# The Cut Cone, $L^1$ Embeddability, Complexity and Multicommodity Flows

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*ABSTRACT*

A finite *metric* (or more properly semimetric) on $n$ points is a non-negative vector $d = (d_{ij})$ $1 \le i < j \le n$ that satisfies the triangle inequality: $d_{ij} \le d_{ik} + d_{jk}$. The $L^1$ (or *Manhattan* )distance $\|x - y\|_1$ between two vectors $x = (x_i)$ and $y = (y_i)$ in $R^m$ is given by $\|x - y\|_1 = \sum_{1 \le i \le m} |x_i - y_i|$. A metric $d$ is $L^1$ – embeddable if there exist vectors $z_1, z_2, \ldots, z_n$ in $R^m$ for some $m$, such that $d_{ij} = \|z_i - z_j\|_1$ for $1 \le i < j \le n$. A *cut metric* is a metric with all distances zero or one and corresponds to the incidence vector of a cut in the complete graph on $n$ vertices. The *cut cone* $H_n$ is the convex cone formed by taking all non-negative combinations of cut metrics. It is easily shown that a metric is $L^1$ – embeddable if and only if it is contained in the cut cone. In this expository paper we provide a unified setting for describing a number of results related to $L^1$ – embeddability and the cut cone. We collect and describe results on the facial structure of the cut cone and the complexity of testing the $L^1$ – embeddability of a metric. One of the main sections of the paper describes the role of $L^1$ – embeddability in the feasibility problem for multicommodity flows. The Ford and Fulkerson theorem for the existence of a single commodity flow can be restated as an inequality that must be valid for all *cut metrics*. A more general result, known as the Japanese theorem, gives a condition for the existence of a multicommodity flow. This theorem gives an inequality that must be satisfied by all *metrics*. For multicommodity flows involving a small number of terminals, it is known that the condition of the Japanese theorem can be replaced with one of the Ford-Fulkerson type. We review these results and show that the existence of such Ford-Fulkerson type conditions for flows with few terminals depends critically on

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the fact that certain metrics are $L^1-$embeddable.

## 1. Introduction

This paper is concerned with sets of distances between all pairs of a finite set of $n$ points. Such distances must be non-negative and satisfy the triangle inequality. Formally, a *metric* on $n$ points is a non-negative vector of length $\binom{n}{2}$, $d = (d_{ij})$, $1 \le i < j \le n$ satisfying the triangle inequality:

$$d_{ij} \le d_{ik} + d_{kj} \qquad \text{for distinct } \ i, j, k \in \{1, \ldots, n\}. \tag{1.1}$$

Here and throughout the paper we define

$$d_{ji} = d_{ij} \quad \text{when } j > i, \qquad d_{ii} = 0 \quad 1 \le i \le n.$$

Strictly speaking we should refer to $d$ as a *semimetric* since we allow $d_{ij} = 0$ when $i$ and $j$ are distinct. For brevity, we will use the term *metric*. It is convenient to consider $d$ as a vector in $R^{\binom{n}{2}}$. For example, any vector of length $\binom{n}{2}$ consisting of 1's and 2's is a metric, since (1.1) is always satisfied. The vector $d = (0, 0, 1, 1, 1, 1)$ is not a metric on 4 points, since $1 = d_{23} > d_{12} + d_{13} = 0$.

One of the fundamental problems in the theory of metrics is the isometric embedding problem, which we now describe for $L^1$ distances. The $L^1$ distance $\|x - y\|_1$ between two vectors $x = (x_i)$ and $y = (y_i)$ in $R^m$ is given by $\|x - y\|_1 = \sum_{1 \le i \le m} |\, x_i - y_i \,|$. This is sometimes referred to as the Manhattan distance, or Hamming distance. A metric $d$ is $L^1-$embeddable if there exist vectors $z_1, z_2, \ldots, z_n$ in $R^m$ for some $m$, such that $d_{ij} = \|z_i - z_j\|_1$ for $1 \le i < j \le n$. For example, consider first the metric on 4 points obtained from the complete bipartite graph $K_{2,2}$ by considering the usual graph metric : distances are given by the length of the shortest paths in the graph. If vertices 1 and 2 are in one part, the metric is $d = (2, 1, 1, 1, 1, 2)$. This metric is $L^1-$embeddable: consider the embedding into $R^2$ with $z_1 = (1, 0)$ $z_2 = (0, 1)$ $z_3 = (1, 1)$ $z_4 = (0, 0)$. Next consider the metric obtained from the complete bipartite graph $K_{2,3}$. It turns out that this metric is not

$L^1$ – embeddable, no matter how high the emdedding dimension $m$ is chosen. This can be shown by ad hoc methods. One of the purposes of this paper is to find inequalities that are satisfied for all $L^1$ – embeddable metrics, but violated by metrics which are not $L^1$ – embeddable, such as that induced by $K_{2,3}$.

A metric $d$ is called a *cut metric* if for some non-empty $S \subseteq \{1, \ldots, n-1\}$

$$d_{ij} = \begin{cases} 1 & i \in S,\ j \notin S \text{ or } i \notin S,\ j \in S \\ 0 & \text{otherwise}. \end{cases} \tag{1.2}$$

A cut metric $d$ is just the incidence vector of a cut in the complete graph $K_n$ with $S$ denoting the subset of vertices on one side of the cut. That is, $d_{ij}$ is one whenever edge $(i, j)$ crosses the cut, and zero otherwise. It follows from the triangle inequality that all metrics with distances zero and one are cut metrics, unless all of the distances are uniformly either all zero or all one.

The set of all metrics $d$ on $n$ points is a polyhedral cone $M_n$ in $R^{\binom{n}{2}}$ called the *metric cone*. The facets of this cone are the triangle inequalities (1.1). The cone generated by all non-negative combinations of cut metrics is called the *cut cone* (also known as the *Hamming Cone*), $H_n$. For a fixed subset $S$, the corresponding cut metric gives us (by multiplication by positive scalars) an *extreme ray* or *generator* of $H_n$. Since $S$ and its complement give the same metric, there are $2^{n-1} - 1$ extreme rays of $H_n$. It is easy to show that a metric on $n$ points is $L^1$ – embeddable if and only if it is contained in the cut cone, $H_n$. An important problem that we will discuss in this paper is the characterization of facets of $H_n$. A complete description of the facets of $H_n$ gives us a set of inequalities for testing whether or not a metric is $L^1$ – embeddable. Such a list must necessarily include the triangle inequalities, and indeed these are all facets of $H_n$.

