Octahedrites

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Abstract

Call *octahedrite* oc_n any 4-valent n-vertex polyhedron, whose faces are 3- and 4-gons only. The edges of an octahedrite, as of any Eulerian plane graph, are partitioned by its *central circuits*, i.e. those, which are obtained by starting with an edge and continuing at each vertex by the edge opposite the entering one. So, any octahedrite is a projection of an alternating link, whose components correspond to its central circuits.

Call an octahedrite *irreducible*, if it has no an *rail-road*, i.e. a circuit of 4-gonal faces, in which every 4-gon is adjacent to two of its neighbors on opposite edges. Examples of results below:

(i) Any irreducible octahedrite has at most six central circuits.

(ii) All irreducible octahedrites without self-intersecting central circuits are listed.

(iii) The octahedrites with highest symmetry, O_h and O, are classified.

(iv) The alternating 3- and $(k+2)$-links, whose projections are, respectively, an octahedral $oc_{96k^2+6}(O)$ and (generalizing Tait series of $3t$-sided antiprisms) Conway graph $((k+1)\times m)^*$ (for $k+2$ odd and $m=(k+2)t$) are Borromean.

(v) All *4-hedrites*, i.e. 4-valent plane graphs, whose faces are 2- and 4-gons only, are classified.

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1 Introduction

See [Grün67] for terms used here for polyhedra (a *polyhedron* means belowa only convex 3-polytope). It is well-known that the p-vector of any 4-valent polyhedron satisfies to $p_3 = 8 + \sum_{i=5}(i-4)p_i$. Some examples of applications of plane 4-valent graphs are *projections* of links, *rectilinear embedding* in VLSI and *Gauss crossing problem* for plane graphs.

Call *octahedrite* and denote by oc_n any 4-valent polyhedron with n vertices, whose faces are 3- and 4-gons only. We use this term because the octahedron is the minimal polyhedron of the family. The Euler relation $n - 2n + (p_3 + p_4) = 2$ implies that for octahedrites holds $p_3 = 8$ and $n = p_4 + 6$. It is given in [Grün67], page 282, that an oc_n exists if and only if $n \geq 6$, $n \neq 7$. The family of octahedrites is unique case of k-valent, $k > 3$, polyhedra with only a- and b-gonal faces, for which p_a is fixed for given (k, b). For 3-valent polyhedra, there are three such cases, all with $b = 6$ and $(a, p_a) = (5, 12), (4, 6), (3, 4)$; the first of them is well-known in Organic Chemistry family of *fullerenes*.

The octahedrites oc_n with $n \leq 9$ are: the octahedron oc_6, the 4-antiprism $oc_8 = APrism_4$ and the oc_9, smallest convex 4-valent convex polyhedron with odd number of vertices. The number of octahedrites oc_n for all $n \leq 50$ (18972 altogether) is computed in [Heid98]. All 46 oc_n with $n \leq 17$ are given on Figures 3, 8, 10 below. There are 25, 13, 30, 23, 51, 33, 76, 51, 109, 78, 144, 106, 218, 150, 274, 212, 382, 279, 499, 366, 650, 493, 815 of them for $n = 18, ..., 40$, respectively. All known space groups of symmetry of octahedrites are: O_h, O, D_{4d}, D_{4h}, D_4, D_{3d}, D_{3h}, D_3, D_{2d}, D_{2h}, D_2, C_{2h}, C_i, C_{2v}, C_2, C_s, C_1. They appear starting from $n = 6, 30, 8, 10, 18, 12, 9, 18, 14, 22, 10, 26, \leq 46, 11, 12, 14, 16$, respectively.

The dual of 4-valent polyhedra are exactly the case d=3 of *cubical* d-polytopes, i.e. convex d-polytopes, having, as facets, only combinatorial $(d-1)$-cubes. In particular, dual octahedrites are exactly *almost simple* (i.e. having only d and $d+1$ as vertex-degrees) cubical d-polytopes with $d = 3$. Amongst almost simple cubical d-polytopes there are *liftable* ones (i.e. with the boundary complex being a subcomplex of the boundary complex of $(d+1)$-cube) and *m-pillars of d-cubes*, i.e. those having the 1-skeleton $K_2^{d-1} \times P_m$ (the direct product of the $(d-1)$-cube and of the path with m edges, $m \geq 1$). [BlBl98] proves that *all* almost simple cubical n-vertex d-polytopes with $d \geq 4$ are:

(i) if $n \leq 2^{d+1}$: $\lfloor \frac{d^2}{4} \rfloor + d + 1$ combinatorial types of liftable d-polytopes, and

(ii) m-pillars of d-cubes, otherwise.

It is not difficult to see that all almost simple cubical 2-dimensional complexes are given by five liftable ones with at most 8 vertices (m-pillars $P_1 \times P_m$ of squares with $m = 1, 2, 3$ and the cube with deleted vertex or edge) and the m-pillars of squares $P_1 \times P_m$ with $m \geq 4$, otherwise. But for the remaining case $d = 3$, besides of the liftable 3-polytopes (dual Nrs. 6-1, 10-1, 12-1, 12-2, 14-1, 14-2 from Figures 8, 10 below) and m-pillars of cubes, there is rich variety of other dual octahedrites, which are the object of present study. A similar example of exceptionality of the 3-dimensional case is provided by perfect matroid-designs of rank $d+1$ with lines and planes of size 3, 9 (see [DeSa90]): for $d \geq 4$ this variety consists only of truncations of the affine geometry $AG(3, m)$, while for $d = 3$ it corresponds to the finite commutative Moufang loops.

2 Central circuits partition

In this Section we consider a connected plane graph G with all vertices of even degree, i.e. an Eulerian graph. Clearly, such graph has no cut-edges and we can, without loss of generality, suppose the absence of cut-vertices. Call a circuit in G *central* if it is obtained by starting with an edge and continuing at each vertex by the edge opposite the entering one; such circuit is called also *traverse* ([GaKe94]), *straight ahead* ([Harb97]), *straight-ahead* ([PTZ96]), *straight Eulerian* (Chapter 17 of [GoRo01]), *cut-through* ([Jeo95]), *intersecting* etc. Clearly, the edge-set of G is partitioned by all its central circuits; call this edge-partition *CC-partition*.

Denote by $CC(G) := (..., a_i^{\alpha_i}, ...; ..., b_j^{\beta_j}, ...)$ its *CC-vector*, where $..., a_i, ...$ and $..., b_j, ...$ are increasing sequences of lengths of all its central circuits, simple ones and self-intersecting, respectively, and α_i, β_j are their respective multiplicities. Clearly, $\sum_i a_i\alpha_i + \sum_j b_j\beta_j = 2n$, where n is the number of vertices of G; let $r(= r(G)) := \sum_i \alpha_i + \sum_j \beta_j$. Each of a_i, b_j is even if $r \geq 2$, because any two different circuits intersect in even number of vertices, and trivially if $r = 1$.

For a central circuit C, denote by $Int(C) := (c_0; ..., c_k^{\gamma_k}, ...)$, the *intersection vector of* C, where c_0 is the number of self-intersections of the circuit C and $(..., c_k, ...)$ is decreasing sequence of sizes of its intersection with other $r - 1$ circuits, while the numbers γ_k are respective multiplicities.

As an example of above notation, consider Nr. 18-1 of Figure 3 below. The CC-vector of this 18-vertex octahedrite oc_{18} is $(6, 8^2; 14)$ and the intersection vector is $(0; 2^3)$, $(0; 2^2, 4)$ and $(2; 2, 4^2)$ for the circuits of length $6, 8$ and 14, respectively.

2.1 Intersection of central circuits

The following Theorem is a local version (for "parts" of the sphere) of the Euler type equality $p_3 = 8 + \sum_{i=5}(i - 4)p_i$ for p-vector of any 4-valent polyhedron P.

Let A be a patch of P bounded by t arcs (paths of edges) belonging to central circuits (different or coinciding). So, A can be seen as a t-gon; we admit also 0-gonal A, i.e. just the interior of a simple central circuit. Suppose that the patch A is *regular*, i.e. the continuation of any of bounding arc (on the central circuit to which it belongs) lies outside of the patch; in other words, the patch contains, around each its vertex, only one (not three) amongst four angles, obtained when two central circuits intersect in this vertex. Let $p'(A) := p'_3, ...$ be the p-vector enumerating the faces of the patch A.

Theorem 1 *Let A be a patch, described above, of a polyhedron P. Then we have $p'_3 = (4 - t) + \sum_{i=5}(i - 4)p'_i$.*

In fact, denote the t-gonal boundary of A by C. Let us add along C and from each side of it, a ring of 4-gons of small width ϵ. Let us introduce on so modified polyhedron a metric, such that all faces became regular Euclidean polygons. We obtain a cell complex of an abstract polyhedron, which is homeomorph to the disc. The curvature of any vertex on C is zero, since it is now surrounded by squares. So, we can apply to the geodesic t-gon, bounded by C the following discrete analog (see [Alex50]) of Gauss-Bonnet's formula

$\sum_{1\leq i\leq t}\alpha_i + (t-2)\pi = \sum_j \omega_j$,

where α_i are angles of above t-gon (each of them is $\frac{\pi}{2}$ by construction) and all ω_j are curvatures in all vertices of P, which are interior with respect to our t-gon. Each ω_j is 2π minus the sum of four angles around the vertex; so an angle (say, α) contributes $\frac{\pi}{2}-\alpha$ to the curvature at the vertex. Therefore, an angle of a regular k-gon contributes $\frac{\pi}{6}$, 0 or a negative value for $k=3$, 4 or $k\geq 5$, respectively. Let us count the curvature $\sum_j \omega_j$ not by vertices j, but by regular k-gons in area, bounded by C. The angles of any such k-gon, $k\geq 3$, contribute

$\sum_{1\leq i\leq k}(\frac{\pi}{2}-\gamma_i) = \frac{k\pi}{2} - \sum_{1\leq i\leq k}\gamma_i = \frac{k\pi}{2} - (k-2)\pi = \frac{4-k}{2}\pi$,

where γ_i are angles of our k-gon.

So, $\frac{4-t}{2}\pi = \sum_{k\geq 3}\frac{4-k}{2}\pi p'_k$ and Theorem 1 is proved.

For octahedrites we have $0\leq t\leq 4$ and Theorem 1 implies $p'_3 = 4-t$.

We gave a metric proof of Theorem 1, but it can be proved combinatorially and even in more general form, with 1- and 2-gons permitted:

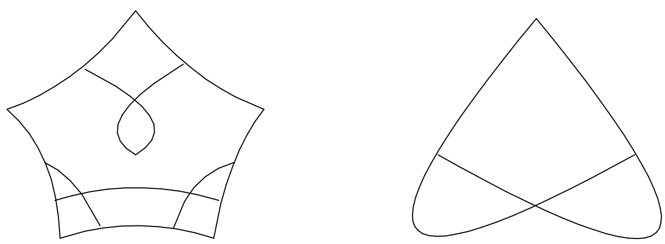
$$\sum_{i=1}(4-i)p'_i = 4-t$$

.

In fact, the patch A can be seen as a polyhedron with $v = v_2+v_3+v_4$ vertices, e edges and $f = 1+\sum_{i=1}p'_i$ faces, counting also the exterior face. Here v_i is the number of i-valent vertices and $v_2 = t$, v_3 is the number of remaining boundary vertices of the patch A, v_4 is the number of interior vertices of A. Clearly,

$2e = 2v_2+3v_3+4v_4 = 1(v_2+v_3)+\sum_{i=1} ip'_i$. Using Euler formula, one has

$8 = 4(v-e+f) = (4v-2e)+(4f-2e) = (2v_2+v_3)+(4+\sum_{i=1}(4-i)p'_i-(v_2+v_3))$, i.e. desired equality $4-t(=4-v_2) = \sum_{i=1}(4-i)p'_i$. See on the left-hand side of Figure below an example of regular patch A with $(v_2,v_3,v_4) = (5,8,3)$; on the right-hand side is given an irregular patch.



