast & \bullet & \cdot & \star & \star \\
\bullet & \bullet & \diamond & \ast & \ast \\
\cdot & \cdot & \star & \ast & \bullet \\
\cdot & \star & \star & \bullet & \bullet
\end{array}$$

Chapter 1

The hexagonal number theorem:

$$S_6(n) = S_3(2n - 1).$$

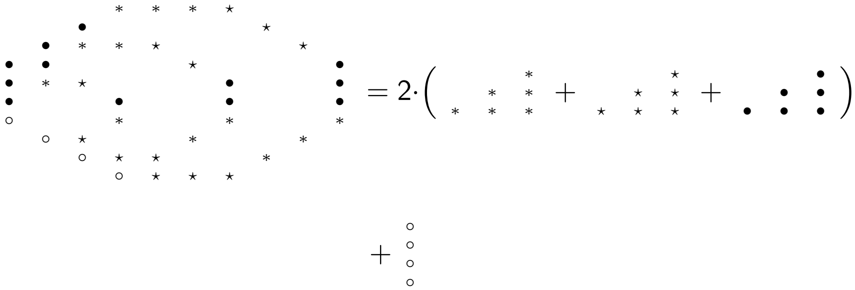
It means, that every hexagonal number is a triangular number.
The geometrical illustration of this property for $n = 3$ is thus.



The octagonal number theorem shows a connection between octagonal and triangular numbers:

$$S_8(n) = 6S_3(n-1) + n.$$

The geometrical interpretation of this property is given for $n = 4$:



Chapter 1

One has the following Nicomachus formula:

$$S_m(n) = S_{m-1}(n) + S_3(n-1).$$

The geometrical illustration of it is given below for $m = 4$, $n = 4$.

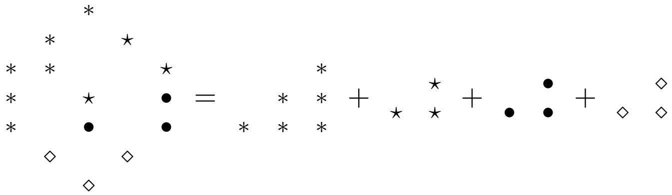
$$\begin{array}{cccc} * & * & * & * \\ * & * & * & \cdot \\ * & * & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \end{array}$$

Chapter 1

One has the following Bachet de Méziriac formula:

$$S_m(n) = S_3(n) + (m-3)S_3(n-1).$$

The geometrical illustration of it is given below for $m = 6$, $n = 3$.



Chapter 1

We consider also the highly polygonal numbers, i.e., positive integers, which are polygonal in two or more ways.

The most known class of such numbers form square triangular numbers 1, 36, 1225, 41616, 1413721, ... (Sloane's A001110). The indices of the corresponding square numbers are 1, 6, 35, 204, 1189, ... (Sloane's A001109), and the indices of the corresponding triangular numbers are 1, 8, 49, 288, 1681, ... (Sloane's A001108).

Chapter 1

Since every triangular number is of the form $\frac{1}{2}u(u+1)$, and every square number is of the form v^2, in order to find find all square triangular numbers, one seeks positive integers u and v, such that

$$\frac{1}{2}u(u+1) = v^2.$$

The problem of finding square triangular numbers reduces to solving the Pell's equation

$$x^2 - 2y^2 = 1.$$

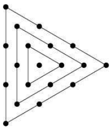
Chapter 1

Beyond classical polygonal numbers, centered polygonal numbers can be constructed in the plane from points (or balls).

Each centered polygonal number is formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than any side of the previous layer.

Chapter 1

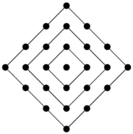
So, a centered triangular number represents a triangle with a dot in the center and all other dots surrounding the center in successive triangular layers.



The first few centered triangular numbers are
1, 4, 10, 19, 31, 46, 64, 85, 109, 136, ... (Sloane's A005448).