Other definitions of $L^1$ – embeddability have appeared. The *hypercube* or *Hamming $N$ – cube* is the set of all binary vectors of length $N$ in $R^N$. It is not hard to show that a rational metric $d$ is $L^1$ – embeddable if and only if $kd$ embeds isometrically into a hypercube for some $N$ and some $k > 0$ [3]. Above, we in fact gave a hypercube embedding of the metric induced by $K_{2,2}$ with $k = 1$. Consider the metric induced by $K_4$. It is easy to see that there is no hypercube embedding with $k = 1$. There is an embedding with $k = 2$: (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1). A similar embedding exists for each $K_n$.

Closely related to these so-called *hypercube embeddings* are intersection patterns. We review the connection here as several results for intersection patterns have application in $L^1$ – embeddability. An *intersection pattern* is specified by an integer matrix $C = (c_{ij})$ of dimension $n \times n$. $C$ is an intersection pattern if there exist finite sets $A_i$, $i = 1, \ldots, n$ such that

$$|A_i \cap A_j| = c_{ij} \qquad 1 \le i \le j \le n.$$

The sequence $A_i$, $i = 1, \ldots, n$ is called a *realization*. Define $N = |\bigcup_{i=1}^{n} A_i|$ to be the size of the realization. For example, consider the pattern with $n = 3$ and

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

A realization of size 2 for $C$ is given by $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{a, b\}$.

The relationship between intersection patterns and hypercube embedding is given by the following.

**Proposition 1.1**

(a) If $C = (c_{ij})$ is an $n \times n$ intersection pattern with given $N$, then $d$ is a metric on $n + 1$ points embeddable in a Hamming $N$-cube, where

$$d_{ij} = c_{ii} + c_{jj} - 2c_{ij}, \;\; 1 \le i < j \le n, \qquad d_{i,n+1} = c_{ii}, \;\; 1 \le i \le n.$$

(b) If $d$ is a metric on $n + 1$ points that is embeddable in a Hamming $N$-cube, then $C = (c_{ij})$ is an $n \times n$ intersection pattern, with a realization of size $N$, where

$$c_{ij} = \frac{1}{2}\,(\, d_{i,\,n+1} + d_{j,\,n+1} - d_{ij}\,), \quad 1 \le i \le j \le n.$$

□

For example, consider the metric $d = (2, 1, 1, 1, 1, 2)$ induced by $K_{2,2}$ with the usual graph metric.

Performing the transformation of part (b) of the proposition we obtain the $3 \times 3$ matrix $C$ given above. The hypercube embedding of $K_{2,2}$ given previously can be seen to be a 0-1 encoding of the realization of size two given for $C$.

An important class of inequalities that give facets of $H_n$ are the $k-gonal$ inequalities introduced by Deza[36] [18] and later, independently, by Kelly[47] [48]. For an integer $k$ the $k-gonal$ inequalities have the form:

$$\sum_{1 \le i < j \le n} b_i b_j d_{ij} \le 0 \tag{1.3}$$

for integers $b_1, \ldots, b_n$ satisfying

$$\sum_{i=1}^{n} b_i = 1, \quad \sum_{i=1}^{n} | b_i | = k. \tag{1.4}$$

The triangle inequality (1.1) is 3-gonal, and is obtained by the vector $b$ with all components zero except $b_i = b_j = 1$ and $b_k = -1$. The first interesting new inequality is the so called *pentagon inequality* and is obtained by the vector $b$ with all components zero except $b_i = b_j = b_k = 1$ and $b_s = b_t = -1$, for distinct indices $i, j, k, s, t$. This yields the inequality

$$d_{ij} + d_{ik} + d_{jk} + d_{st} \le d_{is} + d_{js} + d_{ks} + d_{it} + d_{jt} + d_{kt}. \tag{1.5}$$

The reader may verify that the metric induced by $K_{2,3}$ violates a pentagon inequality: Let $\{i, j, k\}$ be the vertices in one part and $\{s, t\}$ be the vertices in the other. The left hand side of (1.5) is 8 and the right hand side is 6.

A metric $d$ is $k-gonal$ if it satisfies all of the $k$-gonal inequalities. It is easy to verify that a metric space that is $k$-gonal is also $(k-1)$-gonal. A more surprising result is that if a metric space is $(2k+1)-gonal$ then it is also $(2k+2)-gonal$[36] [18] [47]. However, $2k$-gonal metrics are *not* necessarily $(2k+1)$-gonal. A metric is *hypermetric* if it is $(2k+1)-gonal$ for all $k$.

An inequality is *valid* over a set if it is satisfied for every element in the set. An important property of the $k-gonal$ inequalities is that they are valid over the cut cone[36] [18] [47]. In other words, if a metric is $L^1-$embeddable then it is hypermetric. In our example, since the pentagon inequality is valid over the cut cone, we have a proof that the metric induced by $K_{2,3}$ is not in the cut cone, ie. it is not $L^1-$embeddable.

Some, but not all, of the $k-gonal$ inequalities form *facets* of $H_n$ and for $n \le 6$ hypermetricity is equivalent to $L^1-$embeddability. This was proved for $n \le 5$ in[36] and for $n = 6$ in[9]. For larger $n$ there exist facets of $H_n$ which are not $k-gonal$ for any $k$. In Section 2 of this paper, we collect these and other known results on the facial structure of $H_n$. This section is rather technical and may be skipped by readers interested in applications.

In Section 3 of the paper we discuss some complexity issues related to $L^1-$embeddability. Most of the problems turn out to be computationally intractable, although efficient algorithms exist in some special cases. We also discuss the relationship of some results on the complexity of edge factoring of graphs to hypercube embeddings.

In Section 4 of the paper we show how results about $L^1-$embeddability are at the heart of the feasibility problem for multicommodity flows. The Ford and Fulkerson theorem for the existence of a single commodity flow can be restated as an inequality that must be valid for all *cut metrics*. A more general result, known as the Japanese theorem, gives a condition for the existence of a multicommodity flow. This theorem gives an inequality that must be satisfied by all *metrics*. For multicommodity flows involving a small number of terminals, it is known that the condition of the Japanese theorem can be replaced with one of the Ford-Fulkerson type. We review these results and show that the existence of such Ford-Fulkerson type conditions for flows with few terminals depends critically on the fact that certain metrics are $L^1-$embeddable. Integral feasibility is related to embeddability into hypercubes.

Finally in Section 5 we briefly mention some polytopes related to the cut cone.