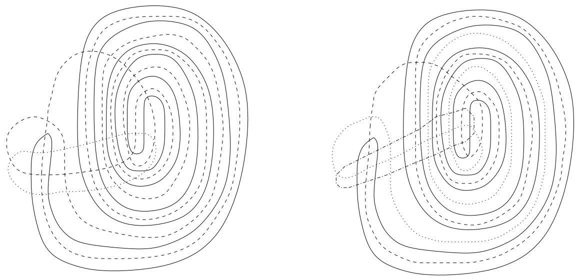


Figure 1: Octahedrites P_7 and P_8

Theorem 2 *Let C_1, C_2 be any two central circuits of an octahedrite. Then they are disjoint if and only if they are simple and there exist a ring of 4-gons separating them.*

In fact, if both C_1, C_2 are simple circuits, Theorem is evident: there are four triangles inside each of two disjoint circuits and so, two

circuits are separated by 4-gons only. Suppose that C_1 is self-intersecting. Then it has at least three regular patches and each of them contains at most three triangles. (For example, the form "eight", which is the second one on Figure 2 below, has two 1-gonal and one 2-gonal regular patches, containing, respectively, 3, 3 and 2 triangles.) The circuit C_2, being disjoint with C_1, lies entirely inside one of those patches, say, A. So, all its triangles, except, possibly, those from its exterior patch, lie in A. But the number of those triangles is at least four, since the exterior patch contains at most four triangles. It contradicts to the fact that A contains at most three triangles. So, C_2 intersects with C_1 and Theorem 2 is proved.

Claim 1 *For any $t \geq 1$ there exist an octahedrite P_t with $n = 8t-2$ vertices, having two simple central circuits of length $4t$ each and exactly $2t$ common vertices. The octahedrites P_1, P_2 are Nos. 6, 14-1. The triangles of $P_t, t > 1$, are organized into four isolated pairs of adjacent ones.*

If t is even, then remaining central circuits are C_3, C_4, having $Int(C_3) = Int(C_4) = (\frac{t}{2}-1; t^3)$.

If t is odd, then only one central circuit C_3 remains; it has $Int(C) = (2t - 2; (2t)^2)$.

In fact, Claim 1 can be verified by inspection of Figure 1.

Any $APrism_m^k$ (defined in Section 5 below) with gcd(m,k+2)=2 has two central circuits of length $m(k+1)$ with $\frac{mk}{4}$ self-intersections each and intersecting in $\frac{m(k+2)}{2}$ vertices. In particular, two central 20-circuits of $APrism_4^4 = oc_{20}$ intersect in 12 vertices, i.e. in $\frac{3}{5}$ of all of them. Two central 18-circuits of $APrism_6^2$ intersect in 12, i.e. in $\frac{2}{3}$ of all vertices.

2.2 Central circuits: self-intersections

Fix a central circuit C in given octahedrite. Denote by $k(C)$ the number of vertices of self-intersection of C. Call C *simple* if $k = 0$, i.e. if C is a cycle. Then C can be seen as a Jordan curve, i.e. a plane curve, which is topologically equivalent to the unit circle. Call a simple central circuit *equatorial* if its interior is isomorphic to its exterior.

All plane curves with at most three self-intersections and without k-gons having $k \geq 5$, are listed in Figure 2 and denoted, by G_0, G_1, G_2, G_2', $G_{3,1}$, $G_{3,2}$, respectively; cf. Figure 3.36 of [Grün72]. The pictures G_2, G_2' are different realizations on the sphere S^2 of the same graph. The number of other (on the plane, but not on the sphere) realizations is one for G_1, G_2, $G_{3,1}$ and two for G_2', $G_{3,2}$. In fact, the largest central circuits of Nos. 18-1 and 13-1 on Figure 3 correspond to such other plane realizations for G_2' and $G_{3,1}$, respectively. Figure 3 gives octahedrites oc_n, containing, respectively, central cycles of six forms listed on Figure 2, and having minimal number n of vertices.

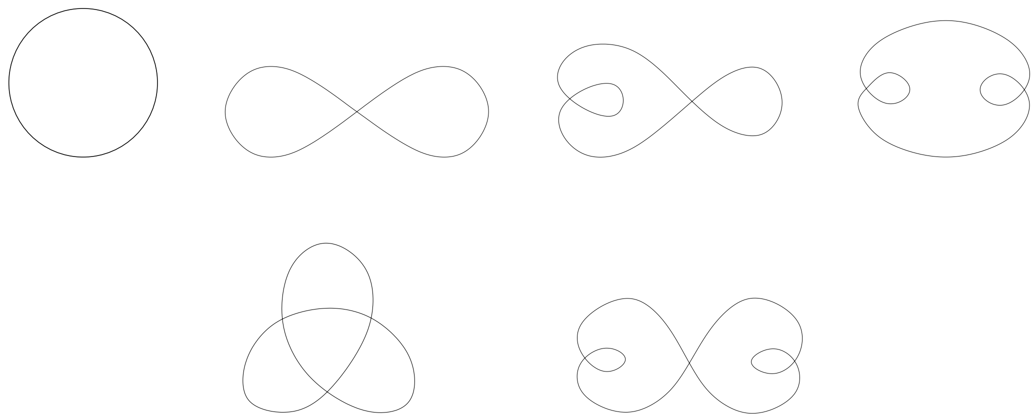
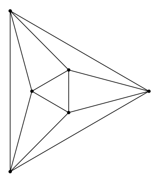
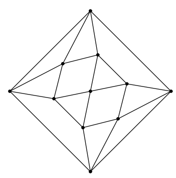


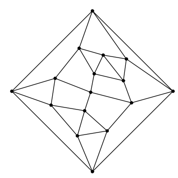
Figure 2: The curves with at most 3 self-intersections and without k-gons having $k \geq 5$: G_0, G_1, G_2, G_2', $G_{3,1}$, $G_{3,2}$



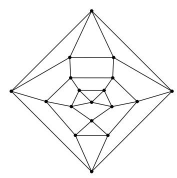
Nr.6-1 (4^3)
Group: O_h



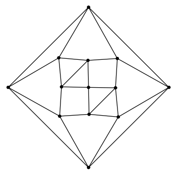
Nr.11-1 $(6^2; 10)$
Group: C_{2v}



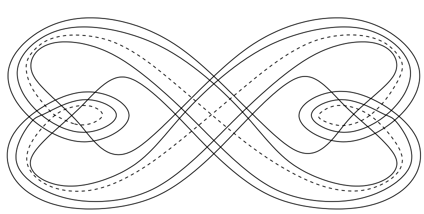
Nr.16-1 $(8^2; 16)$
Group: C_2



Nr.18-1
$(6, 8^2; 14)$
Group: C_{2v}



Nr.13-1 $(6^2; 14)$
Group: C_{2v}



Nr.69-1 $(30, 54^2)$
Group: C_{2v}

Figure 3: Minimal octahedrites, containing central cycles with at most 3 self-intersections (of form G_0, G_1, G_2, G'_2, $G_{3,1}$, $G_{3,2}$, respectively)

2.3 Equilinked octahedrites

Call an Eulerian graph G *balanced*, if all its central circuits of same length have the same intersection vector. Any oc_n with $n \leq 21$ is balanced, but Nr. 22-2 is not.

Call *equilinked* any balanced graph, such that deletion of any of its central circuits produces the same graph on the sphere. (For example, two central circuits of equilinked Nr. 14-3 are different as plane graph.) It will be interesting to characterize all irreducible (see the definition in the next Section) equilinked octahedrites. They generalize the case, when there is only one central circuit, i.e. Gaussian octahedrites.

Call an Eulerian graph G *Gaussian* if its CC-vector consists of only one circuit. Clearly, this unique central circuit is an Eulerian trail (i.e. it contains each edge exactly once), which self-intersects once in every vertex. It is well-known that any connected 4-regular graph admits an Eulerian trail but, in general, it is not central.

For any $n \geq 8$, except of 10 and 11, there exists a Gaussian octahedrite oc_n. The number of all Gaussian oc_n is 18 (see Figures 3, 8, 10) for $n \leq 17$ and $6, 6, 13, 7$ for $n = 18, 19, 20, 21$, respectively. It will be interesting to estimate the magnitude of the part of Gaussian oc_n amongst all oc_n (or all irreducible oc_n), when n increase.

Examples of non-Gaussian equilinked irreducible octahedrites are five ones (Nos. 6-1, 12-1, 12-2, 20-1, 30-1) from Figure 8, six with 2 central circuits and at most 20 vertices (Nos. 14-3, 18-8, 18-9, 20-2, 20-3), all irreducible octahedrites with symmetry O or O_h (see last Section). For irreducible $oc_n(O)$, the group O acts transitively on the central circuits (it acts regularly on 24 edges of 8 triangles); so, they are congruent. On the other hand, there are equilinked irreducible octahedrites (for example, Nr. 20-3) with trivial group.

Any $APrism_m^k$ (see Section 5) is equilinked irreducible graph, having $gcd(m, k+2)$ central circuits; it is an octahedrite $oc_{3(k+1)}(D_{3d})$ if $m = 3$, or $oc_{4(k+1)}$ (of symmetry D_{4h} or D_{4d} for k even or odd, respectively) if $m = 4$; see Nr. 15-1=$APrism_3^4$ and Nr. 20-2=$APrism_4^4$. Again the group is transitive on central circuits.

2.4 Pure octahedrites

We will say that an Eulerian graph G is *pure* if any of its central circuits simple, i.e. has no self-intersections. Easy to check that any pure octahedrite oc_n has an even number n of vertices.

An interesting example of such 4-valent plane graphs of polyhedra are those (with r central circuits, all simple ones), that the skeleton of the dual polyhedron is a spanning subgraph of the r-cube; such polyhedron corresponds to a *class of simple Venn r-diagrams* (see [Rus97] for a good survey). The only such classes for $r \leq 4$ are given by the octahedrites Nos. 6 and 14-1. No other octahedrite have this property, but (see Theorem 8.1 in [CHP96]) for any $r \geq 5$ there exist a 4-valent polyhedron with only 3-, 4- and 5-gonal faces, corresponding to a class of simple Venn r-diagrams.

Grünbaum considered in [Grün72] *arrangements of simple curves*, i.e. those that every two of them meet in precisely two points. Nos. 6-1, 12-1 (and 12-2), 20-1, 30-1 (all from Figure 8) give such arrangements for 3, 4, 5, 6 curves, respectively. The non-existence of other octahedrites, such that their central circuits form an arrangements of simple curves, is a corollary of Theorem 6 below.

Some examples of arrangements of r simple curves with other face-vector p, are:

(i) projections of the links 2^2_1 and 6^3_1 with, respectively, $r = 2$, $p = (p_2 = 4)$ and $r = 3$, $(p_2 = 3, p_3 = 2, p_4 = 3)$;

(ii) for any $r \geq 3$, the Conway graph $((r-1) \times r)^*$=$APrism_r^{r-2}$ (considered in Section 5.2 below) with $p = (p_3 = 2r, p_4 = r(r-3), p_r = 2)$;

(iii) medial graph (see definition in the next subsection) of fullerenes $F_{20}(I_h)$, $F_{28}(T_d)$, $F_{48}(D_3)$, $F_{60}(I_h)$, $F_{60}(D_3)$, $F_{88}(T)$, $F_{88}(D_{2h})$, $F_{140}(I)$ with, respectively, $r = 6, 7, 9, 10, 10, 12, 12, 15$ and $p = (p_3 = 20 + 2p_6, p_5 = 12, p_6)$ having $p_6 = 0, 4, 14, 20, 20, 34, 34, 60$. Here $F_n(G)$ denotes n-vertex fullerene (i.e. any simple 3-polytope with only 5- and 6-gonal faces) having G as the group of symmetry.