Chapter 1

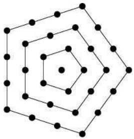
A centered square number is consisting of a central dot with four dots around it, and then additional dots in the gaps between adjacent dots.



The first few centered square numbers are 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, ...(Sloane's A001844).

Chapter 1

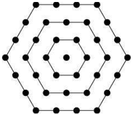
A centered pentagonal number represents a pentagon with a dot in the center and all other dots surrounding the center in successive pentagonal layers.



The first few centered pentagonal numbers are
1, 6, 16, 31, 51, 76, 106, 141, 181, 226, ... (Sloane's A005891).

Chapter 1

A centered hexagonal number represents a hexagon with a dot in the center and all other dots surrounding the center dot in a hexagonal lattice.



The first few centered hexagonal numbers are
1, 7, 19, 37, 61, 91, 127, 169, 217, 271, ... (Sloane's A003215).

Centered hexagonal numbers are most known among centered polygonal numbers. Usually, they are called hex numbers.

Chapter 1

In Chapter 1, we consider above and other plane figurate numbers with multitude of their properties, interconnections and interdependence.

II. Chapter 2. Space figurate numbers

Chapter 2

Putting points in some special order in the space, instead of the plane, one obtains space figurate numbers.
They, first of all, correspond to classical polyhedra.

Chapter 2

The most known are pyramidal numbers, corresponding to triangular, square, pentagonal, and, in general, m-gonal pyramids. They are given as sums of corresponding polygonal numbers.

$$S_m^3(n) = S_m(1) + S_m(2) + ... + S_m(n).$$

Chapter 2

In particular, the sequence of the tetrahedral numbers

$$S_3^3(n) = \frac{n(n+1)(n+2)}{6}$$

is obtained by consecutive summation of the sequence $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$ of triangular numbers, and begins with elements

1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ... (Sloane's A000292).

Chapter 2

Cubic numbers

$$C(n) = n^3$$

correspond to cubes, which are constructed from balls.

They form the sequence 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ... (Sloane's A000578).

Chapter 2

The octahedral numbers

$$O(n) = \frac{n(2n^2+1)}{3},$$

dodecahedral numbers

$$D(n) = \frac{n(3n-1)(3n-2)}{2},$$

and icosahedral numbers

$$I(n) = \frac{n(5n^2-5n+2)}{2}$$

correspond to the three remaining Platonic solids.

Chapter 2

Often one considers centered space figurate numbers.
Their construction is similar to the one for the centered polygonal numbers.

Chapter 2

Are considered also the numbers, which can be obtained by adding or subtracting pyramidal numbers of smaller size.
It corresponds to truncation of corresponding polyhedra or to putting pyramids on their faces, as in constructing of star polyhedra.

Examples of such numbers are:

truncated tetrahedral numbers,
truncated octahedral numbers,
rhombic dodecahedral numbers,
and stella octangula numbers.

Chapter 2

These and other classes of space figurate numbers are collected in Chapter 2.

III. Chapter 3. Multidim. figurate numbers

Chapter 3

Similarly, one can construct multidimensional figurate numbers, i.e., figurate numbers of higher dimensions k, but for $k \geq 4$ it loses the practical sense.

Chapter 3

In the dimension four, the most known figurate numbers are:

pentatope numbers,
and biquadratic numbers.

Chapter 3

The pentatope numbers

$$S_4^3(n) = \frac{n(n+1)(n+2)(n+3)}{24}$$

are 4-dimensional analogue of triangular and tetrahedral numbers and correspond to an 4-dimensional simplex, forming the sequence 1, 5, 15, 35, 70, 126, 210, 330, 495, 715, ... (Sloane's A000332).

Chapter 3

The biquadratic numbers

$$C^4(n) = n^4,$$

starting with 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ... (Sloane's A000583), are 4-dimensional analogue of square numbers and cubic numbers.

