Face-regular polyhedra and tilings with two combinatorial types of faces

Michel Deza
CNRS and LIENS-DMI
Ecole Normale Supérieure
75005 Paris, France *

1. Abstract

We present in detail the list of all 71 face-regular bifaced polyhedra, found in [**BD00**] and give for them symmetry groups and constructions as decorations. All 41 2-isohedral bifaced polyhedra (so, among those 71) are identified, as well as all polyhedra P among those 71, such that (the 1-skeleton of) P or P^* embeds isometrically into (the 1-skeleton of) a hypercube H_m or of a half-hypercube $\frac{1}{2}H_m$. The list of all face-regular bifaced tilings of the Euclidean plane is also presented and compared with the list of all 39 2-homeohedral types of such tilings of ([**GLST85**]); there are 33 realisable sets of parameters and a continuum of face-regular tilings for 11 of them. All those tilings are decorations of regular tilings (6^3), (4^4) or their truncations.

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2. Introduction

We consider here *face-regular bifaced* polyhedra, i.e. such k-valent convex polyhedra with only a- and b-gonal faces ($3 \leq a < b$), that each a-gonal (respectively, b-gonal) face has the same number t_a (respectively, t_b) of a-gonal (respectively, b-gonal) neighbors. For a given polyhedron P, let $v, p_a, p_b, P^*, AutP$ denote the number of vertices, the number of a-gonal faces, the number of b-gonal, the dual polyhedron and the group of symmetry. See [**Grün67**], [**Joh66**], for example, for the notions on polyhedra.

It was proved in [**BD00**] that all face-regular bifaced polyhedra are organized in 68 sporadic one and 3 infinite families: prisms $Prism_b$ ($b \geq 5$), anti-prisms $APrism_b$ ($b \geq 4$) and *barrels* $Barrel_b$ ($b \geq 6$), i.e. simple $4b$-vertex polyhedra with two b-gonal faces, separated by two b-rings of 5-gons; clearly, $Barrel_b$ is a decoration of $Prism_b$.

The results are presented in the form of Table 1 (illustrated by drawings of polyhedra on Figure 2) for 71 polyhedra and, for tilings, Table 2; they are commented in the sections below.

3. 71 polyhedra: constructions and decorations

The 3 infinite families are represented as Nrs. 15, 61 and 44 in Table 1, by their smallest members. Nrs. 2, 18 of Table 1 can be seen as $Barrel_3$ (the cube, truncated on two opposite vertices, is also called the *Durer octahedron*), $Barrel_4$. $Barrel_5$, $Prism_4$, $APrism_3$ are Platonic polyhedra. $Prism_3$ is given under Nr. 1 and not as a case in Nr. 15 of the Table 1, because it has different (a, b) and its dual is embeddable.

The following operations (decorations) are used for 71 polyhedra (and, in the last two sections, for tilings):

*e-mail: michel.deza@ens.fr

m**-cap:** the m-cap of a polyhedron is obtained by putting a pyramid on all m-gonal faces (it is the dual of the truncation on all vertices of degree m).

4-triakon: the 4-triakon of a polyhedron is obtained by partitioning each triangle into a ring of 3 4-gons, by putting a vertex in the middle and connecting it to the midpoint of every edge in the boundary.

5-triakon: a 5-triakon of a polyhedron is obtained by partitioning each hexagon into a ring of 3 pentagons, by putting a vertex in the middle and connecting it to the midpoint of every second edge in the boundary.

m**-halving:** given an even number m, a m-halving of a polyhedron is obtained by putting new edge, connecting the mid-points of opposite edges, on each m-gon.

Nrs. 50, 58 of Table 1 give examples of two different face-regular bifaced polyhedra, both coming as 5-triakon of Nr. 22 (truncated Octahedron). Also Nrs. 52, 59 come both from Nr. 50 as different decorations of 6 of its hexagons. Nrs. 35, 45 come both from Nr. 3 (truncated tetrahedron)) as 4- and 5-triakon, respectively.

Nrs. 3, 45, 48 come by consecutive 5-triakon; Nrs. 23, 46, 56 come by consecutive halving of some hexagons.

The list of 71 polyhedra consists of (see 8th column of Table 1 for details):

1) 10 semi-regular ones (truncations of all 5 Platonic solids, both quasi-regular (cuboctahedron and icosidodecahedron) and both infinite families ($Prism_3$, $Prism_b$ ($b \geq 5$) and $Aprism_b$ ($b \geq 4$));

2) 13 polyhedra obtained as the duals of b-cap of above 10 polyhedra and of 3 twisted forms (of both quasi-regular ones and of rhombicuboctahedron);

3) 10 partial truncations of the cube (2- (on opposite vertices), 6- (all but two opposite vertices, 4- (4 non-adjacent vertices), 4- (4 vertices of two opposite edges)) and of the dodecahedron (4- (on 4 vertices with pairwise distance 3), 16- (all but above 4 vertices), 8- (8 vertices with pairwise distance at least 2), 12- (all but above 8 vertices) 8- (2 opposite vertices s, s' and 6 vertices of 3 edges, on geodesics (s, s'), with pairwise distance 3 between edges), 12- (all but above 8 vertices));

4) 14 4-triakon of Nrs. 1-14;

5) b-cap of 1-, 2- (on 2 opposite vertices), 4- (all but 2 opposite vertices), 6- (i.e. fully) truncated octahedron;