2.5 Medial graphs: $\frac{3}{2}$-inflation

For a plane graph G, denote by G^* its planar dual and *med-G* denotes its *medial* graph, i.e. the vertices of med-G are the edges of G, two of them being adjacent if the corresponding edges are adjacent and belong to the same face of the embedding of G in the plane. So, med-G=med-G^*. (If only adjacency of edges is required, we get the *line graph* of G.) Clearly, the med-G is a 4-valent plane graph and, for any oc_n, its medial med-oc_n is an oc_{2n} with all eight triangles being isolated. One can call med-G *$\frac{3}{2}$-inflation* of G, since (in terms of t-inflation graph G^t, considered in the next Section), we have $G^1 = G$ and G^2 =med-(med-G)). An example: Nos. 12-1, 12-2 (see Figure 8 below) are the medials for the projection of links 6^3_2 (the octahedron) and 6^3_1, respectively.

Given a polyhedron P, denote by P^* the polyhedron dual to it and by *med-P* the convex hull of the mid-points of all edges of a polyhedron P, i.e. it is the mid-edge truncation of P. Clearly, med-P=med-P^* and its skeleton is the medial graph of the skeletons of P and P^*. Since we are interested mainly in the skeletons of considered polyhedra, we will usually confound polyhedra with their graphs.

Any 4-valent connected planar graph H is med-G_1=med-G_2 for following planar dual graphs G_1, G_2, called *face graphs* of H. A 4-regular graph H partitions the plane into regions which can be 2-colored as on checkerboard. Take the regions of each color as vertices of graphs G_1, G_2, respectively, two of vertices being adjacent if the regions have a common vertex of H. The connection between graphs H and G_1, G_2 was explored in [Kot69]. In particular, the *Petri walks* (i.e. circuits, in which each edge is followed by edges going alternatively in the left and right direction) in G_1, G_2 correspond to the central circuits of H.

In Tables 1 and 3 of [DHL02] were given CC- and intersection vectors for the medials of semiregular polyhedra and of some fullerenes, respectively. In the last line of Table 1 there, should be "$m \equiv 1, 2 \pmod 3$" instead of "$m \neq 3$". Table 3 there gives (amongst examples with 56, 60, 84 and 88 vertices) a pair of 60-vertex fullerenes, having exactly 10 Petri walks being each simple and of length 18; it answer the question (on page 296 [Grün67]) if the p-vector together with the vector, listing types and lengths of Petri walks, determine the combinatorial type of a *simple* 3-polytope.

3 Cutting of octahedrites

Cutting of octahedrites is the main operation, considered in this paper. First, we will introduce associated notions; they are valid, in fact, for any 4-valent plane graph G. Informally, a cutting (of a zone) consists of drawing a closed plane curve C, going transversally through some edges of G (the closed sequence of adjacent faces of G, traversed consecutively by C, will be called a *zone*), so that C have at most double self-intersections and none of those lie on an edge of G. So, we obtain again a 4-valent plane graph, in which C is new central circuit. More exactly, a zone of G is the image of a simple ring of polygons on the plane, under a continuous locally-topologic cell-mapping f into the cell-complex (vertices, edges and faces) of G. The image (under the mapping f) of the middle section of the ring corresponds, if it is in general position (i.e. if it self-intersects and intersects some edges of G only in isolated points) to the cutting curve C. If this image is not in general position, we do small movements of the section of the ring, so that the image comes in general position.

The case when C has no self-intersections corresponds to the case when the cell-mapping f of the interior of the ring is an isomorphism. Call a zone *simple*, if it has no self-intersections. Call a zone *opposite*, if any polygon in it is adjacent to its neighbors on opposite sides. Call an opposite zone *alternating*, if the choice between left-opposite and right-opposite edges is done alternatively for every odd-gon; clearly, an alternating zone contains at most $2e$ faces, where e is the number of edges of G.

We are interested only in *good* cuttings of the plane graph G, i.e. those, which not change the p-vector of G; call corresponding zone *good* also. Clearly, in a good cutting every m-gon with $m \geq 4$ is adjacent to its neighbors on its non-adjacent edges. For octahedrite graph G it means that its cutting produces again an octahedrite only if every 4-gon in the zone is adjacent to its neighbors on opposite edges. But this necessary condition is not sufficient: for example, the cutting of an alternating zone of Nr. 12-2, visiting three times two opposite triangles and twice every 4-gon, produces a 6-gon on one of those triangles. The first Figure in Section 4 indicates other necessary, but still not sufficient, conditions in order to get good cutting of an octahedrite.

A good cutting changes the CC-partition of G only in the following way: new central circuit C is added and the length of any other central circuit increases by one for any intersection with C. The length of the new central circuit can be unlimited (for example, if G is the octahedron). On the other hand, if G is irreducible octahedrite with six central circuits (see Section 3.3. below), then the length of C is equal to the length of a central circuit of G, which is separated from C by appearing rail-road (see below).

In fact, for a fixed central circuit C, we define its *shores* as two circuits of faces, lying on the left and on the right of C. Clearly, each shore is a good zone; call such zones *shore-zones*.

Call *rail-zone* or *rail-road* any zone, which consists only of 4-gons; clearly, each rail-zone is a common shore-zone of two parallel central circuits, separated only by it. A rail-road can be seen as a central circuit in the dual octahedrite, avoiding the vertices of degree 3. On the other hand, all eight 4-gons of dual Nr. 8-1 provide example of a rail-road, which self-intersects in every its 4-gon,

The inverse operation to a cutting is a *deleting of the central circuit* C. It produces a 4-valent polyhedron P' having only k-gonal faces with $k \leq 4$ (but cases $k = 2, 1, 0$ are possible); two shore-zones of C collapse into a zone, which was cut by C.

In a special case, when Z is a shore-zone, we use, for the cutting of a zone Z, special term *inflation* (or, *elongation*) *along* C. In this case new central circuit V is separated from C by new rail-zone, which is the common shore of C and V. The inverse operation is a *deletion of a rail-zone*, i.e. of two (parallel neighboring) central circuits; it always produces an octahedrite.

For example, Nr. 22-2 is the inflation of Nr. 14-1 along a central 8-cycle. Nr. 16-2=med-$APrism_4$ is the cutting of the zone of eight triangles for Nr. 8-1=$APrism_4$ and Nr. 18-1 is a cutting of a zone of 14 triangles (having the form G'_2, see Figures 2, 3) of Nr. 6-1. Figure 8 lists the cuttings of all simple zones of Nr. 6-1, which are not inflation.

3.1 t-inflation

Call *t-inflation along* C the result of $t-1$ times repeated above operation. Call *t-inflation of given octahedrite*, the result of t-inflation done simultaneously along each of its central circuits.

For example, denote by $BPyr_4^t$ t-inflation of the octahedron, i.e. of the bipyramid $BPyr_4$, along fixed central 4-cycle. Its CC-vector is $((4^t), (2t+2)^2)$ with corresponding intersection vectors $(0; 2^2, 0^{t-1})$ and $(0; 2^{t+1})$. Its symmetry is D_{4h} for $t > 1$. Other example is an oc_{t^2+8t+2}, which is t-inflation of Nr. 11 along the central 10-circuit. Its CC-vector is $((4t+2)^2, (2t+8)^t)$ with corresponding intersection vectors $(0; 2, 4^t)$ and $(1; 4^2, 2^t)$. Its symmetry is C_{2v}.

In general, for any 4-valent plane graph G denote by G^t its *t-inflation*, i.e. the plane graph obtained from G by putting instead of each its central circuit, t parallel copies of it equispaced with "small enough" distance between any two neighboring copies. Remark that G^2=med-(med-P), in terms of medial graphs.

Claim 2 *Let C be a central circuit of G with $CC(G) = (..., a_i^{\alpha_i}, ...; ..., b_j^{\beta_j}, ...)$, and let $Int(C) = (c_0; c_1^{\gamma_1}, ..., c_r^{\gamma_r})$. Let C' be one of t parallel copies, which are put instead of the circuit C in G^t. Then $CC(G^t) = (..., ta_i^{t\alpha_i}, ...; ..., tb_j^{t\beta_j}, ...)$, $Int(C') = (c_0; c_1^{t\gamma_1}, ..., c_r^{t\gamma_r}, (2c_0)^{t-1})$.*

The 2-inflation of a Gaussian oc_i (for $i = 8, 9$ or $i \geq 12$) is an oc_{4i} with CC-vector $((4i)^2)$ and both intersection vectors being $(i; 2i)$. But there are also equilinked octahedrites oc_n (consisting of two copies of the same central circuit with i self-intersections) which are not 2-inflation of a Gaussian octahedrite. They not exist for $i = 0, 1, 2$, but exist for $i \geq 3$. The smallest such n for $i = 3, 4$ is $14, 18$, respectively (realized by Nos. 14-3, 18-8, 18-9). Returning to the case $i = 1$: there exists plane 4-valent graph (with p-vector $(p_2 = 2, p_3 = 4, p_4 = 2)$), consisting of two central cycles of form G_1 from Figure 2 (i.e. of form "eight"); it is, in terms of Section 4.1 below, an generalized octahedrite $R_{6,6}$. The octahedrite Nr. 15-1 consists of three central cycles of form G_1.

3.2 Regular-faced and face-regular octahedrites

In this subsection we present two interesting special classes of octahedrites, which were classified by direct computation. They are described using the notions of inflation and cutting. Here we articulate by bold letters the *semi-regular* (i.e. vertex-transitive) octahedrites, when they appear.

There are eleven *regular-faced* octahedrites, i.e. admitting regular polygons as the faces. They are **Nr. 8-1 (the 4-antiprism)**, five with an equatorial central cycle - Nos. **6-1 (the octahedron)**, **12-1 (the cuboctahedron)**, 12-2 (twisted **12-1**), 16-3, 16-2 (twisted 16-3) - and their respective inflation: Nos. 10-1, 18-3, 18-2, **24-1 (the rhombicuboctahedron)**, "14th Archimedean solid" 24-2. All above twists and inflation are around of an equatorial central cycle. Also Nos. **12-1**, 16-2, 18-2 and **24-1** are the medials of Nos. **6-1**, **8-1**, 9-1 and **12-1**, respectively.

Call an octahedrite *face-regular* if there exists a pair of natural numbers (t_3, t_4), such that any 3-gon is adjacent to exactly t_3 3-gons and any 4-gon is adjacent to exactly t_4 4-gons. There are nine face-regular octahedrites (see [DHL02] for details): four regular-faced ones (Nos. **6-1, 8-1, 12-1**, 10-1) and Nos. 14-1, 14-2, 14-3, 22-2, 30-1. All, but Nos. **8-1** and 14-3, are pure and can be obtained by cuttings of simple zones of the octahedron; amongst them Nos. **6-1, 12-1**, 14-1, 30-1 are irreducible.

3.3 Irreducible octahedrites

Call an octahedrite *irreducible*, if it cannot be obtained from another one by an inflation along a central circuit, i.e. if it has no a rail-road.

Call an irreducible octahedrite *maximal* irreducible, if it cannot be obtained from another one by a cutting of a zone; clearly, an octahedrite is maximal irreducible if and only if any of its good zones is a shore-zone (and so, their number is twice of the number of central circuits).

Theorem 3 *Any irreducible octahedrite has at most six central circuits.*

In fact, it is clear that any triangle is adjacent to at most three central circuits; so, altogether no more than 24 central circuits are adjacent to a triangle. Now, any central circuit is adjacent to a triangle, since otherwise it is disjoint with neighboring central circuit (they separated by a rail-road of 4-gons), i.e. the octahedrite is reducible. Moreover, any central circuit is adjacent to at least one triangle on each side, by the same argument. Let x_i denote the number of central circuits of *type* i, i.e. those, on which exactly i edges belong to a triangle; an edge, belonging to triangles from both sides, is counted twice. So, we have

$$\sum_{2\le i\le 24} ix_i = 24$$

and, in special case $x_2 = x_3 = 0$ (including pure irreducible octahedrites), we are done. Above equality implies $x_i \le 6$ for $i = 4$; $x_i \le 4$ for $i = 5, 6$; $x_i \le 3$ for $i = 7, 8$; $x_i \le 2$ for $9 \le i \le 12$; $x_i \le 1$ for $13 \le i \le 24$. We have $x_8 = 3$ for the octahedron, $x_6 = 4$ for the cuboctahedron and $x_4 = 6$ for Nos. 30-1, 32-1.

