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NOTE ON PETRIE DUALS AND HYPERCUBE EMBEDDINGS OF SEMIREGULAR POLYHEDRA

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Abstract: *A non-Platonic convex polyhedron in* $\mathbb{R}^3$ *is* semiregular *if it is vertex-transitive and its faces are regular polygons. The semiregular polyhedra consist of the* 13 *Archimedean solids and two infinite families of* n*-sided prisms* $Prism_n$ *and antiprisms* $APrism_n$*. The skeleton graphs (vertices-edges)* $G(M)$ *of the semiregular polyhedra and their duals are well-known* GEM's *(graph-encoded maps)* M *on the sphere* $\mathbb{S}^2$*. We collect in the Table 1 data on two relatively new embeddings of those maps:*

1. *the Euler characteristic* χ *of the surface, on which their maps* $skew(M)$ *and* $phial(M)$ *embed;*
2. *the* hypercube embedding*, if any, of the path-metric of the graph* $G(M)$*.*

Keywords: partial metrics; quasi-semimetrics; oriented multicuts; convex cones; computational experiments.

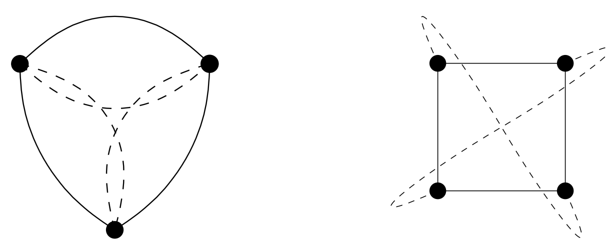
1. INTRODUCTION

In this Note, an *hypercube embedding* of a map M means that $2d$ (doubled path-metric of its skeleton $G(M)$) isometrically embeds into the path-metric of some half-cube $\frac{1}{2}H_m$. Isometric embedding of d in the path-metric of some hypercube H_m implies isometric embedding of $2d$ into a Johnson graph $J(2m, m)$, which implies isometric embedding of $2d$ in the path metric of $\frac{1}{2}H_{2m}$. If a graph admits an hypercube embedding, then it is into a hypercube if and only if the graph is bipartite.

A *zigzag* (or *Petrie walk*, *left-right path*) of a map M is an edge-circuit alternating left and right turns, i.e., any two, but not three, consecutive edges bound the same face. The zigzag-set covers the edge-set exactly twice; so, the number e of edges is the half the sum of the lengths of all zigzags of M. The Z-*vector* of a map enumerates lengths of its zigzags with their multiplicities.

Given a set X and fixed-point-free involutions A, B, C on X with $AB = BA$ and $\langle A, B, C\rangle$ transitive on X, the quadruple $(X; A, B, C)$ defines a *GEM* (graph-encoded combinatorial map) M, with its sets $V(M), E(M), F(M)$, and $Z(M)$ of vertices, edges, faces, and zigzags given by the orbit-sets of the subgroups $\langle A, C\rangle$, $\langle A, B\rangle$, $\langle C, B\rangle$, and $\langle C, AB\rangle$ (acting on X), respectively. For a map $M{=}(X; A, B, C)$, it holds $[\langle A, B, C\rangle : \langle CA, CB\rangle] \leq 2$; the map M is called *orientable* if this rank is 2.

The usual Poincaré duality operation $dual(M)$ interchanges the roles of A and B; so, vertices and faces are interchanged while edges and zigzags are left unchanged. The operation *Petrie dual* (or $skew(M)$) on the map M interchanges B and AB, i.e., faces and zigzags, leaving vertices and edges. The operation $phial(M)$ interchanges vertices and zigzags, leaving faces and edges. Clearly, $phial(M) = (skew(M^*))^*$; so, $\chi(phial(M)) = \chi(skew(M^*))$. The operations $dual$, $skew$, and $phial$ are reflexions. The group $\langle dual, skew\rangle$ of six *trialities* is $S_3 \simeq Sym_3$.