Chapter 3

The 4-dimensional regular polychoral numbers, corresponding to the other regular polychora, are

hyperoctahedral numbers,
hyperdodecahedral numbers,
hypericosahedral numbers,
and polyoctahedral numbers.

Their construction is more complicate, and these numbers rarely appear in the special literature.

Chapter 3

Among multidimensional figurate numbers, constructed for any dimension k, $k \geq 5$, we consider three classes of such numbers, corresponding to classical multidimensional regular polytopes:

multidimensional hypertetrahedron numbers,
multidimensional hypercube numbers,
and multidimensional hyperoctahedron numbers.

Chapter 3

Are considered also:

multidimensional m-gonal pyramidal numbers,
centered multidimensional figurate numbers,
and nexus numbers.

Chapter 3

The elements of the theory of multidimensional figurate numbers are given in Chapter 3.

IV. Chapter 4.

Areas incl. figurate numbers

Chapter 4

Many mathematical facts have deep connections with figurate numbers, and a lot of well-known theorems can be formulated in terms of these numbers.

Chapter 4

In particular, figurate numbers are related to many other classes of special positive integers, such as:

binomial coefficients,
Pythagorean triples,
perfect numbers,
Mersenne and Fermat numbers,
Fibonacci and Lucas numbers, etc.

A large list of such special numbers, as well as other areas of Number Theory, related to figurate numbers, is given in Chapter 4.

Chapter 4

One can easily find the family of figurate numbers in the Pascal's triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Chapter 4

In fact, any triangular number can be written in the form

$$S_3(n) = \binom{n+1}{2},$$

i.e., it is a binomial coefficient.

Therefore, the triangular numbers form 3-th diagonal 1, 3, 6, 10, 15, ... of the Pascal's triangle.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Chapter 4

Similarly,

$$S_3^3(n) = \binom{n+2}{3},$$

i.e., any tetrahedral number is a binomial coefficient.

So, the tetrahedral numbers form 4-th diagonal 1, 4, 10, 20, 35, ... of the Pascal's triangle.

Chapter 4

In general, for a given dimension k, $k \geq 2$, n-th k-dimensional hypertetrahedron number $S_3^k(n)$, representing an k-dimensional simplex, has the form

$$S_3^k(n) = \frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}.$$

Hence, k-dimensional hypertetrahedron numbers form $(k+1)$-th diagonal of the Pascal's triangle.

Chapter 4

Viewing from the opposite perspective, any element $\binom{n}{k}, n \geq 0, 0 \leq k \leq n$, of the Pascal's triangle is a figurate number, representing a k-dimensional analog of triangular numbers. In fact

$$\binom{n}{k} = S_3^k(n-k+1),$$

and we can rewrite the Pascal's triangle in the form

$$\begin{array}{ccccccccc}
 & & & & S_3^0(1) & & & & \\
 & & & S_3^0(2) & & S_3^1(1) & & & \\
 & & S_3^0(3) & & S_3^1(2) & & S_3^2(1) & & \\
 & S_3^0(4) & & S_3^1(3) & & S_3^2(2) & & S_3^3(1) & \\
S_3^0(5) & & S_3^1(4) & & S_3^2(3) & & S_3^3(2) & & S_3^4(1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

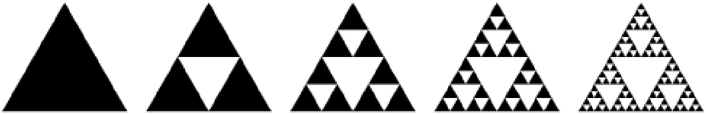
Chapter 4

In the other words, one can rewrite binomial theorem in the form

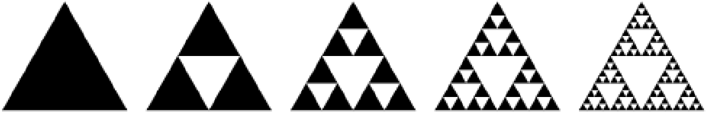
$$(1+x)^n = S_3^0(n+1)+S_3^1(n)x+S_3^2(n-1)x^2+...+S_3^{n-1}(2)x^{n-2}+S_3^n(1)x^n.$$