6) Nrs. 21, 27, 28, 56 are m-halving of, respectively, Nrs. 17, 24, 25, 46; Nrs. 46, 49, 52 are partial (on some 3, 4, 6 hexagons) 6-halving of Nrs. 23, 48, 50; Nr. 16 (dual of the disphenoid) comes as a partial (on two opposite faces) 4-halving of the cube;

7) Nrs. 48, 50 are 5-triakon of Nrs. 45, 22 (Nrs. 45, 48, 60 from 2) above, can be obtained also as 5-triakon of truncated Platonic polyhedra, having hexagons);

8) Nrs. 67, 68, 59 come as decorations of the truncated tetrahedron, the truncated cube and the truncated octahedron;

9) Nrs. 64, 65 are obtained by putting (by two different ways) diagonals on 4 4-gons of dual cuboctahedron, so that those diagonals cover all 8 vertices of degree 3;

10) Nrs. 44, 47, 57 come as decorations of Nrs 15, 44 (the smallest case $b = 6$), 47 (Nrs. 47, 57, 62 are organized into alternated concentric rings of a- and b-gons);

11) Nr. 23 is a decoration of the cube, Nr. 34 can be seen as a decoration of Nr. 20, or of Nr. 1, or of the cube.

Nr. 34 comes from Nr. 20 by putting an "H" on all its 6-gonal faces and a quadrangle on all its 4-gonal faces; Nr. 59 comes from Nr. 22 by putting an "H" on all its 4-gonal faces and 5-triakon of all its 6-gonal faces.

Nrs. $16^*, 17^*, 18^*, 62$ are polyhedra with regular faces, having numbers 84, 51, 17, 15 in the list of all 92 such polyhedra found in [**Joh66**]; Nrs.$1^*, 16^*, 17^*, 18^*$ and $(Prism_5)^*$ (the smallest case of 15^*) are all 5 non-Platonic convex deltahedra.

Nrs. 5, 25, 54 are twisted forms of Nrs. 4, 24, 53, respectively; the last 3 are *chamfered* Platonic tetrahedron, cube, dodecahedron, respectively (i.e. they obtained by putting a prism on each face, followed by deleting all original edges).

4. The symmetry and 41 2-isohedral bifaced polyhedra

All bifaced face-regular polyhedra have symmetry groups D_{nh}, D_{nd} (for $n \geq 2$) or as below (see 6th column of Table 1 for details):

$AutP$:	C_{3v}	D_3	T	T_h	T_d	O	O_h	I	I_h
Number of polyhedra:	2	7	6	4	9	2	8	1	6
Number of 2-isohedral polyhedra:	0	1	3	2	5	2	7	1	5

The remaining 23 sporadic bifaced face-regular polyhedra have symmetry groups:

$AutP$:	D_{2h}	D_{2d}	D_{3h}	D_{3d}	D_{4h}	D_{4d}	D_{5h}
Number of polyhedra:	2	3	7	4	4	2	1
Number of 2-isohedral polyhedra:	0	1	3	3	4	1	0

Using the Euler formula it is easy to check that any k-valent bifaced (a, b)-polyhedron fulfills

$$p_a = \frac{4b - v(2k + 2b - kb)}{2b - 2a}$$

and

$$p_b = \frac{v(2k + 2a - ka) - 4a}{2b - 2a}.$$

By face-regularity,

$$\frac{p_a}{p_b} = \frac{b - t_b}{a - t_a},$$

implying

$$\frac{b - t_b}{a - t_a} = \frac{(k - 2 + \frac{4}{v})b - 2k}{2k - (k - 2 + \frac{4}{v})a},$$

i.e.

$$v = \frac{4(2ab - at_b - bt_a)}{2k(a + b - t_a - t_b) - (k - 2)(2ab - at_b - bt_a)}.$$

A polyhedron P is called *2-isohedral* , if its symmetry group $AutP$ has exactly two orbits of faces. Clearly, any bifaced 2-isohedral polyhedron P should be face-regular. Moreover, it should satisfy the conditions:

(i) p_a, p_b divide the order of $Aut(P)$,

(ii) any face has same 1-*corona*, i.e. same sequence of gonality (i.e. the number of edges) of its neighboring faces.

We found all 2-isohedral polyhedra in our list of 71 polyhedra (see 7th column of Table 1 for details). They are: all three infinite families and 38 sporadic ones. Only Nrs. 25, 46, 52, 65 (marked by the sign ! in the 7th column of Table 1) among of remaining 30 polyhedra (i.e. of those having more than 2 orbits of faces) satisfy (i); their pairs of numbers of orbits of a- and b-gons are (1,2), (2,1), (1,2), (2,2), respectively. For example, Nr. 30 satisfy (ii), but it has two orbits (of sizes 6 and 12) of 4-gonal faces, which differ only on 2-corona (see Figure 1).

Comparing with the partition of the list of 71 polyhedra into groups 1)-9), one can see that the list of 41 2-isohedral bifaced polyhedra consists of:

1') all 10 polyhedra 1),

2') all but 4 polyhedra 2), i.e. all but dual b-cap of rhombicuboctahedron and of 3 twisted Archimedean polyhedra,

3') 5 out of 10 polyhedra 3),

4') 6 out of 14 polyhedra 4) (4-triakon of the truncated tetrahedron, of the truncated cube, of $Prism_3$ and of 3 polyhedra of 3') above,

5') all 4 polyhedra 5),

6') Nrs. 21, 27 (4-halving) and 16 out of polyhedra 6),

7') Nrs. 44, 48, 64, 68.