Lemma 1 *Any irreducible octahedrite satisfies to* $6x_2 + 5x_3 \le 24$.

In fact, call a triangle (denote it T) *lonely*, if it is adjacent to a central circuit (denote it C), but no other triangle is adjacent to the same circuit C from the same side. Denote $f(C)$ the central circuit, containing two sides of our lonely triangle T, which does not belong to C. Clearly, $f(C)$ self-intersects in the vertex of the triangle T, which does not belong to C. The side of T, which belongs to C, is a side of the sequence (possibly, empty) of adjacent 4-gons, which is concluded by a 3-gon (denote it T'). So $f(C)$ has at least two self-intersections and at least four edges belonging to the triangles T and T', all four from the side of T. At least one edge of $f(C)$ belongs to a triangle from other side; so $f(C)$ is of type i with $i \ge 5$. Also C is unique central circuit, which is adjacent to T, but not any other triangle from the same side. It implies that we can:

(i) associate to any central circuit of type 2 the six edges of two adjacent lonely triangles,

(ii) associate to any central circuit of type 3 the following five edges: three edges of unique adjacent lonely triangle and edges of adjacency with remaining two adjacent triangles,

(iii) observe that no edge can be associated by this way to two different central circuits of type 2 or 3.

Lemma is proved. This Lemma implies the bound $x_2, x_3 \leq 4$.

See on Figure 4 two examples with large x_2, x_3:

(a) an irreducible $oc_{66}(D_3)$, having three simple central circuits, any of which is adjacent to two lonely triangles, and three twice self-intersecting central circuits of type 6 each; so all non-zero x_i are $x_2 = x_6 = 3$;

(b) an irreducible $oc_{44}(D_{2d})$, having five simple central circuits, four of which are adjacent to one lonely triangle, and one (four times) self-intersecting central circuit of type 8; so all non-zero x_i are $x_3 = 4$ and $x_4 = x_8 = 1$.

There exists also an irreducible $oc_{64}(D_{4h})$, having four simple central circuits (of length 14), any of which is adjacent to two lonely triangles, and two 4 times self-intersecting central circuits (of length 36) of type 8 each; so all non-zero x_i are $x_2 = 4$, $x_8 = 2$.

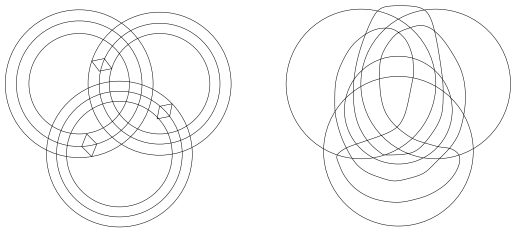


Figure 4: Irreducible octahedrites $oc_{66}(D_3)$ and $oc_{44}(D_{2d})$ with CC-vectors $(14^3; 30^3)$ and $(12^5; 28)$

Returning to the proof of the Theorem 3, we extract from the Lemma 1 the list of all possible pairs x_2, x_3. In each case, except trivial one $x_2 = x_3 = 0$, there exists (see the proof of Lemma 1) at least one central circuit of some type i, having $i \geq 5$. So we have $2x_2 + 3x_3 \leq 19$ and easy scanning of the cases shows that the number of central circuits is at most seven. Simple, but tedious analysis of the cases permit reduce this number to desired six. The main tool is to get, for each case, more information (than simple "there exist at least one") about central circuits of type i with $i \geq 5$.

3.4 Extendibility

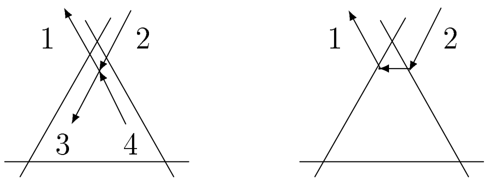
Call an irreducible octahedrite *non-extendible*, if it cannot be obtained from another irreducible octahedrite by deleting a central circuit.

Theorem 3 above implies that any irreducible octahedrite with six central circuits is non-extendible. We conjecture that any irreducible octahedrite with $r, r \leq 5$, central circuits is extendible and prove it partially in Theorem below.

Theorem 4 *Any octahedrite with unique central circuit is extendible.*

In fact, fix a triangle in given Gaussian octahedrite G. Let us add, on a small enough distance from its edges, an rail-road of 4-gons. If this rail-road goes, inside of fixed triangle, along of only *one* side, then take rail-road, which goes along this side, but outside of the triangle. Then the new rail-road goes along two other sides inside of the triangle. So, without loss of generality, we can suppose that our rail-road goes, inside of the triangle, along *two* sides. So, new central circuit has a vertex of self-intersection inside of the triangle. Let us move from this vertex (say, A) traversing the rail-road, along new central circuit.

Consider the left-hand side of Figure below. We cannot come back to the vertex A (during our first comeback to triangle) from 4, since, otherwise, our octahedrite is not Gaussian. We cannot come to A from 3, since we left by 1 to the left side. So, after leaving triangle to 1, we come back to A from 2, continue to 3 and return again to triangle from 4. Now let us cut out all the part of our walk from A (to 3) and back (from 4) and observe the result of this surgery on the right-hand side of Figure below. We got new octahedrite, which will be desired extension of G, if we show that it has no rail-roads. There are two possibilities to have such a rail-road: 1) if a central circuit goes through our triangle in parallel with its lower side, cutting the trapezoid on the right-hand side of the Figure into two 4-gons, and 2) this trapezoid is not cut, but it is a part of some rail-road. An rail-road of type 1) is not possible, since parallel sides of the trapezoid belong to the same central circuit; the existence of an rail-road of type 2) contradicts to the fact that the unique circuit of G and its copy on the small distance have the same length. So, new octahedrite is irreducible and an extension of G.



Now from Gaussian octahedrites we turn attention to the opposite (in terms of self-intersecting) case of pure ones.

4 Classification of pure octahedrites and 4-hedrites

The main result of this Section is that *any* pure irreducible octahedrite is one of eight given on Figure 8; so, any octahedrite without self-intersecting central circuits can be obtained from one of those eight by some inflation (adding rail-roads of 4-gons). Remark first, that five of them - Nos. 6-1, 12-1, 12-2, 20-1, 30-1 - have any two distinct central circuits intersecting exactly in two vertices, while for the remaining three - Nos. 14-1, 22-1, 32-1 - occur, in addition, exactly one intersection in four vertices. Moreover, all eight come by some cuttings from the octahedron. In Figure 9 they are mentioned, together with projections (given in Figure 7) of five well-known links 0_1^1, 2_1^2, 4_1^2, 6_1^3, 8_6^3 and of some 4-link, denoted below by $R_{6,14}^1$. Here and below all links are given in Rolfsen's notation (see the table in [Rol76] and also, for example, [Kaw96], [Weis99]) for links with at most 9 crossings and knots with 10 crossings, or, otherwise, in Dowker-Thistlewhaite's numbering (see [T]). We use notation $A \to B$ if graph A can be obtained from B by deleting one central circuit; the Figure 9 includes all such relations involving the pure irreducible octahedrites.

4.1 Generalized octahedrites

Call an *generalized octahedrite* or, more precisely, call an *i-hedrite* and denote by $R_{i,n}$ any plane 3-connected n-vertex 4-valent graph, such that the number p_j of its j-gonal faces is zero for any j, different from $3, 4$ and 2 (i.e. now 2-gons are permitted), and such that $p_2 = 8 - i$. So, an i-hedrite has $p_4 = n + 2 - i$ 4-gons and $p_2 + p_3 = i$ other faces. Clearly, $(i; p_2, p_3) = (8; 0, 8)$ (octahedrites) and $(i; p_2, p_3) = (7; 1, 6)$, $(6; 2, 4)$, $(5; 3, 2)$, $(4; 4, 0)$ are all possibilities.

The i-hedrites with at most 11 vertices are reduced projections of following alternating links (we write $11^2_?$ for two undecided 6-hedrites and $\approx n^a_b$ if the projection, given in the table in [Rol76], i.e. for $n \le 9$ crossings and $(n, a) = (10, 1)$, or, otherwise, in Dowker-Thistlewhaite's list, was different plane graph) –

for $i = 4$: 2^2_1, 4^2_1, 6^2_2, $\approx 8^2_4$, 8^3_4, 8^4_1, $\approx 10^2_{120}$, 10^2_{121};

for $i = 5$: 3_1, 5_2, 6^2_3, 6^3_1, $\approx 7_5$, 7^2_5, 7^3_1, $\approx 8_{15}$, $\approx 9_{18}$, 9_{38}, 10^2_{85}, 10^3_{155}, 10^4_{173}, $\approx 11_{124}$, $\approx 11_{236}$, 11^2_{226}, 11^3_{500}, 11^4_{547};

for $i = 6$: 4_1, 5^2_1, 6_3, $\approx 6^2_3$, $\approx 7_7$, $\approx 8_{12}$, 8_{17}, $\approx 8^2_8$, 8^2_{14}, 8^3_6, $\approx 9_{31}$, 9_{33}, 9^2_{38}, 9^3_{11}, 9^3_{12}, $\approx 10_{45}$, $\approx 10_{88}$, 10_{115}, 10_{120}, 10^2_{43}, 10^2_{86}, $\approx 10^2_{87}$, 10^3_{136}, $\approx 10^3_{136}$, 11_{297}, 11_{332}, $\approx 11_{125}$, 11^2_{317}, 11^2_{351}, two $11^2_?$;

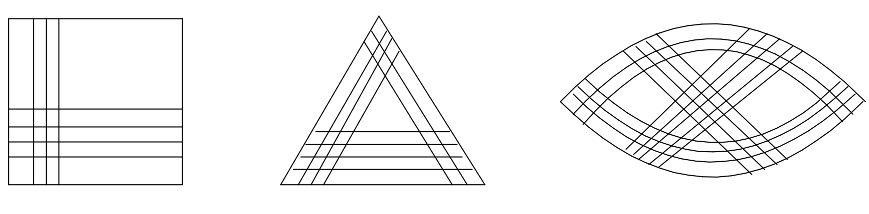
for $i = 7$: 7^2_6, 8^2_{13}, 9_{34}, 10_{121}, $\approx 10^2_{69}$, 10^2_{111}, 11_{288}, 11_{301}, 11^2_{150}, 11^3_{487};

and for $i = 8$: 6^3_2, 8_{18}, 9_{40}, 10^2_{56}, 10^4_{169}, 11^3_{520}.

So, amongst all alternating links with at most 11 crossings and except of two undecided 6-hedrites (being reduced projections of at most two 2-links) only two, 6^2_3 and 10^3_{136}, admit more than one i-hedrite as reduced projection.

There are $4, 3, 14, 5, 5$ 12-vertex i-hedrites with $i = 4, 5, 6, 7, 8$, respectively; amongst projections of 1288 alternating knots with 12 crossings, given in [T], there are only three i-hedrites: octahedrites Nos. 12-4, 12-3 for 12_{868}, 12_{1019} and a 6-hedrite for 12_{1102}. Remark also that the projections of alternating links 8^3_4 and 8^4_1, given in citeR, are 4-hedrites, containing an rail-road.

See on Figure below all ways of intersection of 4-, 3- and 2-gons by new central circuits, i.e. all ways how any octahedrite can be obtained by cutting. Using, for example, this Figure for unique 2-gon of any $R_{7,n}$, one can see that it *always* contains a self-intersection. On the other hand, any $R_{4,n}$ *never* contains a self-intersection.



It turns out that the class of 4-hedrites admits relatively easy full classification. First, any of them is pure and has even number of vertices. In fact, a self-intersection of some central circuit will imply that the 1-gonal patch, formed by it, contains a 3-gon, a contradiction.