## 2. $\mathbf{L^1}$ – embeddability, Hypermetricity and Facets of $H_n$.

In this section we survey known results on the relationship of $L^1-$embeddability to hypermetricity and on the facets of $H_n$. As we remarked in the introduction, all $k-gonal$ inequalities are valid over $H_n$ and some form facets. We call such facets *hypermetric* facets. We begin by defining an equivalence between such facets, which will simplify the task of describing the facet producing inequalities.

Integer vectors $b = ( b_1 , \ldots , b_n )$ and $c = ( c_1 , \ldots , c_n )$ are *switching equivalent* if there exists an index set $J \subset \{1, \ldots, n\}$ such that

$$c_i = - b_i \qquad i \in J \tag{2.1}$$

$$c_i = b_i \qquad i \notin J \tag{2.2}$$

$$\sum_{i \in J} b_i = 0. \tag{2.3}$$

It is easy to verify that if $a$ is switching equivalent to $b$ and $b$ is switching equivalent to $c$, then $a$ is switching equivalent to $c$. Integer vectors $b = ( b_1 , \ldots , b_n )$ and $c = ( c_1 , \ldots , c_n )$ are *permutation equivalent* if there exists a permutation $\pi = (\pi(1), \ldots, \pi(n))$ of the integers $1, \ldots, n$ such that $c_i = b_{\pi(i)}$, $i = 1, \ldots, n$. We say that $b$ is *equivalent* to $c$ if $b$ can be transformed to $c$ by a switch and or permutation. We say that $b = ( b_1 , \ldots , b_n )$ *defines a facet* of $H_n$ if it satisfies (1.4) and the inequality (1.3) determines a facet of $H_n$. Deza [20] proved the following.

**Theorem 2.1** Let $b = ( b_1 , \ldots , b_n )$ and $c = ( c_1 , \ldots , c_n )$ be two integer sequences satisfying (1.4).

(a) If $b$ is equivalent to $c$ then $b$ defines a facet of $H_n$ if and only if $c$ defines a facet of $H_n$.

(b) $b$ defines a facet of $H_n$ if and only if $b' = (b_1, \ldots, b_n, 0)$ defines a facet of $H_{n+1}$.

In view of Theorem 2.1, we may assume that integer sequences determining facets are labeled such that

$$b_1 \geq b_2 \geq \cdots \geq b_f > 0 > b_{f+1} \geq \cdots \geq b_n . \tag{2.4}$$

A list of hypermetric facets for $H_n$ is *canonical* if all hypermetric facets of $H_n$ can be obtained from the list by permutation and/or switching and/or the addition of zeroes. An important problem is to find such lists of canonical hypermetric facets of $H_n$. For large $n$, a complete list of such facets is not known. However, an important classes of facet-producing integer sequences are known and will be described in Section 2.1. In Section 2.2 we give a complete list of all canonical facets for $H_n$, $n \le 6$. We also list some known facets for $H_7$ and $H_8$. From the point of view of the complexity of $H_n$, there is interest in the largest integer $g(n)$ such that there is a $g(n) - gonal$ facet of $H_n$. Since $H_n$ is polyhedral, $g(n)$ is finite. The set of *hypermetrics* also form a cone but it was unknown until recently whether or not this cone is polyhedral. This has been settled affirmatively by Deza, Grishukhin and Laurent[30]. Since the cone is polyhedral it means for each $n$ there is a smallest integer $f(n)$ such that an $n$ point metric is hypermetric if and only if it is $f(n)$-gonal. No tight bounds on $f(n)$ are known. In Theorem 3 of [31], Deza and Maehara show that for each $n$ there exists a maximal $c(n)$, $0 < c(n) < n$, such that $d^c$ is hypermetric for every $n$ point metric $d$. In Section 2.3 we give a quadratic lower bound and exponential upper bound for $g(n)$. Finally in Section 2.4 we discuss the problem of finding the variety of realization of metrics. The study of non-hypermetric facets is beyond the scope of this paper. For results on these facets the reader is referred to[4] [23] and the references given in the Footnote at the end of the paper.

### 2.1. Special and Linear Facets

When $k$ is close to $n$, a complete characterization of facet producing $k - gonal$ inequalities is known. Call a vector $b$ satisfying (2.4) *special* if $b = (1, 1, -1)$ or $b_{n-1} = -1$. This is equivalent to saying that all but possibly one of the negative components of $b$ are $-1$. We call $b$ *linear* if $b_n = -1$. Special and linear facets were completely characterized by Deza[20], although in the terminology of intersection patterns. This result is restated in the following theorem.

**Theorem 2.2**

(a) If $b$ defines a facet then either $b = (1, 1, -1)$ ;or $f = n - 2$, $b_1 = 1$; or $3 \le f \le n - 3$.

(b) If $b$ is special and $3 \le f \le n - 3$ then it defines a facet if and only if

$$n - f - 1 \ge ( b_1 + b_2 ) - sign \mid b_1 - b_f \mid . \qquad (2.5)$$

(c) If $b$ is linear then it defines a facet if and only if it satisfies the condition in (a). □

We call an integer vector $b'$ $special - like$ (respectively $linear - like$) if it is equivalent to a special (respectively linear) vector $b$. All known hypermetric facets of $H_n$ are special-like. This indicates the power of the previous theorem, and the need to effectively characterize special-like facets. The next simple lemma is useful for such a characterization. If two vectors are switching equivalent then their components must sum to the same value. With (2.1) and (2.2) this obvious necessary condition is also sufficient.

**Lemma 2.3** $b$ and $c$ are switching equivalent if and only if

$$b_i = \pm c_i \quad \text{and} \quad \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i .$$

**Proof:** We observed the necessity above. For sufficiency, let

$$J = \{ \, i : b_i = - c_i \} .$$

Then

$$0 = \sum_{i=1}^{n} (\, b_i - c_i \,) = \sum_{i \in J} (\, b_i - c_i \,) + \sum_{i \notin J} (\, b_i - c_i \,)$$

$$= 2 \sum_{i \in J} b_i .$$

Therefore $J$ satisfies (2.1)-(2.3). □

The next result characterizes special-like and linear-like sequences. Let $\alpha = | \, \{ b_i : |b_i| = 1 \, \} \, |$ .

**Theorem 2.4** An integer vector $b$ satisfying (1.4) is

(a) linear-like if $\alpha \geq \dfrac{k-1}{2} \geq 1$; and

(b) special-like if $\alpha \geq \dfrac{k-1}{2} - | \, b_i \, | \geq 1$ for some $i$.