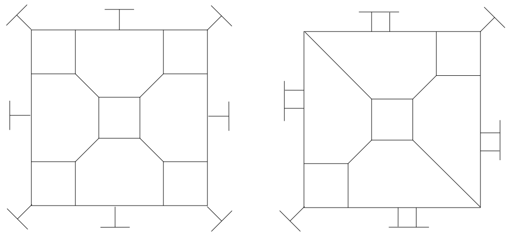


Figure 1: Two 2-coronas of 4-gons of the polyhedron Nr. 30

Remarks:

(i) It is easy to see that the skeletons of all 71 polyhedra are Hamiltonian.

(ii) The duals of all 5 non-Platonic convex deltahedra, as well as all 3 infinite families - Nrs. 15, 44, 61 - are 2-isohedral.

(iii) Among largest 3-valent (Nrs. 43, 55, 60 with $v = 140$) and 4-valent (Nrs. 68, 70, 71 with $v = 30$) polyhedra of our list of 71, only Nr.43 is not 2-isohedral.

(iv) Nrs. 9, 13, 28, 33, 34, 36, 38, 43, 47, 54, 57, 65, for example, have > 3 orbits of faces.

5. Embedding

Since the skeleton of any polyhedron is a planar graph, its possible embedding, isometric up to a scale, into the skeleton of a hypercube, is (using a result from [**CDGr97**]) up to a scale 1 or 2, i.e. it is the isometric embedding into a hypercube or a half-hypercube. Embeddable graphs embeds into a hypercube if and only if it is bipartite, i.e. in the case of bifaced a, b-polyhedra, if and only if both a and b are even. Remind that the *half-hypercube* $\frac{1}{2}H_m$ is (in coordinate terms) the set of all binary m-tuples with even number of ones, two of them being adjacent if their Hamming distance is 2. The notation $P \to H_m$ (or $P \to \frac{1}{2}H_m$) means that the skeleton of polyhedron P embeds isometrically into m-cube (or m-half-cube); see last two columns of Table 1 for details. Among polyhedra of 3 infinite families $Prism_b$ ($b \geq 5$), $Barrel_b$ ($b \geq 6$), $APrism_b$ ($b \geq 4$) and their duals, all embeddings are:

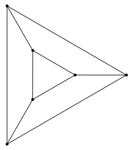
$Prism_b \to \frac{1}{2}H_{b+2}$ (moreover, $Prism_b \to H_{\frac{b}{2}+1}$ for even b), $APrism_b \to \frac{1}{2}H_{b+1}$. Among 71 polyhedra P, only Nrs. 1, 2, 62, 66 have both P and P^* embeddable. Exactly 9 (respectively, 17) polyhedra P have only P (respectively, only P^*) embeddable. All 14 polyhedra with $(k, a) = (3, 3)$ (but none of other polyhedra with $k = 3$) have embeddable P^*. All 108 non-embeddable P, P^* (except *hypermetric* 17^* and 18^*, marked by the sign ! in the last column of Table 1) does not satisfy already to *5-gonal inequality*, necessary for embedding (see [**DGr97a**], [**DGr99**], [**DL97**], [**DS96**] for related notions and results). All embeddings of P, P^* among 68 sporadic polyhedra P are:

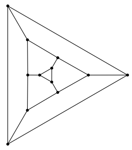
$Nr.1^* \to \frac{1}{2}H_4$; $Nr.1 \to \frac{1}{2}H_5$; $Nrs.2^*, 62 \to \frac{1}{2}H_6$; $Nrs.3^*, 45^* \to \frac{1}{2}H_7$;
$Nrs.2, 4^*, 5^*, 16, 66 \to \frac{1}{2}H_8$; $Nrs.62^*, 63^* \to H_4$; $Nrs.6^*, 7^*, 51^* \to \frac{1}{2}H_{10}$;
$Nr.66^* \to H_5$; $Nrs.10^*, 71 \to \frac{1}{2}H_{12}$; $Nrs.22, 70^* \to H_6$; $Nrs.8^*, 9^* \to \frac{1}{2}H_{14}$;
$Nrs.24, 25, 69^* \to H_7$; $Nr.48 \to \frac{1}{2}H_{16}$; $Nrs.11^*, 12^* \to \frac{1}{2}H_{18}$;
$Nrs.13^*, 53 \to \frac{1}{2}H_{22}$; $Nr.14^* \to \frac{1}{2}H_{26}$.

Bifaced polyhedra with weaker notion of face-regularity - only c-gonal (where c is a *or* b) faces have the same number t_c of c-gonal neighbors - were considered in [**DGr97b**] and [**DGr99**]. For example, all *fullerenes* (i.e. simple polyhedra with with $(a, b) = (5, 6)$) with fixed t_6 or fixed $t_5 \geq 2$ were found. Simple bifaced polyhedra, such that c-gonal faces form a ring (so $t_c = 2$) are considered in [**DGr01**]. Face-regular polyhedra with more than two types of faces were considered in [**BD00**].