Theorem 5 *Let R be any 4-hedrite $R_{4,2m}$.*

(i) R is irreducible if and only if it has exactly two central circuits; there exist pair of integers $t_1, t_2 \geq 0$ and unique irreducible 4-hedrite R_0 with $2m_0$ vertices, such that R can be obtained from R_0 by simultaneous t_1- and t_2-inflation, respectively, along two central circuits of R_0 (so, $m = m_0(t_1+1)(t_2+1)$).

(ii) The group of symmetry of R is one of five groups (see Figure 6.2) D_{4h}, D_{2d}, D_{2h}, D_4, D_2; it is D_2 unless it is:

(ii1) D_{4h}, if $t_1 = t_2$ and $m_0 = 1, 2$;

(ii2) D_{2d}, if $t_1 \neq t_2$, $m_0 = 2$ or if $t_1 = t_2$, m_0 is odd and R_0 has two isolated pairs of vertex-intersecting 2-gons;

(ii3) D_{2h}, if $t_1 \neq t_2$, $m_0 = 1$ or if $t_1 = t_2$, m_0 is even and R_0 has two isolated pairs of vertex-intersecting 2-gons;

(ii4) D_4, if $m = a^2 + b^2$, for some integers $a > b > 0$, and R is organized as illustrated on Figure 6.3; in terms of R_0, $t_1 = t_2 = gcd(a,b) - 1$, R_0 is of symmetry D_4 and $m_0 = (\frac{a}{gcd(a,b)})^2 + (\frac{b}{gcd(a,b)})^2$. Above case (ii1) can also be seen as the sub-case $b = 0, a$ of the case (ii4).

(iii) R is, up to a t-inflation along a central circuit, defined (see example for $2m = 14$ vertices on Figure 5) by the shift by $i, i = 0, 1, 2, ..., [\frac{m}{2}]$ vertices between the pair of boundary 2-gons of the horizontal circuit and the remaining pair of 2-gons. Moreover:

(iii1) R with shift $i = 1$ is irreducible and has, for $m \geq 3$, two isolated pairs of vertex-intersecting 2-gons (see examples of this series on Figure 6.1);

(iii2) All R with shift $i, 1 < i < [\frac{m}{2}]$ have four isolated 2-gons; amongst them are possible isomorphisms (for example, between 3rd and 4th on Figure 5) and not irreducible R (for example, for any shift $i, i \geq 2$, and $m \equiv 0 \pmod{2i}$ there exists R having $i+1$ central circuits (all, but the horizontal one, are concentric).

(iii3) The symmetry of irreducible R is D_2 if and only if interchange of central circuits changes the value of shift.

The number of irreducible 4-hedrites $R_{4,n=2m}$ is 1 for $m = 1, 2, 3, 4, 6$ (of symmetry D_{4h}, D_{4h}, D_{2d}, D_{2h}, D_{2h} and with shift $0, 1, 1, 1, 1$, respectively). It is 2 for $m = 5, 7, 8, 9, 10, 12$; is 3 for $m = 11$; is 4 for $m = 13$, respectively. Two $R_{4,10}$ have symmetry D_{2d}, D_4 and shift $i = 1, 2$, respectively; two $R_{4,14}$ have symmetry D_{2d}, D_2 and shift $i = 1, 2$ (or, interchanging the circuits, 3), respectively. The cases $m = 1, 2$ are depicted as 2nd, 3rd on Figure 7; the cases $m = 2, 3, 4$ and $R_{4,10}(D_4)$, $R_{4,14}(D_2)$ are given on Figure 6.2; the cases $m = 4$ and $R_{4,10}(D_{2d})$ are on Figure 6.1.

Remark that, for any $m > 1$, the result of m-inflation (see Section 3.1.) of the $R_{4,2}$ (i.e. minimal projection of 2^2_1) along one of its two circuits, is 2-connected, but not 3-connected plane map with symmetry D_{2h}; it is minimal projection of a *composite* alternating (m+1)-link. Remind also, that the groups above are space (point) groups of symmetry of realizations of graph on the sphere; so, the full automorphism group of the graph can be larger.

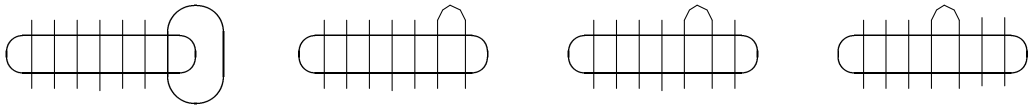


Figure 5: The shifts $i = 0, 1, 2, 3$ for 4-hedrites $R_{4,14}$: 3rd and 4th are isomorphic; the number of central circuits and the symmetry group is 8, 2, 2 and D_{2h}, D_{2d}, D_2 for 1st, 2nd, 3rd, respectively

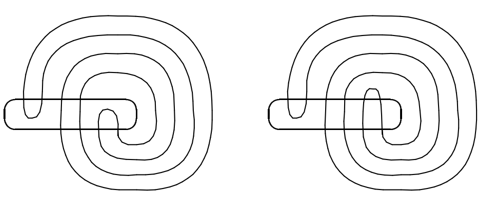


Figure 6.1: The cases $n = 8, 10$ of the series of irreducible 4-hedrites $R_{4,n}$ with two pairs of intersecting 2-gons, i.e. with shift $i = 1$

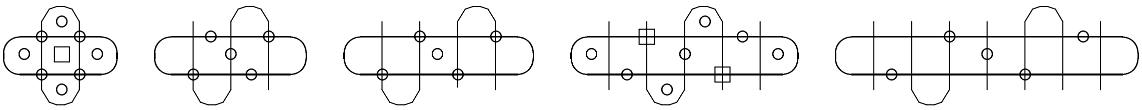


Figure 6.2: Smallest irreducible 4-hedrites $R_{4,2m}$ with $p_4 > 0$ and all possible groups of symmetry: $(2m, Group)$=$(4, D_{4h})$, $(6, D_{2d})$, $(8, D_{2h})$, $(10, D_4)$, $(14, D_2)$

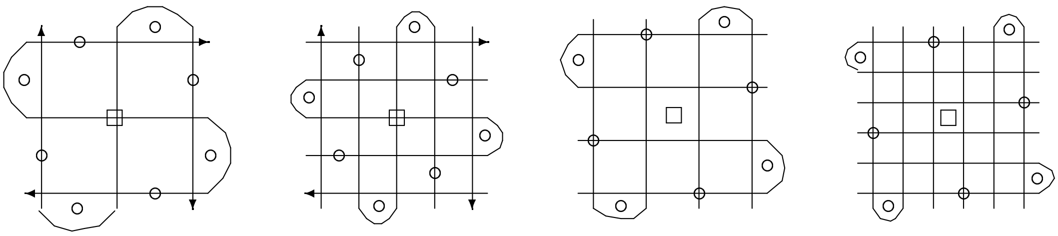
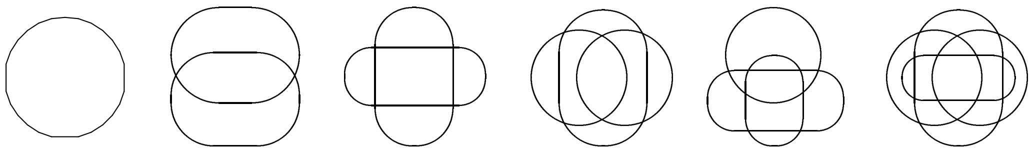


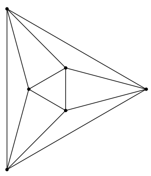
Figure 6.3: The cases (a, b)=$(2, 1)$, $(3, 2)$ and $(3, 1)$, $(5, 1)$, respectively, of the irreducible 4-hedrite $R_{4,2m}$, $m = a^2 + b^2$, with symmetry D_4



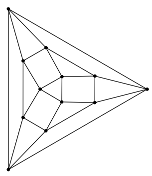
0_1^1 (0^1)	2_1^2 (2^2)	4_1^2 (4^2)	6_1^3 (4^3)	8_6^3 $(4, 6^2)$	$R_{6,14}^1$ $(6^2, 8^2)$
Group: $D_{\infty h}$	Group: D_{4h}	Group: D_{4h}	Group: D_{3h}	Group: D_{2h}	Group: D_{2h}

Figure 7: Projections of all alternating links, to which reduce the octahedrites of Figure 8 and which are not octahedrites

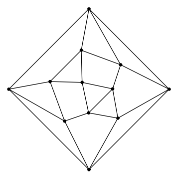
4.2 Classification of pure irreducible octahedrites



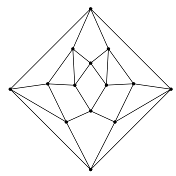
Nr.6-1 (4^3)
Group: O_h



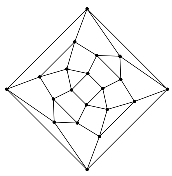
Nr.12-1 (6^4)
Group: O_h



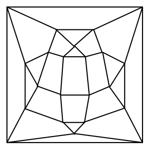
Nr.12-2 (6^4)
Group: D_{3h}



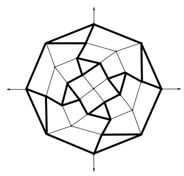
Nr.14-1 $(6^2, 8^2)$
Group: D_{4h}



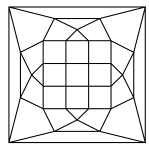
Nr.20-1 (8^5)
Group: D_{2d}



Nr.22-1 $(8^3, 10^2)$
Group: D_{2h}



Nr.30-1 (10^6)
Group: O



Nr.32-1
$(10^4, 12^2)$
Group: D_{4h}

Figure 8: All pure irreducible octahedrites

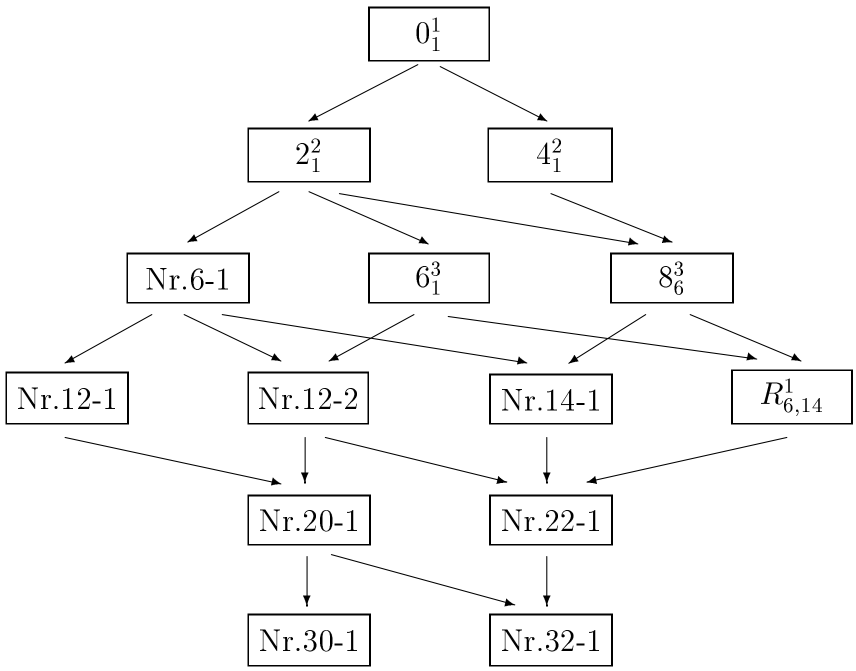


Figure 9: All, ordered by cutting, links, coming by deletion of central circuits from pure irreducible octahedrites

Theorem 6 *Any pure irreducible octahedrite P is one of eight given on Figures 8 and 9; all come by a cutting from the octahedron.*

In fact, let, as above, A be a patch of P bounded by t arcs (paths of edges) belonging to central circuits (different or coinciding). So, A can be seen as a t-gon. Suppose again that the patch A is regular, i.e. the continuation of any of bounding arcs (on the central circuit to which it belongs) lies outside of the patch. The patch A cannot be 1-gon, since P is pure, i.e. have no vertices, in which a central circuit self-intersects. So, as result of deletion of a central circuit from a pure octahedrite, we get an i-octahedrite.