**Proof:** Let $|b_{i_0}|, \ldots, |b_{i_g}|$ be the absolute values of the components of $b$ which are not $\pm 1$ indexed in non-increasing order. For (a), if $b$ is linear-like then it must be equivalent to an integer vector

$$b' = ( \, |b_{i_0}|, \ldots, |b_{i_g}| \, , 1, \ldots, 1, -1, \ldots, -1 \, ).$$

Suppose that there are $m$ minus ones in the above sequence, and $\alpha - m$ $1's$. Then, by Lemma 2.3, $b$ is equivalent to $b'$ if and only if

$$\sum_{j=0}^{g} |\, b_{i_j}| + (\alpha - m) - m = 1$$

and $m = \dfrac{k-1}{2}$. For $b'$ to be a well defined linear sequence we must have that $1 \le m \le \alpha$, so

$$1 \le \frac{k-1}{2} \le \alpha,$$

proving (a).

For (b), $b$ is special-like if, for some index $i_l$, it is equivalent to

$$b^* = (\,|b_{i_0}|, \ldots, |b_{i_{l-1}}|, |b_{i_{l+1}}|, \ldots, |b_{i_g}|\,, 1, \ldots, 1, -1, \ldots, -1, -|b_{i_l}|\,).$$

Let $m$ and $\alpha$ be defined as above. Again by the lemma

$$\sum_{j=0}^{g} |b_{i_j}| + (\alpha - m) - m - 2\,|b_{i_l}| = 1$$

$$k - 2m - 2\,|b_{i_l}| = 1$$

and so

$$m = \frac{k-1}{2} - |b_{i_l}|.$$

Therefore $b$ is equivalent to $b^*$ whenever

$$1 \le \frac{k-1}{2} - |b_{i_l}| \le \alpha.$$

□

**2.2. Hypermetric Facets of $H_n, n \le 8$**

The 3-gonal inequalities are just the familiar triangle inequalities, and they are facets of $H_n$ for all $n \ge 3$. Deza[36] [18] showed that all four point metrics are $L^1$ – embeddable and so $H_4 = M_4$ and that all 5-gonal five point metrics are $L^1$ – embeddable. He also showed that all 4 or 5 point $L^1$-embeddable metrics are embeddable in a hypercube if and only if the perimeter of each triangle is even. Avis and Mutt showed that all 7-gonal six point metrics are

$L^1-$embeddable[9]. These results are summarized in the following theorem.

**Theorem 2.5** For $n \le 6$ the canonical facets for $H_n$ are defined by $(1,1,-1)$, $(1,1,1,-1,-1)$ and $(2,1,1,-1,-1,-1)$.

For seven points, Avis[5] [8] found an example of a seven point hypermetric that was not $L^1-$embeddable, (graph metric of $K_7 - P_3$ ) and dually, a facet of $H_7$ that is not $k-gonal$ for any $k$. This facet was also found by Assouad[2] who later also found several other non-hypermetric facets for $n = 7$ (see Section 2.4). Some partial results for hypermetric facets of $H_7$ can be obtained from Theorem 2.4. Any special-like 7-gonal facet is equivalent to (1,1,1,1,-1,-1,-1) and any special-like 9-gonal facet is equivalent to either (3,1,1,-1,-1,-1,-1) or (2,2,1,-1,-1,-1,-1). There are no 11-gonal, 13-gonal, or 15-gonal facets.

For $n = 8$, we have the following results on hypermetric facets of $H_8$, that are derived from Theorem 2.4. All special-like 9-gonal facets are switching equivalent to linear facet (2,1,1,1,-1,-1,-1,-1). All special-like 11-gonal facets are switching equivalent to linear facets (4,1,1,-1,-1,-1,-1,-1), (3,2,1,-1,-1,-1,-1,-1) or (2,2,2,-1,-1,-1,-1,-1). All special-like 13-gonal facets are switching equivalent to the non-linear facet (3,2,2,-1,-1,-1,-1,-2).

Recall that $g(n)$ is the largest integer such that there is a $g(n)-gonal$ facet of $H_n$. From the above we have that $g(3) = g(4) = 3, g(5) = 5$ and $g(6) = 7$. Recently, Grishukhin [39] showed that $g(7) = 9$. Therefore $g(n) = 2n - 5$ for $n = 4, 5, 6, 7$ (but $g(8) \ge 13$) and all hypermetric facets with $n \le 7$ are linear. $g(n)$ is unknown for all $n \ge 8$, however we will give a some bounds for $g$ in the next section.

## 2.3. Bounds for $g(n)$

In this section we exhibit a class of $k-gonal$ facets of $H_n$ with $k \ge n^2/4 - 4$ giving a quadratic lower bound for $g$. We also give an upper bound that is exponential in $n$. Let

$$m = \left\lfloor \frac{n+1}{4} \right\rfloor$$

and let $b = ( b_1, \ldots, b_n )$ have components

$$b_i = \begin{cases} m & 1 \le i \le n - 2m - 1 \\ m - 1 & i = n - 2m \\ -1 & n - 2m + 1 \le i \le n - 1 \\ m(2 + 2m - n) + 1 & i = n \end{cases}$$

**Theorem 2.6** For $n \ge 7$, $g(n) \ge n^2/4 - 4$ and $b$ defines a facet of $H_n$ with $k = 2m(n - 2m) - 3 \ge n^2/4 - 4$.

**Proof:** We verify that $b$ satisfies (1.4) and condition (b) of Theorem 2.2. First, since $n \ge 7$, $m \ge 2$ and the components of $b$ are ordered as in (2.4) with $f = n - 2m$. To check that the components of $b$ sum to one, we see that

$$\sum_{i=1}^{n} b_i = m(n - 2m - 1) + m - 1 - (2m - 1) + (2 + 2m - n)m + 1 = 1.$$

Now, $3 \le f = n - 2m \le n - 3$ and

$$n - f - 1 = 2m - 1 = b_1 + b_2 - sign(b_1 - b_f).$$

This shows that condition (b) of Theorem 2.2 is satisfied and $b$ defines a facet. Next we observe that

$$k = 2 \sum_{b_i > 0} b_i - 1 = 2(m(n - 2m) - 1) - 1 = 2m(n - 2m) - 3.$$

Finally, let $l = (n + 1)$ $mod$ 4, so that $n + 1 = 4k + l$. Then

$$2m(n - 2m) = 2m(2m + l - 1) = \frac{n^2}{4} - \frac{(l - 1)^2}{4} \ge \frac{n^2}{4} - 1.$$