$phial(Tetrahedron)$ $skew(Tetrahedron)$

Figure 1: *Folded* (antipodal quotients of) Octahedron and Cube on the projective plane

The *folded* maps on Figure 1 below are obtained by identifying opposite vertices of Octahedron and Cube to get maps on the projective plane. $Skew(Cube)$ (see Figure 2) and its dual, $phial(Octahedron)$, are toric maps, while $phial(Cube)$ and

$skew(Octahedron)$ are maps on a non-orientable surface of genus 4, i.e., with $\chi = -2$.

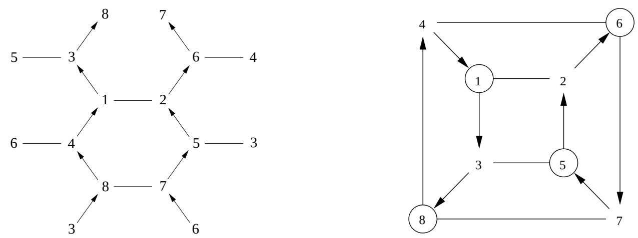


Figure 2: Two representation of $skew(Cube)$: on the torus and as Cube with cyclic order of encircled vertices to their adjacent vertices being reversed

$Skew(Cube)$ (see Figure 2) and its dual, $phial(Octahedron)$, are toric maps, while $phial(Cube)$ and $skew(Octahedron)$ are maps on a non-orientable surface of genus 4, i.e., with $\chi = -2$.

For bipartite oriented maps, the skew operation consists of reversing orientation of one of the parts of the bipartition. It is shown (Nedela *et al.*, 2001) that $skew(M)$ of an orientable map M is orientable if and only if M is bipartite.

For a map M, denote by D, v, f, z, and e, respectively, the diameter of M, and the numbers of vertices, faces, zigzags, and edges of M. Clearly, $\chi(M){=}v-e+f$, $\chi(skew(M)) = v - e + z$ and $\chi(phial(M)) = f - e + z$. So, for the maps M on the sphere, $\chi(skew(M)) = (z+2) - f$ and $\chi(phial(M)) = (z+2) - v$.

Moreover, for a map M, let χ_s, denote the Euler characteristic of the surface on which the skew map $skew(M)$ (not M itself) embeds. Similarly, we use the notation D^*, v^*, f^*, z^*, e^*, χ_s^* for the dual map M^*. Clearly, $v^* = f$, $f^* = v$, and $z^* = z$, $e^* = e$. Also, the maps M and M^* have the same Z-vectors.

The Table 1 gives, for the maps M of the semiregular polyhedra, their Z-vectors and number of edges; and, for the maps in a dual pair M, M^*, their diameters, (the strongest form of) their hypercube embeddings, the number of their vertices, and the Euler characteristics of the Petrie dual. The numbers in last column are emphasized if (as it was found by a computer) surface is orientable.