Chapter 4

The consideration of Pascal's triangle modulo 2 gives the sequence 1; 1, 1; 1, 0, 1; 1, 1, 1, 1; 1, 0, 0, 0, 1; ... (Sloane's A047999) and leads to an interpretation of the Sierpiński sieve, i.e., a fractal, obtained by consecutive deleting of the "central part triangle" in a given equilateral triangle, in each of three remaining triangles, etc.



Chapter 4



Obviously, the central white triangles, as well as other white triangles, arising in this construction, corresponds to some triangular numbers: the first white triangle has one point, giving $S_3(1) = 1$, the second one has 6 points, giving $S_3(3) = 6$, the third one has 28 points, giving $S_3(7) = 28$, etc.

In fact, n-th central white triangle has $2^{n-1}(2^n - 1)$ points, giving the triangular number $S_3(2^n - 1)$.

Chapter 4

This construction is closely connected with the theory of perfect numbers.

Remind, that a positive integer is called perfect number, if it is the sum of its positive divisors excluding the number itself.

The first perfect numbers are
6, 28, 496, 8128, 33550336, ... (Sloane's A000396).

Chapter 4

It is easy to show that all even perfect numbers are triangular.

In fact, the Euclid-Euler's theorem implies, that any even perfect number has the form

$$2^{k-1}(2^k - 1),$$

where $2^k - 1$ is a Mersenne prime.

Since $2^{k-1}(2^k - 1) = \frac{(2^k-1)2^k}{2}$, it holds

$$2^{k-1}(2^k - 1) = S_3(2^k - 1),$$

i.e., any even perfect number is a triangular number index of which is a Mersenne prime.

Chapter 4

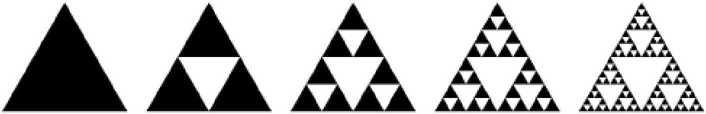
The largest known perfect number is

$$2^{43112608}(2^{43112609}-1)=S_3(2^{43112609}-1).$$

It has 25956377 decimal digits and corresponds to the largest known (in fact, 47-known) Mersenne prime $2^{43112609}-1$ having 12978189 decimal digits.

Chapter 4

So, we can say that any even perfect number is represented in the Pascal's triangle modulo 2.



Chapter 4

A positive integer of the form

$$M_n = 2^n - 1, \ n \in \mathbb{N},$$

is called Mersenne number.

The first few Mersenne numbers are
1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, ... (Sloane's A000225).

Chapter 4

This class of numbers has a long and reach history, which go back to early study of perfect numbers.
The main questions of the theory of Mersenne numbers are related the problem of their primality.

Chapter 4

A positive integer of the form

$$F_n = 2^{2^n} + 1, \; n = 0, 1, 2, 3, ...,$$

is called Fermat number.

The first few Fermat numbers are
3, 5, 17, 257, 65537, 4294967297, ... (Sloane's A000215).

Chapter 4

Like Mersenne numbers, which are connected with the theory of perfect numbers, Fermat numbers also are related to an old and beautiful arithmetical problem of finding constructible polygons, i.e., regular polygons that can be constructed with ruler and compass.

It was stated by Gauss that in order for a m-gon to be circumscribed about a circle, i.e., be a constructible polygon, it must have a number of sides given by

$$m = 2^r \cdot p_1 \cdot \ldots \cdot p_k,$$

where $p_1, \ldots, p_k$ are distinct Fermat primes.