Nr.	k	a,b	v	t_a,t_b	Aut	2 orbits	Polyhedron P	emb. P	emb. P^*
1	3	3,4	6	0,2	D_{3h}	+	$Prism_3$	$1/2H_5$	$1/2H_4$
2	3	3,5	12	0,4	D_{3d}	+	2-truncated cube	$1/2H_8$	$1/2H_6$
3	3	3,6	12	0,3	T_d	+	trunc.tetrahedron	-	$1/2H_7$
4	3	3,6	16	0,4	T_d	+	4-truncated cube	-	$1/2H_8$
5	3	3,6	16	0,4	D_{2h}	-	twisted Nr. 4	-	$1/2H_8$
6	3	3,6	28	0,5	T	+	4-truncated dodec.	-	$1/2H_{10}$
7	3	3,7	20	0,4	D_{3d}	+	6-truncated cube	-	$1/2H_{10}$
8	3	3,7	36	0,5	T_h	+	8-truncated dodec.	-	$1/2H_{14}$
9	3	3,7	36	0,5	D_3	-	twisted Nr. 8	-	$1/2H_{14}$
10	3	3,8	24	0,4	O_h	+	truncated cube	-	$1/2H_{12}$
11	3	3,8	44	0,5	T_h	-	12-truncated dodec.	-	$1/2H_{18}$
12	3	3,8	44	0,5	D_3	-	twisted Nr. 11	-	$1/2H_{18}$
13	3	3,9	52	0,5	T	-	16-truncated dodec.	-	$1/2H_{22}$
14	3	3,10	60	0,5	I_h	+	trunc.dodecahedron	-	$1/2H_{26}$
15	3	4,$\mathbf{b}$	2$\mathbf{b}$	2,0	$D_{\mathbf{b}d}$	+	$Prism_{\mathbf{b}}$, $\mathbf{b} \geq 5$	$1/2H_{\mathbf{b}+2}$	-
16	3	4,5	12	1,2	D_{2d}	+	decorated cube	$1/2H_8$	-
17	3	4,5	14	0,3	D_{3h}	+	(b-cap $Prism_3$)*	-	- !
18	3	4,5	16	0,4	D_{4d}	+	(b-cap $Aprism_4$)*	-	- !
19	3	4,6	14	2,2	D_{3h}	+	4-triakon Nr. 1	-	-
20	3	4,6	20	2,4	D_{3d}	+	4-triakon Nr. 2	-	-
21	3	4,6	20	1,3	D_3	+	4-halved Nr. 17	-	-
22	3	4,6	24	0,3	O_h	+	trunc.octahedron	H_6	-
23	3	4,6	26	1,4	D_{3h}	-	decorated cube	-	-
24	3	4,6	32	0,4	O_h	+	(b-cap cuboct.)*	H_7	-
25	3	4,6	32	0,4	D_{3h}	- !	(b-cap tw.cuboct.)*	H_7	-
26	3	4,6	56	0,5	O	+	(b-cap snub cube)*	-	-
27	3	4,7	44	1,4	T_h	+	4-halved Nr. 24	-	-
28	3	4,7	44	1,4	D_3	-	4-halved Nr. 25	-	-
29	3	4,7	44	2,5	T	+	4-triakon Nr. 6	-	-
30	3	4,7	80	0,4	O_h	-	(b-cap rhombicbct.)*	-	-
31	3	4,7	80	0,4	D_{4d}	-	(b-cap tw.rhombicbct.)*	-	-
32	3	4,8	32	2,4	T_d	+	4-triakon Nr. 4	-	-
33	3	4,8	32	2,4	D_{2h}	-	4-triakon Nr. 5	-	-
34	3	4,8	80	1,4	D_3	-	decorated Nr. 20	-	-
35	3	4,9	28	2,6	T_d	+	4-triakon Nr. 3	-	-
36	3	4,9	68	2,4	T_h	-	4-triakon Nr. 8	-	-
37	3	4,9	68	2,4	D_3	-	4-triakon Nr. 9	-	-
38	3	4,10	44	2,6	D_{3d}	-	4-triakon Nr. 7	-	-
39	3	4,11	92	2,6	T_h	-	4-triakon Nr. 11	-	-
40	3	4,11	92	2,6	D_3	-	4-triakon Nr. 12	-	-
41	3	4,12	56	2,8	O_h	+	4-triakon Nr. 10	-	-
42	3	4,13	116	2,8	T	-	4-triakon Nr. 13	-	-
43	3	4,15	140	2,10	I_h	-	4-triakon Nr. 14	-	-
44	3	5,$\mathbf{b}$	4$\mathbf{b}$	4,0	$D_{\mathbf{b}d}$	+	$Barrel_{\mathbf{b}}$, $\mathbf{b} \geq 6$	-	-
45	3	5,6	28	3,0	T_d	+	(b-cap trunc. tetr.)*	-	$1/2H_7$
46	3	5,6	32	3,2	D_{3h}	- !	decorated Nr. 23	-	-
47	3	5,6	38	2,2	C_{3v}	-	decorated $Barrel_6$	-	-
48	3	5,6	44	2,3	T	+	5-triakon Nr. 45	$1/2H_{16}$	-
49	3	5,6	52	1,3	T	-	decorated Nr. 48	-	-
50	3	5,6	56	2,4	T_d	-	5-triakon Nr. 22	-	-
51	3	5,6	60	0,3	I_h	+	trunc.icosahedron	-	$1/2H_{10}$
52	3	5,6	68	1,4	T_d	- !	decorated Nr. 50	-	-
53	3	5,6	80	0,4	I_h	+	(b-cap icosido.)*	$1/2H_{22}$	-
54	3	5,6	80	0,4	D_{5h}	-	(b-cap tw.icosido.)*	-	-
55	3	5,6	140	0,5	I	+	(b-cap snub dodec.)*	-	-
56	3	5,7	44	3,1	D_{3h}	-	6-halved Nr. 46	-	-
57	3	5,7	92	2,2	C_{3v}	-	decorated Nr. 47	-	-
58	3	5,8	56	3,0	O_h	+	(b-cap trunc.cube)*	-	-
59	3	5,8	92	3,2	T_d	-	decorated trunc.oct.	-	-
60	3	5,10	140	3,0	I_h	+	(b-cap trunc.dodec.)*	-	-
61	4	3,$\mathbf{b}$	2$\mathbf{b}$	2,0	$D_{\mathbf{b}d}$	+	$Aprism_{\mathbf{b}}$, $\mathbf{b} \geq 4$	$1/2H_{\mathbf{b}+1}$	-
62	4	3,4	10	2,2	D_{4h}	+	capp. 1-trunc.oct.	$1/2H_6$	H_4
63	4	3,4	12	0,0	O_h	+	cuboctahedron	-	H_4
64	4	3,4	14	1,2	D_{4h}	+	decorated ($cuboct.$)*	-	-
65	4	3,4	14	1,2	D_{2d}	- !	decorated (cuboct.)*	-	-
66	4	3,4	14	2,3	D_{4h}	+	capp. 2-trunc.oct.	$1/2H_8$	H_5
67	4	3,4	22	1,3	D_{2d}	-	decorated trunc.tetr.	-	-
68	4	3,4	30	0,3	O	+	decorated trunc.cube	-	-
69	4	3,5	22	2,3	D_{4h}	+	4-cap 4-trunc.oct.	-	H_7
70	4	3,5	30	0,0	I_h	+	icosidodecahedron	-	H_6
71	4	3,6	30	2,3	O_h	+	4-cap trunc.oct.	$1/2H_{12}$	-