The proof is complete. □

We conjecture that for integers $k < m$, if $k$-gonal and $m$-gonal facets exist, then there exists a $l$-gonal facet for each $k < l < m$. We remark that this is true for linear-like $k$–gonal and $m$–gonal facets, where in fact there exists a linear-like $l$–gonal facet for $k \le l \le m$. A corresponding statement is true for special-like facets. To get an upper bound on $g(n)$, we show that any integer $b = (b_1, \ldots, b_n)$ defining a facet must be the solution to a set of equations with binary coefficients and a binary right hand side. We call a cut metric $d$ determined by

$S \subseteq \{1, \ldots, n-1\}$, (refer to (1.2)), a *root* for $b$ *based on* $S$ if (1.3) is satisfied as an equation. That is,

$$\sum_{1 \le i < j \le n} b_i b_j d_{ij} = 0 \tag{2.6}$$

If $b$ defines a facet then there must be $N = \binom{n}{2} - 1$ linearly independent roots for $b$, since the facet is defined by $N$ linearly independent generators of $H_n$. The following simple but useful lemma characterizes the roots of $b$.

**Lemma 2.7** $S \subseteq \{1, \ldots, n-1\}$, defines a root $d$ for $b$ if and only if

$$\sum_{i \in S} b_i = 0 \; or \; 1. \tag{2.7}$$

**Proof:** For $d$ defined in the Lemma,

$$0 = \sum_{1 \le i < j \le n} b_i b_j \mathbf{d}(x_i, x_j) = \sum_{i \in S} \sum_{j \notin S} b_i b_j$$

$$= \Big( \sum_{i \in S} b_i \Big) \Big( 1 - \sum_{i \in S} b_i \Big).$$

Since all the $b_i$ are integer,

$$\sum_{i \in S} b_i = 0 \; or \; 1$$

as required. The argument is reversible, proving the lemma. □

We now show that when $b$ defines a facet there are $n-1$ linearly independent equations of the type (2.7).

**Lemma 2.8** Let $b = ( b_1, \ldots, b_n )$ define a facet of $H_n$ with roots $d^1, \ldots, d^N$ based on the subsets $S^1, \ldots, S^N$, $N = \binom{n}{2} - 1$. Then the system of equations

$$\sum_{i \in S^t} b_i = 0 \; or \; 1 \qquad t = 1, \ldots, N \tag{2.8}$$

has rank $n-1$.

**Proof:** Form the $N \times (N+1)$ matrix $A$ with rows $A_i$ corresponding to roots $d^i$, $i = 1, \ldots, N$. In other words

$$A_i = (\ d^i_{12},\ d^i_{13},\ \ldots,\ d^i_{n-1,n}\ ).$$

Let $\bar{b}$ be the $N+1$ vector

$$\bar{b} = (\ b_1 b_2,\ b_1 b_{3,}\ \ldots,\ b_{n-1} b_n).$$

Then we have

$$A\bar{b} = 0$$

and $A$ has rank $N$. By relabeling points if necessary, we can assume that the first column can be deleted to give a $N \times N$ non-singular matrix $A'$. Consider the $N \times (n-1)$ submatrix of $A'$ formed by selecting columns of $A'$ corresponding to $d_{1n}, d_{2n}, \ldots, d_{n-1,n}$. Now $D$ is precisely the coefficient matrix for the left hand side of (2.8). Indeed, in row $t$ of $D$, $1 \le t \le N$,

$$d^t_{in} = \begin{cases} 1 & i \in S^t \\ 0 & otherwise \end{cases}$$

which is the coefficient of $b_i$ in equation $t$ of the system (2.8). Since $D$ has linearly independent columns, being a submatrix of $A'$, there are $n-1$ linearly independent equations in (2.8). □

Using Kramer's rule we can now get a bound on the integers $b_i$. Let $\beta_n$ be the maximum of an $n \times n$ determinant with all binary entries. Williamson[59] gave $\beta_n = 2^{n-1}\gamma_{n-1}$ where $\gamma_{n-1}$ is the maximum determinant of $(n-1) \times (n-1)$ $(\pm 1)$-matrix; so $\gamma_n = n^{n/2}$ if the Hadamard $n \times n$ matrix exists.

**Theorem 2.9** $g(n) \le n\ \beta_{n-1}$.

**Proof:** By Kramer's rule, for $i = 1, \ldots, n-1$ we have

$$b_i = \det(B)/\det(B')$$

for two $(n-1) \times (n-1)$ binary matrices $B$ and $B'$. Since $B'$ is non-singular it $\det(B') \ge 1$ and the theorem follows. □

The above upper bound is probably very weak.

### 2.4. Variety of Realization

Given that a metric $d$ on $n$ points is $L^1$ – embeddable or embeddable in a hypercube, we may ask additional questions about the possible variety of embedding possible. For example, we may ask if a realization is unique and if not, how many distinct realizations are possible, or we may seek an embedding that is minimal in some sense.

An embedding of $d$ that is unique for any $\lambda d$ is called *rigid*. Metrics on at most five points with rigid hypercube embeddings were studied in[22] [24]. Questions concerning the rigidity and number of distinct embeddings for general metrics were discussed in[21] [22] [24] [17]. For example,[22] [24] fully describes the variety of hypercube realizations for metrics on at most 5 points, belonging to a facet of $H_5$. The paper[17] relates the rigidity (actually a property of a facet) to the *support* of the facet (the set of cut metrics generating it) and shows that the number of realizations is at most polynomial.

We look at the question of the dimension of embedding more closely because it has some applications to complexity in the next section. Two natural measures of minimality arise and have been studied. The first is to find the smallest integer $m$ such that $d$ is $L^1$ – embeddable om $R^m$. Let $m(n)$ denote the maximum such $m$ for any metric on $n$ points. A second measure is to find the minimum dimension $q$ such that for some $t > 0$ the metric $td$ embeds into the $q$-dimensional hypercube. To illustrate the difference between these two measures, consider the metric $d$ induced by a path $P_n$ on $n$ points. $d$ can be $L^1$-embedded into $R^1$. However the smallest hypercube into which $d$ can be embedded has dimension $q = n - 1$.

We begin with minimum $L^1$-embedding. It is immediate that $d$ can be embedded into $R^m$ if and only if it can be expressed as a non-negative combination of $m$ metrics $L^1$ – embeddable in $R^1$: simply take each coordinate of an embedding of $d$ as a metric embeddable in $R^1$. Each cut metric can be embedded onto the end points of the unit interval, so is embeddable in $R^1$. If $d$ is $L^1$ – embeddable, then $d \in H_n$ so $d$ can be expressed as a positive combination of at most $\binom{n}{2}$ cut metrics, and hence $d$ is $L^1-$embeddable in $R^{\binom{n}{2}}$. This gives the upper bound in the following theorem, originally due to Witsenhausen. A set of examples giving the lower bound and some exact values for small $n$ are due to Ball[11].