Chapter 4

It is known that a Mersenne number greater than 1 can not be a square number:

$$M_n \neq k^2, \ k \in \mathbb{N} \backslash \{1\}.$$

Moreover, any Mersenne number greater than 1 can not be a non-trivial integer power:

$$M_n \neq k^s \text{ for an integer } k > 1 \text{ and a positive integer } s > 1.$$

Chapter 4

Therefore, there are no square Mersenne numbers, cubic Mersenne numbers, and biquadratic Mersenne numbers, except the trivial 1.

In general, there are no k-dimensional hypercube Mersenne numbers, greater than 1, for any dimension k, $k \geq 2$.

Chapter 4

On the other hand, it is easy to check that there are triangular Mersenne numbers:

$M_1 = S_3(1)$, $M_2 = S_3(2)$, $M_4 = S_3(4)$, and $M_{12} = S_3(90)$.

However, it is proven that the numbers $1, 3, 15$ and 4095 are the only triangular Mersenne numbers, i.e.,

$$M_v = \frac{u(u+1)}{2} \text{ only for } v = 1, 2, 4, 12.$$

Chapter 4

Similarly, a Fermat number can not be a square number, i.e.,

$$F_n \neq k^2, \; k \in \mathbb{N}.$$

Moreover, a Fermat number can not be a cubic number, i.e.,

$$F_n \neq k^3, \; k \in \mathbb{N}.$$

Chapter 4

Moreover, one can prove that a Fermat number can not be a non-trivial integer power:

$$F_n \neq k^s, \; k, s \in \mathbb{N}, \; s > 1.$$

Therefore, there are no biquadratic Fermat numbers, and, in general, there are no k-dimensional hypercube Fermat numbers for any dimension k, $k \geq 2$.

Chapter 4

It is easy to see that the number $F_0 = 3$ is the only triangular Fermat number, i.e.,

$$F_n \neq \frac{k(k+1)}{2}, \; n, k \in \mathbb{N}.$$

Chapter 4

Moreover, it is known, that the number $F_0 = 3$ is the only k-dimensional hypertetrahedron number in any dimension k, $k \geq 2$:

$$F_n \neq S_3^k(m) \text{ for any } k, m \in \mathbb{N} \text{ except } k = 2, m = 1.$$

This result can be interpreted by saying that the Fermat numbers sit in the Pascal's triangle only in the trivial way:

$$\text{if } F_m = \binom{n}{k} \text{ for some } n \geq 2k \geq 2, \text{ then } k = 1.$$

Chapter 4

The Fibonacci numbers are defined as members of the following well-known recurrent sequence:

$$u_{n+2} = u_{n+1} + u_n, \; u_1 = u_2 = 1.$$

The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...(Sloane's A000045).

Chapter 4

The Lucas numbers are defined by the same recurrent equation with different initial conditions:

$$L_{n+2} = L_{n+1} + L_n,\ L_1 = 1, L_2 = 3.$$

The first few Lucas numbers are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123,... (Sloane's A000032).

Chapter 4

It is known, that the numbers 1 and 4 are the only square Lucas numbers, i.e., square numbers which belong to the Lucas sequence.

Moreover, the numbers 2 and 18 are the only doubled square Lucas numbers.

Chapter 4

Similarly, the numbers 0, 1, and 144 are the only square Fibonacci numbers, i.e., square numbers which belong to the Fibonacci sequence,

while the numbers 0, 2 and 8 are the only doubled square Fibonacci numbers.

Chapter 4

We consider also the connections of figurate numbers with other special integer numbers, including:

Catalan numbers,
Stirling numbers,
Bell numbers,
Bernoulli numbers,
Sierpiński numbers, etc.

Chapter 4

The figurate numbers are closely associated with many types of Diophantine equations.

Chapter 4

For instance, the square numbers are involved in a natural way in the theory of Pythagorean triples, i.e., are closely connected with the Diophantine equation of the form

$$x^2 + y^2 = z^2.$$

The cubic numbers and their multidimensional analogues can be used to formulate Last Fermat's theorem.