Table 1: All face-regular bifaced polyhedra


Nr. 1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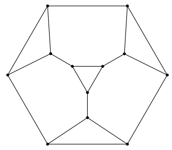
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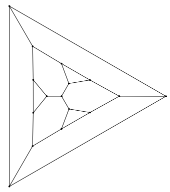
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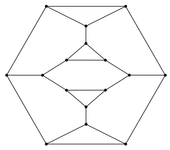
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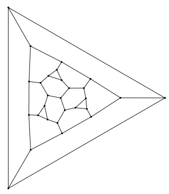
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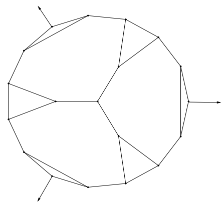
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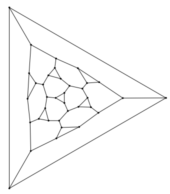
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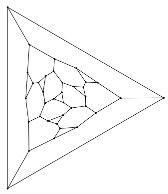
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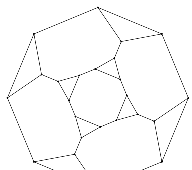
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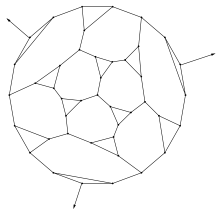
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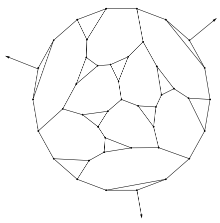
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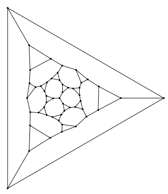
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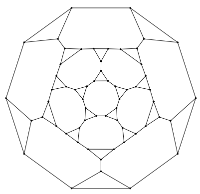
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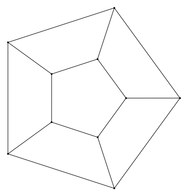
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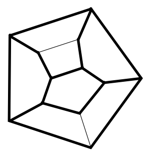
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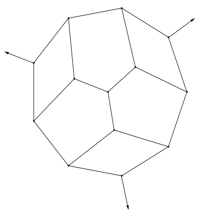
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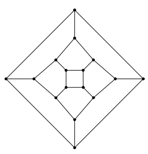
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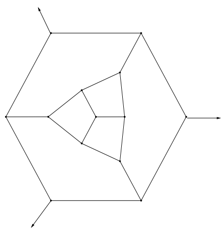
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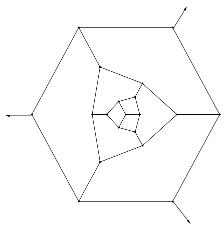
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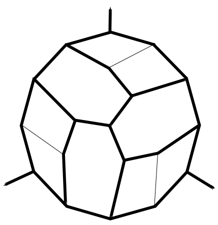
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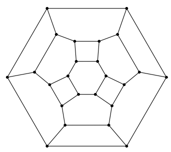
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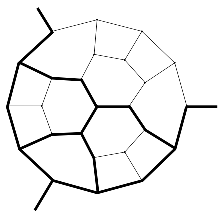
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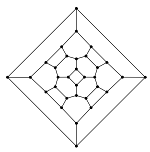
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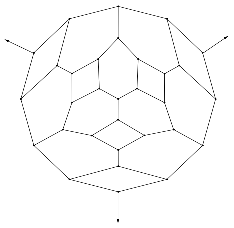
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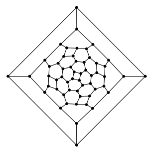
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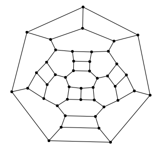
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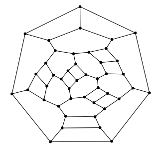
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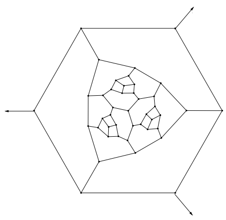
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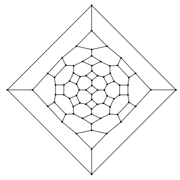
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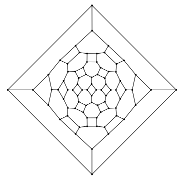
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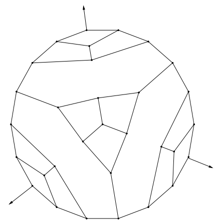
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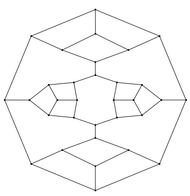
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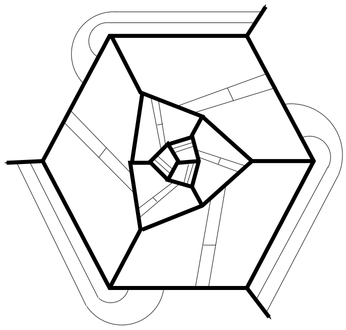
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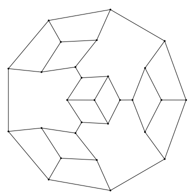
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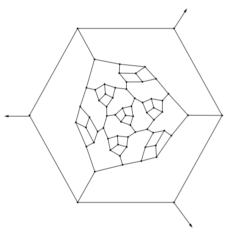
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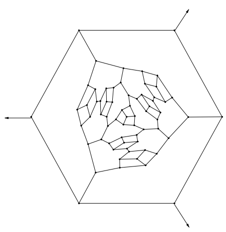
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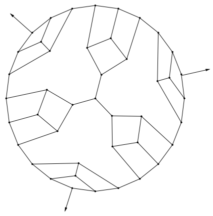
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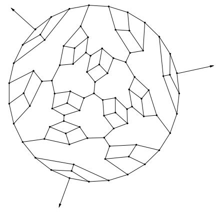
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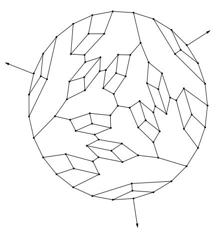
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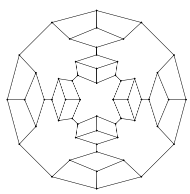
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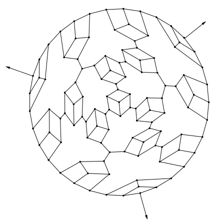
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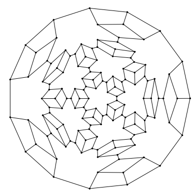
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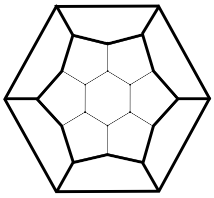