**Theorem 2.10**

$$\binom{n-2}{2} \le m(n) \le \binom{n}{2},$$

$$m(3) = m(4) = 2,\ m(5) = 3,\ m(6) = 6. \quad \square$$

We now turn to minimum dimension hypercube embedding. Suppose that a metric $d$ on $n$ points can be embedded onto the $N$-cube. We may assume that one of the points, say point 1, is mapped to the origin, $(0, 0, \ldots, 0)$, of the hypercube. Suppose we add a new point $n+1$ with distances

$$d_{i,n+1} = t - d_{1i}.$$

for some given integer $t$, large enough so that all of the new distances are non-negative. We call the new metric the $t-$*suspension* of $d$ and denote it $(d, t)$. We see that $(d, N)$ is embeddable in the $N$-cube by embedding point $n+1$ onto vertex $(1, 1, \ldots, 1)$ of the $N$-cube. The implication is reversible, giving the following result of Deza[19] linking the dimension problem to the question of hypercube embeddability.

**Theorem 2.11** $d$ is embeddable in a hypercube of dimension $N$ if and only if the $N$-suspension $(d, N)$ is hypercube embeddable. $\square$

A similar result is true for $L^1-$embeddability.

## 3. Complexity

In this section we survey results about various algorithmic questions related to $L^1-$embeddability. We assume that the given metrics have all integer distances. We begin with the most fundamental question which has been raised in[36] [18] [20] [5] [8] [55] [44] [45].

**P1. Membership in $H_n$ ($L^1$ − embeddability)**

Instance: Integer metric $d$ on $n$ points.

Question: Is $d \in H_n$?

Complexity: NP-Complete

The complexity of P1 follows from recent developments in the equivalence of various algorithmic problems for convex bodies, and the fact that the following dual problem is Co-NP-Complete. The dual problem is testing whether an inequality is valid over $H_n$.

**P2. Valid Inequalities over $H_n$.**

Instance: Integer vector $c$ in $R^{\binom{n}{2}}$.

Question: Is $cx \geq 0$ valid for all $x \in H_n$?

Complexity: Co-NP-complete, Karzanov[45].

The reduction for P2 is from the maximum cut problem for graphs. The equivalence of membership testing and testing for a valid inequality depends on certain additional conditions that we now outline. A complete treatment of this topic is contained in the book by Grötschel, Lovász and Schrijver[40], which the reader should consult for definitions not given explicitly here. Firstly we consider the full-dimensional bounded polyhedron

$$\bar{H}_n = \{ \, x \in H_n \mid 1x \leq \binom{n}{2} \}$$

obtained by truncating $H_n$. Following[40], we say that a polyhedron $P$ has *facet − complexity* at most $\phi$ if there exists a system of inequalities with rational coefficients that has solution set $P$ and such that the encoding length of each inequality of the system is at most $\phi$. Similarly the *vertex − complexity* $v$ of $P$ is a bound on the encoding length of each vertex of $P$. Since the vertices of $\bar{H}_n$ are essentially $0-1$ vectors (multiplied by a scaling factor), $\bar{H}_n$ has vertex-complexity $\binom{n}{2}$. It follows from[40] (6.2.4) that $\bar{H}_n$ has facet-complexity $3\binom{n}{2}^3$.

A further condition is that we need to know a ball contained in $\bar{H}_n$. In [3] it is stated that the set of $\binom{n}{2}$ cut metrics generated by all 2-element subsets of $\{1, \ldots, n\}$ are affinely independent. With the origin, these vectors span a full-dimensional simplex contained in $\bar{H}_n$. By[40] (6.2.6) the barycentre of this simplex is the centre of a ball of radius

$$2^{-7\binom{n}{2}^3 \phi} = 2^{-21\binom{n}{2}^6}$$

completely contained in $\bar{H}_n$.

We can now demonstrate the equivalence of P1 and P2. By a theorem of Yudin and Nemirovskii[60] (See [40] (4.3.2)), the weak membership problem for a polyhedron $P$ satisfying the above conditions is at least as hard as the weak validity problem. Roughly speaking, the weak versions of these problems allow an "error" tolerance of some given $\delta > 0$ in the correctness of the answer. One of the main results of [40], (6.3.2), is that the weak validity problem is as hard as the (strong) validity problem under the conditions given above. Trivially, the (strong) membership problem is at least as hard as the weak membership problem. Applying these results to $\bar{H}_n$, the fact that P2 is NP-hard implies that P1 is also NP-hard. Because of the bound on the facet complexity, we also have that P1 is in NP, so the problem is NP-complete.

Results about the complexity of hypercube embedding follow from the following result on intersection patterns.

**P3. Intersection Pattern.**

Instance: Integer $n \times n$ matrix $C$.

Question: Is $C$ an intersection pattern?

Complexity: NP-complete, even if each $c_{ii} = 3$, $i = 1, \ldots, n$ but solvable in polynomial time if each $c_{ii} = 2$, Chvátal[15].

This result can be reformulated as a graph theory problem. Interpret $C$ to be the adjacency matrix of a multigraph with $n$ vertices and $c_{ij}$ edges between vertices $i$ and $j$, for distinct $i$ and $j$. If $C$ is an intersection pattern, then the edges of the multigraph can be partitioned into cliques.

The sets $A_i$ realizing the intersection pattern correspond to a list of the cliques containing the vertex $i$. The size $N$ of the realization corresponds to the number of cliques in the partition.

As an example, consider the following intersection pattern:

$$C = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

Figure 3.1(a) shows the corresponding multigraph, which in this case is a graph since each $c_{ij} \le 1$ for distinct $i$ and $j$. We can partition the edges into four cliques $a = \{1, 5\}$, $b = \{1, 2, 3\}$, $c = \{3, 4\}$, $d = \{2, 4, 5\}$. Now we obtain the realization $A_1 = \{a, b\}$, $A_2 = \{b, d\}$, $A_3 = \{b, c\}$, $A_4 = \{c, d\}$, $A_5 = \{a, d\}$ of size four for the intersection pattern $C$. Since each $A_i$ has cardinality two, we can regard it as an edge in a graph with vertex set $\{a, b, c, d\}$, as shown in Figure 3.1(b). It is easy to check that the graph in Figure 3.1(a) is just the line graph of the graph in Figure 3.1(b).