Since our octahedrite P is irreducible, any two central circuits intersect in at least two vertices (see Theorem 2). So, the proof is separated in three cases: when the maximal size of pairwise intersection of central circuits is two, four or $2t, t \geq 3$. In fact, any two central circuits form an irreducible pure 4-hedrite $R_{4,2t}$, which extends on another pure irreducible $R_{i,2t}$, $4 \leq i \leq 8$. The proof in each case consists of tedious scanning of all possibilities, from given pure irreducible i-octahedrite with r, $2 \leq r \leq 5$, central circuits, to get, by a cutting, a pure irreducible octahedrite. We use that new central circuit can cut, without creating 5- or 6-gons, any 3-gon only by two edges and any 4-gon only by two opposite edges; but there are two ways to cut a 2-gon: both edges once or one edge twice. Our main tool is the classification of all 4-hedrites, given in previous Section; using it, we show that 2_1^2, 4_1^2 (i.e. $2t = 2, 4$) are only possibilities. It is easy to check then, that 6_1^3, 8_6^3 and Nr. 6-1 (the octahedron) are only acceptable extensions of 2_1^2 and 4_1^2 by new simple central circuit. Larger scanning show that octahedrites Nos. 12-1, 12-2, 14-1 and the 6-octahedrite, which is the 6th on Figure 7 (denote it by $R_{6,14}^1$), are only such graphs with $r = 4$ central circuits. In particular, we show that Nr. 12-2 is the only one irreducible (pure or not) octahedrite B, such that $6_1^3 \to B$ (the proof is easy for pure B) and that Nr. 14-1 is the only one pure irreducible octahedrite B, such that $8_6^3 \to B$.

The remainder of the proof is complete scanning of possibilities for $r = 5$. Since (by Theorem 3) irreducible octahedrite has at most six central circuits, we can stop here.

In fact, we proved more than above Theorem: see on Figure 9 all, ordered by cutting, links (trivial one 0_1 and $2, 1, 2, 8$ of $4-, 5-, 6-, 8-$hedrites, respectively), coming from a pure irreducible octahedrites by deleting some central circuits. Two last theorems imply the following Corollary.

Corollary 1 *Any pure irreducible generalized octahedrite $R_{i,n}$ is, either $R_{4,n=2m}$, $m \geq 1$, with two central circuits, or the $R_{5,6} = 6_1^3$, or one of two 6-hedrites 8_6^3, $R_{6,14}^1$, or one of eight octahedrites of Figure 8.*

Remark to Figure 9. (Remind that a simple circuit is called equatorial if its interior is isomorphic to its exterior.)

All 3 central circuits of Nr. 6-1 are equatorial; deleting of each reduces it to 2_1^2.

All 4 central circuits of Nr. 12-1 are equatorial; deleting of each reduces it to Nr. 6-1.

All 6 central circuits of Nr. 30-1 are equatorial; deleting of each reduces it to Nr. 20-1.

All 4 central circuits of Nr. 14-1 are equatorial; deleting of each 8-circuit reduces it to Nr. 6-1, but deleting of each 6-circuit reduces it to 8_6^3.

All 6 central circuits of Nr. 32-1 are equatorial; deleting of each 12-circuit reduces it to Nr. 20-1, but deleting of each 10-circuit reduces it to Nr. 22-1.

Amongst 4 central circuits of Nr. 12-2 only one is equatorial (and reduces it to 6_1^3); others reduce it to Nr. 6-1.

Amongst 5 central circuits of Nr. 20-1 only one is equatorial (and reduces it to Nr. 12-1); others reduce it to Nr. 12-2.

Amongst 5 central circuits of Nr. 22-1 only three 8-circuits are equatorial (two reduce it to Nr. 14-1 and one to $R^1_{6,14}$); others reduce it to Nr. 12-2.

Amongst 4 central circuits of $R^1_{6,14}$ only two 6-circuits are equatorial (and reduces it to 8_6^3); others (two 8-circuits) reduce it to 6_1^3.

Amongst 3 central circuits of 8_6^3 only one 4-circuit is equatorial (and reduces it to 4_1^2); others (two 6-circuits) reduce it to 2_1^2.

Amongst 3 central circuits of 6_1^3 no one is equatorial; each reduces it to 2_1^2.

So, amongst of all reductions of Figure 9, only five - $2_1^2 \to 6_1^3 \to R^1_{6,14}$, $Nr.6 \to Nr.12-2 \to Nr.22-1$ and $2_1^2 \to 8_6^3$ - can not be done by deletion of an *equatorial* central circuit. Amongst eight octahedrites of Figure 8, those admitting two non-isomorphic equatorial central circuits, are $Nos.14-1, 22-1$ and $32-1$, i.e. exactly those admitting intersection of central circuits in four vertices.

5 Connection to alternating links

See, for example, [Kaw96] and [Rol76] for all notions about knots and links, used here. Any 4-valent plane graph can be seen as a regular alternating projection of an *alternating* knot or link in following way. Choose a vertex and an edge incident to it. Consider that the edge enter in the vertex *under* the line containing two neighboring edges incident to the vertex. Then follow the central path alternating passing "under" and "up". Easy to see that it is always possible (see, for example, Theorem 8 in [Kot69]). But an alternating link or knot can have non-isomorphic alternating projections; for example, 6_3^2 have an $R_{5,6}$ and an $R_{6,6}$ as such projections.

The first five octahedrites Nos. 6-1, 8-1, 9-1, 10-1, 10-2 are *Conway graphs* (see [Kaw96], page 138) $6^*, 8^*, 9^*, 10^{***}, 10^{**}$, respectively; those are all, except of $10^* = APrism_5$, Conway graphs with at most ten vertices. Nos. 6-1 and 8-1, 9-1 are projections of the link 6_2^3 (*the* Borromean rings and of the knots 8_{18}, 9_{40}. Apropos, all three are isometric subgraphs of a m-dimensional half-hypercube (for $m = 4, 5, 6$, respectively) and oc_8, oc_9 are only known such Gaussian octahedrites.

It will be interesting to characterize, by some link invariants, all alternating links having octahedrites amongst their minimal projections and, in general, characterize all alternating links such that amongst their minimal projections, there are 4-valent polyhedra with only i-gonal faces for $2 \le i \le q$ for fixed $q \ge 4$.

5.1 Borromean octahedrites

A *Borromean* link is any link such that any two components form a *trivial* link (i.e. such that each its component can be bounded with non-overlapping spheres, up to isotopy); each component can be non-trivial knot. Clearly, a 4-valent polyhedron with r central circuits, corresponds to Borromean alternating link if and only if for any two its central circuits, the distance (on each of those circuits) between any two consecutive points of their intersection, is even. For example, it is easy to check, using it, that Nr. 6-1 is unique Borromean one on Figure 8. Also amongst the items of Table 1, only three are Borromean.

If C, C' are different central circuits of a Borromean 4-valent polyhedron, then the size of their intersection is no larger than the half of the size of shortest of them, since, otherwise, there are two adjacent (and so on the odd distance 1) vertices in their intersection.

A *Brunnian* link is any link such that any its proper sub-link is trivial; so any Brunnian link is Borromean. Easy to see that the octahedrites Nos. 6-1 and 14-1 (corresponding to unique, on the sphere, simple Venn 3- and 4-diagrams) are projections of Brunnian links. Moreover, [MaCe01] showed that each simple Venn r-diagram, $r \geq 3$, coming by *Edwards construction* (so, including above cases $r = 3, 4$) is Brunnian.

Remind that the $BPyr_4^t$ is the octahedrite oc_{4t+2} (of symmetry D_{4h}), which is the octahedron, t-inflated around fixed central 4-cycle. For example, the $BPyr_4^t$ corresponds, as projection, to a $(t+2)$-component alternating link, which is Borromean if and only if t is odd. In general, easy to check that the pure octahedrite, obtained from the octahedron by simultaneous t_1-, t_2-, t_3-inflation respectively around all its three central 4-cycles, is Borromean if and only if each of numbers t_1, t_2, t_3 is odd. On the other hand, the octahedrite, obtained from any octahedrite, having a *self-intersecting* central circuit, by simultaneous t_1-,..., t_k-inflation (with $t_i > 1$ for some $1 \leq i \leq k$) respectively around all its k central circuits, is *not* Borromean.

We will give below two new series of Borromean links: $(2t+1)$-link $APrism_{i(2t+1)}^{2t-1}$ for any $t, i \geq 1$, and 3-link $oc_n(O)$, $n = 6(16t^2 + 1)$ for any $t \geq 0$.

5.2 Conway graph $((k+1) \times m)^*$

Denote by $APrism_m^k$, a sort of "column of k m-antiprisms", defined by $APrism_m^{2k} = med - BPyr_m^k = med - Prism_m^k$ and $APrism_m^{2k-1} = med - Pyr_m^k$. (Here $Prism_m^k := (BPyr_m^k)^*$ and Pyr_m^k is m-pyramid, elongated $(k-1)$ times.) An $APrism_m^k$ is a Conway graph $((k+1) \times m)^*$. It is an octahedrite if $m = 3, 4$; in this case it is irreducible, unless $k \equiv 2 \pmod 4$.

An $APrism_m^k$ has CC-vector $((2p(tq-1))^t)$ and all its intersection vectors are $(p(q-1); 2pq^{t-1})$, where $t := gcd(m, k+2)$ and $p := \frac{m}{t}$, $q := \frac{k+2}{t}$ (so gcd(p,q)=1).

Theorem 7 *(i) An $APrism_m^k = APrism_{pt}^{qt-2}$ with $t = 1, 2, k+2$ is Borromean if and only if $t = k+2$ and it is odd (i.e. $k+2$ is an odd number, dividing m)*

(ii) The octahedron is unique Borromean octahedrite of form $APrism_m^k$.

In fact, (i) holds for $t = 2$, because in this case two central circuits intersect in more than the half of size of each of them. The case $t = k+2$ correspond to t simple curves, intersecting pair-wisely in $2p$ vertices; moreover, for $p = 1$ it is a Grünbaum's arrangement of t simples curves. The vertices of intersection of a central circuits C, C' are *almost equispaced* on each of those circuits. They are equispaced on even distance $t-1$ if t is odd. Otherwise, there is an odd distance amongst them.

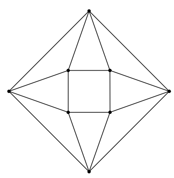
In order to prove (ii), we should only check $APrism_3^k$ with $k \equiv 1 \pmod 3$ and $APrism_4^k$ with $k \equiv 0 \pmod 4$. In the first case one can see on example of Nr. 15-1, that any two central circuits have, in the center of the picture, two neighboring (on one of them) vertices of intersection on the odd distance three. For the second case, on can see on example of Nr. 12-1, that any two opposite central circuits have, in the center of the picture, two neighboring (on one of them) vertices of intersection on the same odd distance three.

The case $k = 1$ of (i) above implies that $APrism_m$ is Borromean if and only if $m \equiv 0 \pmod 3$; those correspond to Tait (1876) series of Borromean 3-links. The case $m = k + 2$ correspond to an arrangement of $m = k + 2$ simple curves. The $APrism_5^3$ corresponds to Borromean 5-link, referred as "Penrose Mandala" in

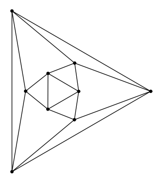
http://jubal.westnet.com/hyperdiscordia/mandala.htm.

Remark also that cuboctahedron $APrism_4^2$ is not Borromean, but deleting any fixed one of its central circuits gives Borromean octahedron $APrism_3$. Perhaps, $APrism_{p(k+2)}^k$ with odd k are only Borromean $APrism_m^k$. For example, $APrism_9^4$ is not Borromean. It has 3 central circuits of length 30, each having 3 self-intersections and intersecting each other in 12 vertices. The distances of consecutive vertices of intersection of circuits C, C' on C are 2, 3 , 2 , 3 , 2 , 3 , 2, 3, 2, 3, 2, 3.