A lot of Diophantine equations, in particular, many Pell-like equations, arise in the theory of highly figurate numbers.

Chapter 4

Due to the theory of the Pythagorean triples, we can establish, that there exist infinitely many square numbers, which are equal to the sum of two other square numbers:

$$S_4(k(m^2 - n^2)) + S_4(2kmn) = S_4(k(m^2 + n^2)).$$

Chapter 4

Fermat's theorem states, that the system

$$\begin{cases} x^2 + y^2 = z^2 \\ x^2 - y^2 = t^2 \end{cases}$$

has no solutions in positive integers.

In other words, there are no square numbers $S_4(x)$ and $S_4(y)$, such that the sum and the difference of them are square numbers.

Chapter 4

Now it is easy to show that there are no square numbers, so that the difference of their squares is a square number:

$$(S_4(x))^2 - (S_4(y))^2 \neq S_4(z).$$

Chapter 4

It implies, that the sum of two squared square numbers is not a squared square number:

$$(S_4(x))^2 + (S_4(y))^2 \neq (S_4(z))^2.$$

Chapter 4

This fact is equivalent to the special case $n = 4$ of the Fermat's last theorem: for any $n > 2$ the equality

$$x^n + y^n = z^n$$

has no positive integer solutions.

Chapter 4

It means (for $n = 3$) that a sum of two cubic numbers can not be a cubic number:

$$C(x) + C(y) \neq C(z).$$

Similarly, one obtains, for $n = 4$, that a sum of two biquadratic numbers can not be a biquadratic number:

$$BC(x) + BC(y) \neq BC(z).$$

In general, for $k \geq 3$, a sum of two k-dimensional hypercube numbers can not be a k-dimensional hypercube number:

$$C^k(x) + C^k(y) \neq C^k(z), \ k \geq 3.$$

V. Chapter 5. Fermat's polygonal number theorem

Chapter 5

Figurate numbers were studied by the ancients, as far back as the Pythagoreans, but nowadays the interest in them is mostly in connection with the following Fermat's polygonal number theorem.

In 1636, Fermat proposed that every number can be expressed as the sum of at most m m-gonal numbers.
In a letter to Mersenne, he claimed to have a proof of this result but this proof has never been found.

Chapter 4

Lagrange (1770) proved the square case:

$$N = S_4(m) + S_4(n) + S_4(k) + S_4(l).$$

Gauss proved the triangular case in 1796.

$$**E\Upsilon RHKA \qquad num = \Delta + \Delta + \Delta.$$

In 1813, Cauchy proved the proposition in its entirety.

Chapter 5

In Chapter 5, we give full and detailed proof of the Fermat's polygonal number theorem and related results in full generality, since it was scattered in many, often difficult to find publications. and never united in some book.

For example, the proofs by Cauchy and Pepin were only in extinct french mathematical journals, accessible only in several libraries.

Chapter 5

The Chapter 5 contains:

the entire proof of Lagrange's four-square theorem,
the entire proof of Gauss' three-triangular number theorem,
the Cauchy's and Pepin's proofs of general case, dealing with m-gonal numbers for $m \geq 5$,
as well as some generalizations made by Dickson, 1927-1929.

Chapter 5

We give the Landau's proof of the Gauss three-triangular number theorem, which is based on the theory of quadratic forms and the Dirichlet's theorem about primes in an arithmetic progression.

Moreover, we consider a short and elementary proof of the three-square theorem, based on the Minkowski convex body theorem.

VI. Chapter 6. Zoo of figurate-related numbers

Chapter 6

In a small Chapter 6, we collected some remarkable individual figurate-related numbers.

VII. Exercises

Chapter 7

The Chapter 7 contains more than 150 exercises, as well as the hints for their solutions.

Index

Finally, the huge Index lists all classes of special numbers mentioned in the text.