Figure 3.1(a) Figure 3.1(b)

By Chvátal's result, the clique partition problem is still hard even if each vertex is contained in exactly 3 cliques. If each vertex is contained in exactly 2 cliques (as in the example), then the problem reduces to line graph recognition, which is solvable in polynomial time. Using the equivalence between intersection patterns and hypercube embedding given in Proposition 1.1 we obtain the following.

**P4. Hypercube Embedding**

Instance: Integer metric $d$ on $n$ points.

Question: Is $d$ embeddable in a Hamming $N$-cube?

Complexity: NP-complete, even if some point is distance 3 from all other points and all other distances are chosen from $\{0, 2, 4, 6\}$, but solvable in polynomial time if some point is distance 2 from all other points and all other distances are chosen from $\{0, 2, 4\}$[15]. Also solvable in polynomial time for metrics arising from bipartite graphs, Djokovic[37], and if each distance is chosen from $\{0, 1, 2, 3\}$, Avis[10].

Using the graph theoretic interpretation given above, other complexity results for edge partitions of graphs can be restated in terms of hypercube embedding or intersection patterns. For example, Hoyler[41] showed that it is NP-complete to determine if the edges of a graph can be partitioned into complete subgraphs of fixed size $t$, for all $t \geq 3$. In terms of intersection patterns, this implies that it is NP-complete to decide if a matrix is an intersection pattern realizable by subsets $A_i \subseteq \{1, \ldots, N\}$ such that each element is contained in exactly $t$ subsets.

Next we look at the problem of finding an embedding of minimum dimension. As was shown in Section 2, a metric $d$ on $n$ points can be embedded in a cube of dimension $N$ if and only if the $n+1$ point suspension $(d, N)$ of $d$ is hypercube embeddable. Using P4 we can show that testing whether a suspension is hypercube embeddable is NP-complete. Indeed, consider a metric $d$ with distances chosen from $\{0, 2, 3, 4, 6\}$. If it can be embedded in a hypercube, it can be embedded in a cube of dimension at most $6n-6$. This follows from the fact that we may take any point of the embedding as the origin, and each other point being distance at most 6 from the origin can use at most 6 additional dimensions. It follows from Section 2.5 that the suspension

$(d, 6n - 6)$ is hypercube embeddable if and only if $d$ is hypercube embeddable. This proves the following.

**P5. Embedding in Given Dimension.**

Instance: An integer metric $d$ and an integer $N$.

Question: Can $d$ be embedded in a Hamming $N$-cube?

Complexity: NP-complete.

It follows that it is NP-hard to determine the minimum dimension that admits an embedding. In conclusion we mention that very little is known about testing a metric for hypermetricity.

**P6. Hypermetricity**

Instance: An integer metric $d$.

Question: Is $d$ hypermetric?

Complexity: Unknown. For the class of connected hypermetric spaces (the graph with edges between all pairs of points with distance one is connected) Deza and Terwilliger have found a characterization that yields an exponential time algorithm[58].

## 4. Connections with Multicommodity Flows

Let $V$ denote a set of $n$ points and let $E$ denote the set of all unordered pairs of points in $V$. It is convenient to consider $E$ to be the set of edges of a complete graph on $V$. An instance of the multicommodity flow problem is given by two non-negative vectors indexed by $E$: a capacity $c(e)$ and a requirement (or demand) $r(e)$ for each $e \in E$. Let $U = \{e \in E: r(e) > 0\}$. If $T$ denotes the subset of $V$ spanned by the edges in $U$, then we say that the graph $G = (T, U)$ denotes the *support* of $r$. For each edge $e = (s, t)$ in the support of $r$, we seek a flow of $r(e)$ units between $s$ and $t$ in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a flow exists we call $c, r$ *feasible*.

A necessary and sufficient condition for feasibility is given by the 'Japanese Theorem' of Iri[43] and Kakusho and Onaga[51]. It was stated in this form by Lomonosov[49] [50].

**Theorem 4.1** A pair $c, r$ is feasible if and only if

$$(c - r)\ x \geq 0 \tag{4.1}$$

is valid over $M_n$.

This condition can easily be checked by solving the linear program:

$$\min\ (c - r)\ x$$

$$x \in M_n$$

This LP has $3\binom{n}{3}$ linear constraints in $\binom{n}{2}$ variables, each constraint being a triangle inequality. A second way to test the condition of feasibility would be to test (4.1) for each of the extreme rays (generators) of $M_n$. The extreme rays of $M_n$ have been studied in[6] [7] [49] [50]. These studies show that there does not appear to be any easy way to characterize all of the extreme rays of $M_n$, and so the LP approach is preferred. A different direction is to consider weakening the condition of (4.1) to only test over a subcone of $M_n$ for which the generators are well characterized. A natural question is to ask when it is sufficient to verify (4.1) over $H_n$ the cone of $L^1$ embeddable metrics. It turns out that the celebrated Ford-Fulkerson theorem[38] for single commodity flows gives the first partial answer to this question. In this case, the support of $r$ is just a single edge, that is, $K_2$. The condition "being valid over $H_n$" translates into the condition "being valid over all cuts" and the inequality (3.1) is then just the Ford-Fulkerson condition.

**Theorem 4.2** A pair $c, r$, where the support of $r$ is $K_2$, is feasible if and only if

$$(c - r)\ x \geq 0$$

is valid over $H_n$. □

Papernov[53] has obtained a complete characterization of supports for which (4.1) can be weakened to a result like (4.2). Let $S^2$ denote the family of all graphs which are a union of two stars. Papernov proved:

**Theorem 4.3** A pair $c, r$, where the support of $r$ is a graph of $\{K_4, C_5\} \cup S^2$, is feasible if and only if

$$(c - r)\ x \geq 0$$

is valid over $H_n$. □

Furthermore, if $r$ has support outside of the class of graphs in (4.3), Papernov has constructed examples of pairs satisfying (4.3) which are not feasible. We show later in this section how Theorem 4.3 is closely related to the fact that all 5-gonal metrics on five points are $L^1$-embeddable.

From a complexity point of view, how easy is it to verify if an inequality is valid over $H_n$? As we saw in Section 3, this is Co-NP-complete in general. For the test in (4.3) however, the vector $(c - d)$ has sufficient structure to enable the test to be performed in polynomial time[44].