M: name, vertex configur.	D, D^*	hyp. emb. M	hyp. emb. M^*	v, v^*	e	Z-vector	χ_s, χ_s^*
tetrahedron (3.3.3)	1,1	$=\frac{1}{2}H_3, J(4,1)$	$=\frac{1}{2}H_3, J(4,1)$	4,4	6	4^3	1,1
cube (4.4.4)	3,2	$=H_3$	$=J(4,2)$	8,6	12	6^4	***0***,-2
dodecahedron (5.5.5)	5,3	$\to \frac{1}{2}H_{10}$	$\to \frac{1}{2}H_6$	20,12	30	10^6	***-4***,-12
cuboctahedron (3.4.3.4)	3,4	no	$\to H_4$	12,14	24	8^6	-6,***-4***
icosidodecahed. (3.5.3.5)	5,6	no	$\to H_6$	30,32	60	10^{12}	-18,***-16***
tr.tetrahedron (3.6.6)	3,2	no	$\to \frac{1}{2}H_7$	12,8	18	12^3	-3,-7
tr.octahedron (4.6.6)	6,3	$\to H_6$	no	24,14	36	12^6	***-6***,-16
tr.cube (3.8.8)	6,3	no	$\to J(12,6)$	24,14	36	18^4	-8,-18
tr.icosahedron (5.6.6)	9,5	no	$\to \frac{1}{2}H_{10}$	60,32	90	18^{10}	***-20***,-48
tr.dodecahed. (3.10.10)	10,5	no	$\to \frac{1}{2}H_{26}$	60,32	90	30^6	-24,-52
rhombicubocta. (3.4.4.4)	5,5	$\to J(10,5)$	no	24,26	48	12^8	-16,***-14***
rhombicosidod. (3.4.5.4)	8,8	$\to \frac{1}{2}H_{16}$	no	60,62	120	20^{12}	-48,***-46***
tr.cuboctahedron (4.6.8)	9,4	$\to H_9$	no	48,26	72	18^8	***-16***,-38
tr.icosidodecah. (4.6.10)	15,6	$\to H_{15}$	no	120,62	180	30^{12}	***-48***,-106
snub cube (3.3.3.3.4)	4,7	$\to \frac{1}{2}H_9$	no	24,38	60	$30^4_{3,0}$	-32,-18
snub dodec. (3.3.3.3.5)	7,15	$\to \frac{1}{2}H_{15}$	no	60,92	150	$50^6_{5,0}$	-84,-52
$Prism_3$	2,2	$\to J(5,2)$	$\to \frac{1}{2}H_4$	6,5	9	$18_{3,6}$	-2,-3
$Prism_{n>3,\ odd}$	$\frac{n+1}{2}$,2	$\to J(n+2, \frac{n+1}{2})$	no	2n,n+2	3n	$6n_{n,2n}$	1-n,3-2n
$Prism_{n\equiv 2\,(mod\,4)}$	$\frac{n+2}{2}$,2	$\to H_{\frac{n+2}{2}}$	no	2n,n+2	3n	$(3n_{\frac{n}{2},0})^2$	***2-n***,4-2n
$Prism_{n\equiv 0\,(mod\,4)}$	$\frac{n+2}{2}$,2	$\to H_{\frac{n+2}{2}}$	no	2n,n+2	6n	$(\frac{3n}{2})^4$	***4-n***,6-2n
$APrism_{n\equiv 3\,(mod\,6)}$	$\frac{n+1}{2}$,3	$\to J(n+1, \frac{n+1}{2})$	no	2n,2n+2	4n	$(2n)^4$	4-2n, ***6-2n***
$APrism_{n\equiv 0\,(mod\,6)}$	$\frac{n}{2}$,3	$\to \frac{1}{2}H_{n+1}$	no	2n,2n+2	4n	$(2n)^4$	4-2n,***6-2n***
$APrism_{n\equiv 1,5\,(mod\,6)}$	$\frac{n+1}{2}$,3	$\to J(n+1, \frac{n+1}{2})$	no	2n,2n+2	4n	$2n; 6n_{0,2n}$	2-2n, ***4-2n***
$APrism_{n\equiv 2,4\,(mod\,6)}$	$\frac{n}{2}$,3	$\to \frac{1}{2}H_{n+1}$	no	2n,2n+2	4n	$2n; 6n_{0,2n}$	2-2n,***4-2n***

Table 1: Hypercube embeddings, Z-vectors and surfaces of the Petrie duals of the semiregular polyhedra and of their duals

It turns out that in Table 1, the surfaces of $skew(M)$ and $skew(M^*) = (phial(M))^*$ are never both orientable. If one is orientable, it is always one with larger Euler characteristic, i.e., with smaller number of faces.

Apart from the Tetrahedron and the $Prism_3$, exactly one of the maps M, M^* of semiregular polyhedra is hypercube embeddable. Apart for the Tetrahedron, all the hypercube embeddings are essentially unique. More details on hypercube embedding of semiregular polyhedra and of their duals are given in Tables 4.1 and 4.2 in Deza (Deza *et al.*, 2004).

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