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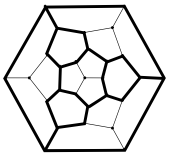

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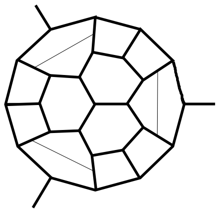

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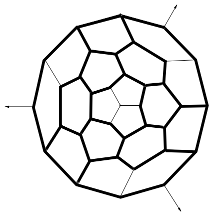

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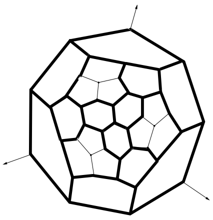

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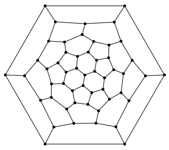

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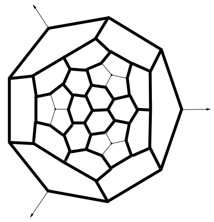

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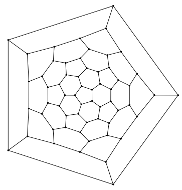

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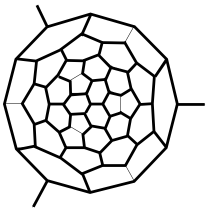

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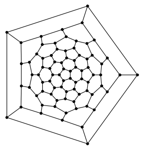

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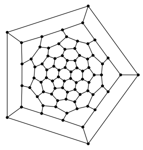

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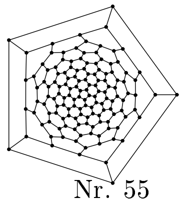

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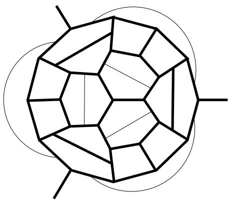

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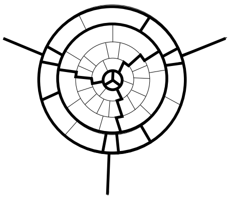

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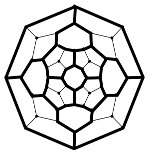

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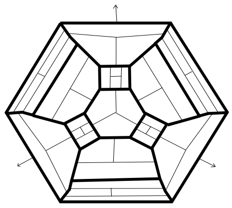

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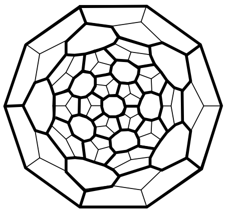

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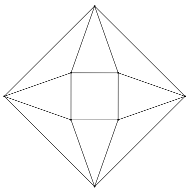

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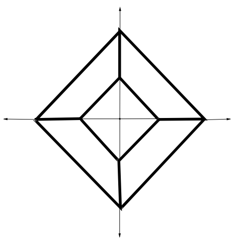

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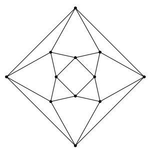