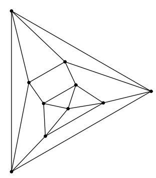
The link corresponding to any $APrism_{p(k+2)}^k$ with odd k has, moreover, the property that any its sub-link with odd number of components is Borromean. We checked the smallest case: any 3-component sub-link of $APrism_5^3$ is Borromean (there are two types up to symmetry), but no one of its 4-component sub-links is Borromean (there is only one type up to symmetry).



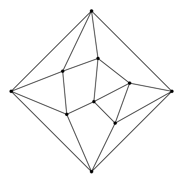
Nr.8-1 (16)
Group: D_{4d}



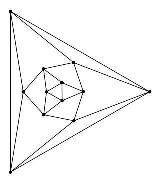
Nr.9-1 (18)
Group: D_{3h}



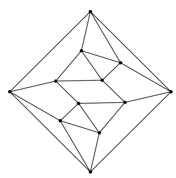
Nr.10-1 $(4^2, 6^2)$
Group: D_{4h}



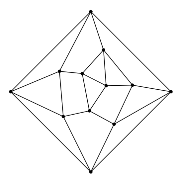
Nr.10-2 $(6; 14)$
Group: D_2



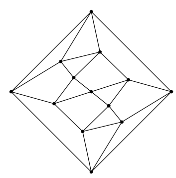
Nr.12-3 (24)
Group: D_{3d}



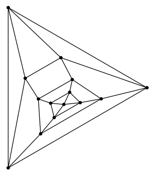
Nr.12-4 (24)
Group: D_2



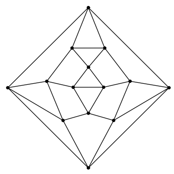
Nr.12-5 $(6; 18)$
Group: C_2



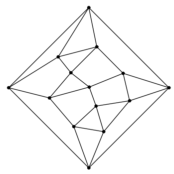
Nr.13-2 (26)
Group: C_2



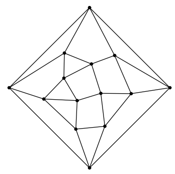
Nr.14-2 $(4^3, 8^2)$
Group: D_{4h}



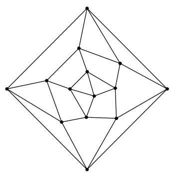
Nr.14-3 $(14, 14)$
Group: D_{2d}



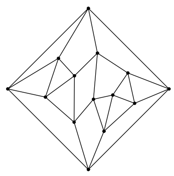
Nr.14-4 (28)
Group: C_2



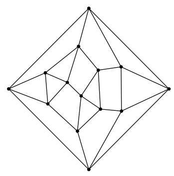
Nr.14-5 $(6; 22)$
Group: D_2



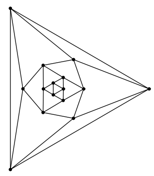
Nr.14-6 $(6; 22)$
Group: C_s



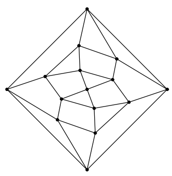
Nr.14-7 $(8; 10^2)$
Group: D_2



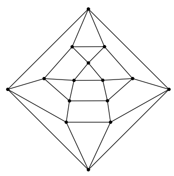
Nr.14-8 $(6^2; 16)$
Group: C_2



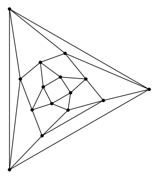
Nr.15-1 (10^3)
Group: D_{3h}



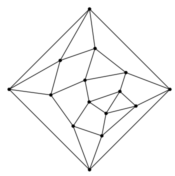
Nr.15-2 (30)
Group: C_2



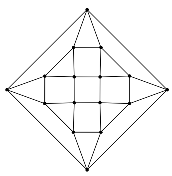
Nr.15-3 $(6; 24)$
Group: C_s



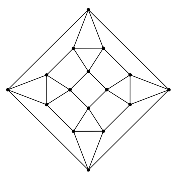
Nr.15-4 $(6; 24)$
Group: C_s



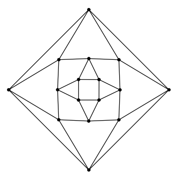
Nr.15-5 $(8; 22)$
Group: C_2



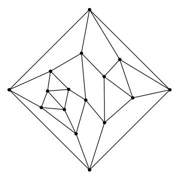
Nr.16-2 $(8; 24)$
Group: D_{4d}



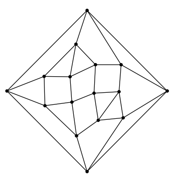
Nr.16-3 $(6^4, 8)$
Group: D_{4h}



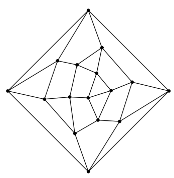
Nr.16-4 (32)
Group: D_{4d}



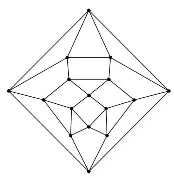
Nr.16-5 (32)
Group: D_2



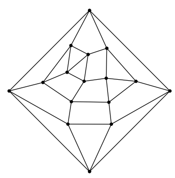
Nr.16-6 (32)
Group: C_2



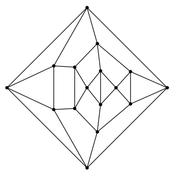
Nr.16-7 (32)
Group: C_2



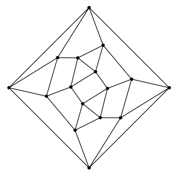
Nr.16-8 (32)
Group: C_s



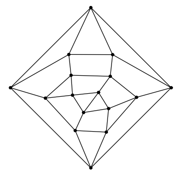
Nr.16-9 (32)
Group: C_2



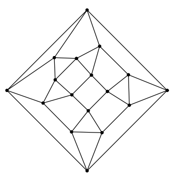
Nr.16-10 (6; 26)
Group: C_s



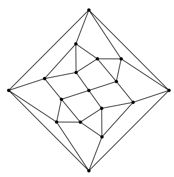
Nr.16-11
(6^2; 20)
Group: D_2



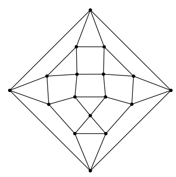
Nr.16-12
(6, 8; 18)
Group: C_1



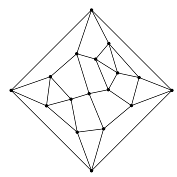
Nr.17-1
(6^2, 8; 14)
Group: C_s



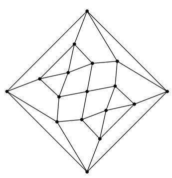
Nr.17-2 (8; 26)
Group: C_2



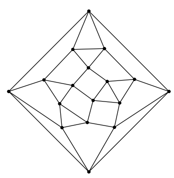
Nr.17-3 (8; 26)
Group: C_s



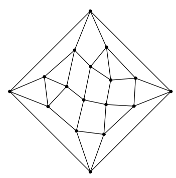
Nr.17-4 (8; 26)
Group: C_2



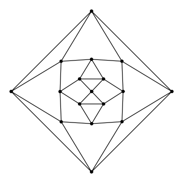
Nr.17-5 (10, 24)
Group: C_{2v}



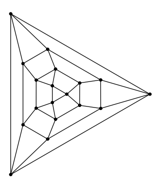
Nr.17-6 (14, 20)
Group: C_2



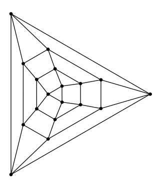
Nr.17-7 (16, 18)
Group: C_1



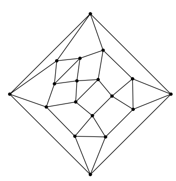
Nr.17-8 (34)
Group: C_{2v}



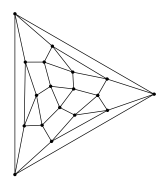
Nr.18-2 ($6^2, 8^3$)
Group: D_{3h}



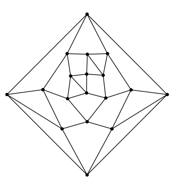
Nr.18-3 ($6^2, 8^3$)
Group: D_{3d}



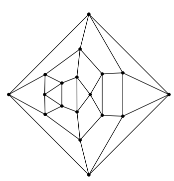
Nr.18-4 ($6^2, 8^3$)
Group: C_{2v}



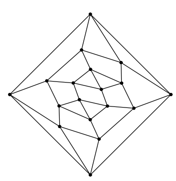
Nr.18-5 (36)
Group: D_3



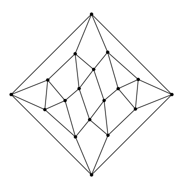
Nr.18-6 (10; 26)
Group: D_2



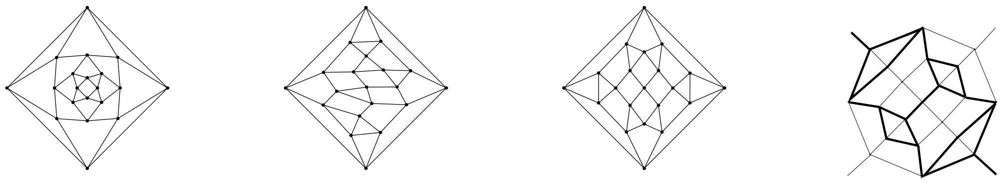
Nr.18-7 (10; 26)
Group: C_{2v}



Nr.18-8 (18, 18)
Group: D_4



Nr.18-9 (18, 18)
Group: C_2



Nr.20-2 $(20, 20)$ Group: D_{4h} **Nr.20-3** $(20, 20)$ Group: C_1 **Nr.21-1** (42) Group: C_{2v} **Nr.22-2** $(8^4, 12)$ Group: D_{2d}

Figure 10: Small octahedrites

6 Octahedrites with symmetry O_h or O

The notion of t-inflation, considered in Section 3, permits to classify octahedrites of the highest possible symmetry O_h.

Theorem 8 *Any octahedrite oc_n with symmetry O_h can be obtained from the octahedron or cuboctahedron by t-inflation (i.e. replacing each central circuit by $t-1$ parallel rail-roads). Then $n = 6t^2, 12t^2$ and CC-vector is $(4t^{3t}), (6t^{4t})$, respectively. The octahedrite $oc_n(O_h)$ is Borromean if and only if $n = 6t^2$, t is odd (i.e. if 4 not divides n).*

More general, similar characterization holds for the octahedrites with symmetry O_h *or* O. In fact, any such octahedrite has $6(a^2 + b^2)$ vertices for some integers a, b with $0 \le b \le a$; the case of symmetry O_h corresponds to $b = 0$ or $b = a$. Any $oc_n(O)$ also can be obtained by t-inflation from irreducible one (with symmetry O), i.e. one having gcd(a,b)=1. See on Figure 11 an example with $(a, b) = (3, 2)$, i.e. an $oc_{78}(O)$. The pair (a, b) is defined by the shortest path on the square lattice (of length $a + b$) between neighboring vertices of the cube.