Interest in multicommodity flows usually centres around the problem of finding a set of integral flows. If the condition in (4.3) is strengthened, a condition for the existence of a set of integral flows is obtained. We say that a vector $c$ indexed by $E$ is even if for each cut metric $x$, $cx$ is even.

**Theorem 4.4** A pair $c, r$, where $c - r$ is even and the support of $r$ is a graph of $\{K_4, C_5\} \cup S^2$, is feasible with integral flows if and only if

$$(c - r)\ x \geq 0$$

is valid over $H_n$.

The result in this form is due to Lomonosov[50]. It was independently established for support $K_4$ by Seymour[56]. Dinits, see[1], had previously reduced the case where the support is a subgraph of a graph in $S^2$ to the two commodity flow problem. The two commodity flow problem had been solved by Rothschild and Whinston[54] strengthening an earlier result of Hu[42]. A polynomial time algorithm for testing the condition in (4.4) and for finding an integral flow when it is satisfied was given by Karzanov[44]. The most general result of this type is also due to Karzanov[46]:

**Theorem 4.5** A pair $c, r$, where $c - r$ is even and the support of $r$ is a subgraph of $K_5$ ( including $K_5$ itself), is feasible with integral flows if and only if

$$(c - r)\, x \geq 0$$

is valid over $M_n$. □

The above results are intimately connected to the results about $L^1$ – embeddability described in section 2, as we will now show. For a metric $d$ on $n$ points $V$, an *extremal graph* of $d$ is a minimal graph, $H = (V_0, W)$ such that, for each $x, y \in V$ there exists $st \in W$ (not necessarily distinct from $x$ and $y$) satisfying

$$d_{sx} + d_{xy} + d_{yt} = d_{st}.$$

$V_0$ is the set of endpoints of edges in $W$. If all $d_{ij}$ are positive we say that $d$ is positive, and the extremal graph is unique. The extremal graph of a positive metric was introduced by Lomonosov[49] [50], where it was called the *antipode* graph. For a positive metric, the complement of $H$ is the graph consisting of all edges $xy$ that lie on some shortest path of length at least two between a pair of vertices in $V_0$. The importance of the extremal graph is demonstrated in the following key theorem. A stronger half-integral version of the theorem is contained in[45].

**Theorem 4.6** If $d$ has an extremal graph in the family $\{K_4, C_5\} \cup S^2$, then $d$ is $L^1$ – embeddable. □

The result of the theorem is not true for graphs $H$ outside of the specified class. The theorem is proved by proving the intermediate result that a metric $d$ satisfying the conditions of the theorem is $L^1$ – embeddable if and only if its restriction to $V_0$ is $L^1$ – embeddable. Since the original metric $d$ can have any number of points, the importance of this intermediate result is that the $L^1$ – embeddability of $d$ can be checked by just considering a small subgraph. As we remarked earlier, the case of multicommodity flow problems with supports in $S^2$ was reduced to problems with support $2K_2$ which is a subgraph of $K_4$. Using Deza's result that all metrics on 4 points and all 5-gonal metrics on five points are $L^1-$embeddable[36], one can show that all metrics which have an extremal graph that is a subgraph of $K_4$ or $C_5$ are $L^1-$embeddable, completing the proof of Theorem 4.6.

Lomonosov[49] [50] showed that extremal graphs also play an important role in testing feasibility.

**Theorem 4.7** A pair $c, r$ is feasible if and only if

$$(c - r)\ x \geq 0$$

is valid for all metrics $x$ having an extremal graph $H = (V_0, W)$ such that $W$ a subset of the support of $r$. □

It the support of $r$ is very sparse, this theorem greatly limits the metrics for which we have to test the Japanese condition (4.1). In fact, if the support is a subgraph of the class $\{K_4, C_5\} \cup S^2$, by Theorem 4.6 we need only consider cut metrics. This gives Theorem 4.3.

## 5. Other Related Polytopes

In this section, we briefly describe some other polytopes that are related to the cut cone. A detailed treatment of these polytopes is beyond the scope of this survey.

The convex hull of the incidence vectors of all cuts in a graph is called the *cut polytope*, and was introduced by Barahona[12]. When the graph under consideration is the complete graph, this is the convex hull of all cut metrics, and is closely related to the cut cone. The cut polytope was studied by Barahona and Mahjoub[13]. They showed a remarkable feature of the cut polytope, namely that all its facets can be obtained via the switching operation (see Section 2) from facets of the cut cone (defined in an analagous way for arbitrary graphs.) Therefore, from the facial structure point of view, it is enough to consider the cut cone. Barahona and Mahjoub also gave a partial list of facets of the cut cone, although they were unaware of the earlier work [20].

The class of hypermetric facets for the cut polytope of the complete graph given in Theorem 3.4 of[13] are in fact equivalent to linear facets with $b_{f-1} = 1$, $b_f = 1$, $b_j = -1$, $j > f$, using the notation of Section 2. They also give a class of "bicycle" facets which are non-hypermetric. The study of the cut polytope for graphs other than the complete graph is a rich subject, but is beyond the scope of this survey. For a descrition of the application of these polytopes to problems as diverse as spin glasses and VLSI layout, a description of computational experience, and many related references, the reader is referred to the paper by Barahona et al.[14].

De Simone also showed the close relationship between the cut polytope of a graph and another polytope, called the *Boolean quadric polytope*, defined by Padberg[52]. She showed that every Boolean quadric polytope is the image of a cut polytope under a bijective linear transformation[57].

Other relatives of the cut cone and cut polytope of the complete graph may be obtained by considering subsets of cut metrics. An *equicut* is a cut in $K_n$ in which one part has precisely $\lfloor n/2 \rfloor$ vertices. The other cuts are called *inequicuts*. The convex hull of all incidence vectors of equicuts, or *equicut polytope*, has been studied by Conforti, Rao and Sassano[16]. The *inequicut cone*, formed by taking all non-negative combinations of inequicuts, was studied by Deza, Fukuda and Laurent[25]. The *even* (resp., *odd*) *cut polytope*, studied by Deza and Laurent[32], is defined for even $n$ by taking the convex hull of those cut metrics with an even (resp., odd) number of points on each side of the cut.

Another generalization of the cut polytope for complete graphs is obtained by considering cuts into more than two parts. These so-called multicut polytopes have been studied by Deza, Grötschel and Laurent in[26] and[33].

## 6. Footnote

Since this paper was submitted, there have been many new results obtained on the facets of the cut cone, especially the series of three papers[23] [27] [34]. Other new results on the cut cone are contained in[28] [29] [35].

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