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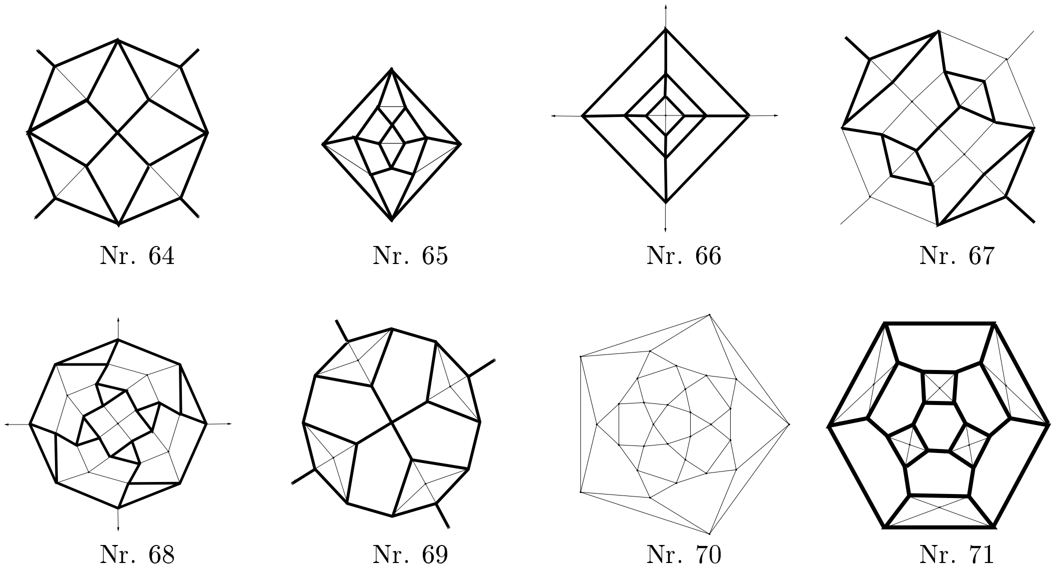


Figure 2: The 71 face-regular bifaced polyhedra

6. Infinite face-regular bifaced polyhedra

We present now all k-valent *infinite* a, b-polyhedra (i.e. tilings of Euclidean plane), which are face-regular.

Of course, our k-valent tiling of Euclidean plane by a- and b-gons only, should be normal and balanced, in the sense of [**GLST85**], [**GS85**]. So the limits $v = lim\frac{v(r,P)}{t(r,P)}$ and $e = lim\frac{e(r,P)}{t(r,P)}$ for $r \to \infty$ exist (and are finite) and the Euler's relation for tilings, $v - e + 1 = 0$, holds. Here $v(r,P), e(r,P), t(r,P)$ denote the number of vertices, edges, tiles, respectively, in the patch of tiles, corresponding to the circular disk in the plane with center P and radius r. This Euler formula implies that any k-valent tiling by a- and b-gons only, fulfills (cf. formulas in Section 4)

$$\frac{p_a}{p_b} = \frac{(k-2)b - 2k}{2k - (k-2)a}$$

and, by face-regularity,

$$\frac{p_a}{p_b} = \frac{b - t_b}{a - t_a}.$$

So, we have the equation (in fact, this is equation (4.6.11) in [**GS85**], given there for k-valent 2-homeohedral tilings)

$$\frac{b - t_b}{a - t_a} = \frac{(k-2)b - 2k}{2k - (k-2)a}$$

(it is the case $v \to \infty$ of the corresponding equation from Section 4).

Following to [**GLST85**] and [**GS85**], call a tiling *2-homeohedral* (or topologically 2-tile-transitive) if the faces form two transitivity orbits under the group of *combinatorial* (or topological) self-transformations of the Euclidean plane, that map the tiling onto itself. Such a tiling is called, moreover, *2-isohedral* if its symmetry group is isomorphic to the group of combinatorial self-transformations.

The list of all 39 2-homeohedral types of k-valent tilings is given in [**GLST85**], pages 135-136, and (with some errors, corrected in Remark below)) in [**GS85**], Figure 4.6.3; see also the pioneering thesis [**L68**] and the extension of the results in [**DHZ90**]. Each of 39 types represented there by a 2-isohedral tiling. (The tilings with two orbits of *edges* were considered in [**GS83**]; they have at most three orbits of tiles and vertices each.)

Remarks:

(i) Grünbaum ([**Gr99**]) indicated that Figure 4.6.3 in [**GS85**] (the list of 39 tilings) contains the following errors, with respect of corrected list, given in [**GLST85**]:

a) the diagrams 4_3 8_3 I and 4_3 8_3 III have the same 2-homeohedral type, and so do the diagrams 5_3 8_6 I and 5_3 8_6 II; so only one of two should be included;

b) two 2-homeohedral types (3_2 5_2 10 and 4_3 8_3 1, in notations of [**GLST85**]) should be added;

c) on page 188, the second diagram in the second row and the left-most diagram in the bottom row should be interchanged.

(ii) Grünbaum ([**Gr99**]) also asked for enumeration of non-convex face-regular polyhedra (thus giving up the 3-connectedness, or the requirement that the polyhedra be of genus 0, or both) and also of those, in which the two kinds of faces differ only by their "color", and, possibly, by the number of neighbors of the same color.

(iii) One can show that the p-vector of any k-valent tiling of Euclidean plane, which does not contain i-gons with $i > i_0$, satisfies to $3p_3 + 2p_4 + p_5 \leq 6$ for $(k, i_0) = (3, 6)$ and to $p_3 \leq 4$ for $(k, i_0) = (4, 4)$.