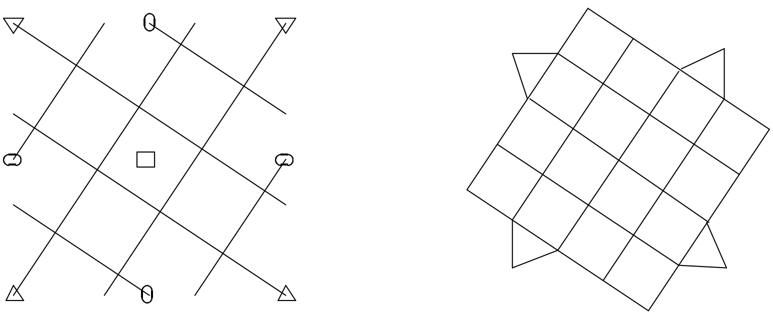


Figure 11: Fragments of dual $oc_n(O)$ and of $oc_n(O)$ for $n = 6(a^2 + b^2)$ with $(a, b) = (3, 2)$. The picture on the left-hand side can be completed to the dual $oc_{78}(O)$, using 4-, 3- and 2-axes of symmetry indicated there

The group O_h on the cube has four 3-axes through (pairs of opposite) vertices, three 4-axes through (pairs of opposite) centers of faces and six 2-axes through (pairs of opposite) centers of edges. So, on an octahedrite $oc_n(O)$ four 3-axes go through (pairs of opposite) centers of triangles and the net of squares is turned, with respect of the net of squares of the cube, by angle $arctang\frac{a}{b}$. (For example, this angle is 0 for the octahedron and $\frac{\pi}{4}$ for the cuboctahedron.) The group O of order 24 acts regularly on 24 edges of 8 triangles. Because of the action of 3-axes, any of eight triangles is either *1-colored*, or *3-colored*, i.e. all its three sides belong to the same or to three different central circuits. By a symmetry argument one can check that all triangles are 3-colored if $r = 3, 6$ and 1-colored if $r = 1, 2$; for $r = 4$ all triangles are either 3-colored, or any two opposite ones are 1-colored by the same color. By a tedious scanning of sub-cases we excluded all possibilities of 1-colored triangles in an $oc_n(O)$.

Since our $oc_n(O)$ is irreducible, any or r its central circuit contains an edge of triangle. So, the group O is transitive on central circuits, i.e. an irreducible $oc_n(O)$ is not only equilinked: its central circuits are congruent. (Remark that the group O is transitive on pairs of central circuits for $r = 3$, but not for the cases, all with $r = 6$, $(a, b) = (6, 1), (5, 4), (9, 2), (10, 1), (8, 7)$ of Table 1. So, pairwise intersection of two central circuits have same size, if $r = 3$; we expect it also for $r = 4$.) Remind that x_i denotes the number of central circuits, which contains i edges of triangles; an edge of two adjacent triangles is counted twice. So, for an irreducible $oc_n(O)$, having r central circuits, $r = x_i$ for some $2 \leq i \leq 24$ and $ir = 24$, which exclude $r = 5$. In two theorems below, using the symmetry approach outlined above, we show that $r = 3, 4$ or 6 only and, for the case $b = 1$, indicate when each value of r occurs.

Theorem 9 *Given an irreducible octahedral octahedrite $P = oc_n(O)$, $n = 6(a^2 + b^2)$ with r central circuits. Let, for each of them, l denotes the length $\frac{2n}{r}$ and i_0 denotes the number of vertices of self-intersection. Then we have:*

either (i) CC-vector is (l^4) with $l \equiv 2(mod\,4)$, $i_0 \equiv 0(mod\,3)$ and both a, b are odd;

or (ii) CC-vector is (l^3) with $l \equiv 0(mod\,4)$, $i_0 \equiv 0(mod\,2)$;

or (iii) CC-vector is (l^6) with $i_0 \equiv 0(mod\,2)$.

First, suppose that $r = 2$. Because of the action of four 3-axes, any of eight triangles is 1-colored and exactly four triangles (corresponding, in fact, to four vertices of one of two tetrahedra in cube) have all sides from each of two central circuits. Each of those two circuits is incident, from both its sides, to a triangle, because our octahedrite P is irreducible, i.e. has no rail-roads. Let us add, on small enough same distance, rail-roads on both sides of each central circuit. Each of four new central circuits is incident (on one side) to a triangle; so, the group O is transitive on four new central circuits. Delete now two original central circuits; we get new octahedrite P' with same symmetry O. Moreover, O is transitive on four central circuits of P' and its subgroup T is the group of symmetry of each of two rail-roads. Each rail-road is incident to its four triangles, which had same color in P. In P'

each of four triangles, incident to fixed rail-road, are again 1-colored, but two by one and two by the second color (i.e. central circuit). But now no 3-axes can act on four triangles of fixed rail-road, since T has no normal subgroup of index two. This contradiction proves that $r \neq 2$.

If $r = 3$, then $l = 4(a^2+b^2)$ and so, divisible by 4. If $r = 4$, then both a, b are odd (they cannot be both even since $gcd(a,b) = 1$) and $l = 3(a^2+b^2) = 3(a+b)^2 - 2ab) \equiv 2(mod\,4)$, because $(a+b)^2 \equiv 0(mod\,4)$ and $2ab \equiv 2(mod\,4)$. We conjecture that $r = 4$ if and only if both a, b are odd. We know that $r = 6$ for all $(a,b) = (4t+2, 1)$ and expect $r = 6$ for $(a,b) = (8t+1, 2), (8t-1, 2), (8t+3, 4), (8t-3, 4)$ and (3t+2,3t+1), where $t \geq 1$ is any integer.

On each central circuit C acts group $D_{\frac{12}{r}}$. So, the size of orbit of any vertex of self-intersection of C under this group is $\frac{24}{r}$, if the vertex is in general position with respect of $D_{\frac{12}{r}}$. Otherwise, it is 4 or 2 for $r = 3$, 3 or 2 for $r = 4$ and 2 for $r = 6$. The existence of orbit of size 2 in the case $r = 4$ will imply that the vertex of self-intersection lies on a 3-axe. The group D_3, but not D_4 and D_2, contains 3-axes, but in any octahedrite, admitting a 3-axe, this 3-axe goes through centers of triangles and so not through a vertex. So, $i_0 \equiv 0(mod\,3)$, if $r = 4$, and i_0 is even if $r = 3$ or 6. (In all known examples we have, moreover, $i_0 \equiv 0(mod\,4)$ if $r = 6$ and $i_0 \equiv 0(mod\,8)$ if $r = 3$.

Theorem 10 *Given an irreducible octahedral octahedrite $oc_n(O)$, $n = 6(a^2+b^2)$ with $b = 1$. Then we have:*

(i) in the case $a \equiv 0(mod\,4)$, our $oc_n(O)$ is Borromean, its CC-vector is $((4a^2+4)^3)$ and each intersection vector is $(\frac{a^2}{2}; (\frac{3a^2}{2}+2)^2) = (\frac{l-4}{8}; (\frac{3l+4}{8})^2)$;

(ii) in the case $a \equiv 2(mod\,4)$, our $oc_n(O)$ has CC-vector $((2a^2+2)^6)$ and each central circuit has $i_0 = \frac{a^2-4}{8}$ self-intersection vertices;

(iii) in the case $a \equiv 1, 3(mod\,4)$, our $oc_n(O)$ has CC-vector $((3a^2+3)^4)$ and each central circuit has $i_0 = \frac{3(a^2-1)}{8}$ self-intersection vertices.

In terms of the number r of central circuits and the length l of each of them, let us denote $a_{r,l} := \frac{l-2(r-1)}{8}$ if $r = 3, 4$ and $a_{r,l} := \frac{l-2(r-1)}{16} = \frac{l-10}{16}$ if $r = 6$. Above Theorem gives that $i_0 = a_{r,l}$ for any $oc_n(O)$ with $n = 6(a^2+b^2)$, if $b = 1$. For all known $oc_n(O)$ with $n = 6(a^2+b^2)$, gcd(a,b)=1 and $b > 1$, we have:

(i) $i_0 \geq a_{3,l} = \frac{l-4}{8}$ if $r = 3$, with equality if $b = 1$, $a \equiv 0(mod\,4)$, and

(ii) $i_0 \leq a_{4,l} = \frac{l-6}{8}$ if $r = 4$ with equality if $b = 1$.

It is well-known that a natural number $k = \prod_i p_i^{\alpha_i}$ admits a representation $k = a^2+b^2$ if and only if any α_i is even, whenever $p_i \equiv 3(mod\,4)$. Such representation is not unique, in general; for example, $25 = 5^2+0^2 = 4^2+3^2$, and (the first case with $gcd(a,b) = gcd(a_1,b_1) = 1$) $65 = 8^2+1^2 = 7^2+4^2$. It can be deduced from Theorem 6 on page 136 of [Gon85], that the number of pairs (a,b), such that $n = a^2+b^2$ and $a \geq b \geq 0$, is $\lfloor \frac{x-y+1}{2} \rfloor$, where x and y are the number of distinct divisors of n of the form $4k+1$ and $4k+3$, respectively. For prime $n \geq 3$, it is just Fermat's two-squares theorem; in fact, the primes of the form a^2+b^2 are the primes that factor in *Gaussian integers*.

The construction of all $oc_n(O)$, $n = 6(a^2 + b^2)$, can be translated in terms of multiplication in the ring of Gaussian integers; so, in principle, their CC-vectors can be expressed in those terms.

In Table 1 we computed, for all irreducible (i.e. with gcd(a,b)=1) $oc_n(O)$ with $n = 6(a^2 + b^2) < 750$, their CC-vectors and intersection vectors. The deletion of all but one central circuit in each octahedrite of Table 1 reduces it to projection of an alternating knot (the trivial knot 0_1, the trefoil 3_1, the knot 4_1, a Gaussian octahedrite). We specify it, together with its highest symmetry, in the last column of this Table for $\frac{n}{6} \leq 65$. Two largest such octahedrites are given on Figure 12; on the other hand, for $(a, b) = (9, 2)$ such projection is Nr.12-4(D_2).

Figure 12: Gaussian octahedrites Nos. 32-2(D_4) and 40-1(D_4) from Table 1

Table 1: **All irreducible oc_n with symmetry O_h or O**
(so, $n = 6(a^2 + b^2)$ and, for $n \neq 6$, gcd(a,b)=1) with $n < 750$

$\frac{n}{6} = a^2 + b^2$	a,b	CC-vector	int. vector	Borromean?	reduces to
1	1,0	(4^3)	$(0; 2^2)$	**yes**	proj. of 0_1
2	1,1	(6^4)	$(0; 2^3)$	no	proj. of 0_1
5	2,1	(10^6)	$(0; 2^5)$	no	proj. of 0_1
10	3,1	(30^4)	$(3; 8^3)$	no	proj. of 3_1
13	3,2	(52^3)	$(8; 18^2)$	no	Nr. 8(D_{4d})
17	4,1	(68^3)	$(8; 26^2)$	**yes**	Nr. 8(D_{4d})
25	4,3	(100^3)	$(16; 34^2)$	no	Nr. 16-4(D_{4d})
26	5,1	(78^4)	$(9; 20^3)$	no	Nr. 9(D_{3d})
29	5,2	(116^3)	$(16; 42^2)$	no	Nr. 16-4(D_{4d})
34	5,3	(102^4)	$(9; 28^3)$	no	Nr. 9(D_{3d})
37	6,1	(74^6)	$(4; 14^4, 10)$	no	proj. of 4_1
41	5,4	(82^6)	$(4; 18, 14^4)$	no	proj. of 4_1
50	7,1	(150^4)	$(18; 38^3)$	no	Nr. 18-5(D_3)
53	7,2	(106^6)	$(8; 18^5)$	no	Nr. 8(D_{4d})
58	7,3	(174^4)	$(18; 46^3)$	no	Nr. 18-5(D_3)
61	6,5	(244^3)	$(40; 82^2)$	no	Nr. 40-1(D_4)
65	7,4	(260^3)	$(40; 90^2)$	no	Nr. 40-1(D_4)
65	8,1	(260^3)	$(32; 98^2)$	**yes**	Nr. 32-2(D_4)
73	8,3	(292^3)	$(48; 98^2)$	no	
74	7,5	(222^4)	$(24; 58^3)$	no	
82	9,1	(246^4)	$(30; 62^3)$	no	
85	7,6	(340^3)	$(56; 114^2)$	no	
85	9,2	(170^6)	$(12; 30^4, 26)$	no	
89	8,5	(356^3)	$(56; 122^2)$	no	
97	9,4	(388^3)	$(56; 138^2)$	no	
101	10,1	(202^6)	$(12; 38^4, 26)$	no	
106	9,5	(318^4)	$(30; 86^3)$	no	
109	10,3	(436^3)	$(72; 146^2)$	no	
113	8,7	(226^6)	$(12; 50, 38^4)$	no	
122	11,1	(366^4)	$(45; 92^3)$	no	

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