7. The list of face-regular bifaced tilings

The main result of this section is the list of face-regular bifaced tilings given in Table 2. We hope that this list is complete, but (as well as [**GLST85**] say on page 132 about their enumeration of the normal 2-homeohedral tilings) we cannot be certain, because of the large number of cases to be considered, that some possibilities have not been overlooked. The sketch of proof follows.

First, one should prove that the 33 parameter sets $(k;\ a, b;\ t_a, t_b)$ of Table 2 are the only realizable ones. Main tool is the equation above; it gives a finite list of possible parameter sets. Further, one cane remove many admissible parameter sets by some combinatorial and geometric ad hoc considerations (usually, in terms of possible 1- and 2-corona of faces). This messy, case by case enumeration was used in [**GLST85**]), [**GS85**] (and illustrated there by examples) in order to obtain the list of 2-homeohedral tilings; see also the Appendix in [**DFSV00**]. For example, for $k = 3, a = 5$, all possibilities for parameters $(b;\ t_a, t_b)$ are $(7;\ 1, 3), (7;\ 2, 4), (8;\ 2, 2), (12;\ 3, 0)$ (all realized by seven 2-isohedral tilings; see Figure 3) and, possibly, $t_a = 3, t_b = 12 - b$ with $8 \leq b \leq 11$(the cases $b = 8, 10, 11$ will occur; see Figure 6).

The second part is to classify all face-regular (combinatorial types of) tilings for each of realizable 33 parameter sets. All non-existence and unicity results come by the same ad hoc considerations as above. We list now all existing tilings. All 39 2-homeohedral types (except Nr. 12, which is simply 4-triakon of Archimedean (3.12^2)) are represented on Figures 4, 5. Now we will list all others: first the sporadic ones and then, in detail, 11 continuums.

Nrs. 4', 4", 4"', 5 are truncations of (6^3) on all vertices, which were not truncated in Nrs. 2', 2", 2"', 1, respectively.

See tilings Nrs. 2', 2", 2"'on Figure 6. Nr. 2' is 2-homeohedral; Nr. 2"comes from (6^3) by truncation of pairs of vertices of edges from the non-extendable set of edges with minimal distance 3; Nr. 2"'comes from (6^3) by taking each third zone of 6-gons and truncating pair of opposite vertices of each 6-gon of the zone (on the diameter perpendicular to the direction of the zone).

Nrs. 7-12 are simply 4-triakon (see the definition in Section 3 above) of Nrs. 1-6, respectively.

See tilings Nrs. 20-22 on Figure 6. Nr. 20 comes from (6^3) by decoration, by the letter H, of each 6-gon of each third zone of 6-gons; each of two vertical lines of H goes in the direction perpendicular to the direction of the zone and connect the midpoints of 2 edges. Nr. 21 comes from (6^3) by the same decoration of each 6-gon of each second zone of 6-gons. Nr. 22 comes from (6^3) by the following decoration of each zone of 6-gons: 2 non-decorated 6-gons are followed by 5-triakon of 6-gon, above H-decorated 6-gon, 5-triakon of 6-gon, 2 non-decorated 6-gons and so on; the decorations are shifted on two 6-gons on each second zone.

Finally, we describe the 11 continuums. Each continuum of such tilings is represented by all infinite (in both directions) words over letters u, v, spanned between words $(u)_\infty := ...uuuu...$ and $(uv)_\infty := ...uvuv...$ (cf. the packings f.c.c. and h.c.p. within continuum of Kelvin partitions of 3-space by regular tetrahedra and octahedra). For example, the description of (the continuum) Nr. 17 comes by following steps: 1) only two types of 2-corona are possible, 2) each of those two motives should propagate in both directions on the tiling, 3) denote resulting two sequences by letters u, v and see the tiling as an infinite word over u, v. For each of 11 parameter sets Nr. i, realized by a continuum, we denote by $Nr.i_A, Nr.i_B$ the tilings corresponding to words $(u)_\infty, (uv)_\infty$; all such tilings, which are not 2-homeohedral, are Nrs. $13_B, 18_B, 26_A, 26_B$ (for example, Nr. 18_B has 4 orbits of faces). Denote by $Nr.i_c$ the unique tiling outside of the continuum, if it exists; those are Nrs. $13_c, 15_c, 18_c, 29_c$. Each of 11 continuums can be visualized using definition of letters u,v below and Figure 5, representing the extremal cases $(u)_\infty, (uv)_\infty$.

Nr. 13 (except Nr. 13_c): for each zone of 8-gons in (4.8^2), each 8-gon is cut in half by a new edge, all new edges being parallel. Letters u, v correspond to zones in which the direction of cutting edges was SW-NE or NW-SE. This is a continuum of different metric realizations of the same topological type; only 13_A and 13_B are 2-isohedral. Apropos, Nr. 13_c corresponds to cutting in half all "black" 8-gons of (4.8^2) (in a chess-board coloring of all 8-gons) by edge N-S (North-South) and all "white"8-gons by edge W-E (West-East).

Nr. 15 is nothing but cutting in half each 4-gon of each sequence of 4-gons (alternated by linking edges) in any fixed tiling Nr. 13. The cutting edges for any sequence are in the same direction; the choice of the direction does not affect combinatorial type.

Nr. 16 comes from (4.8^2) by partitioning it into parallel zones of 8-gons and decorating each sequence of 4-gons (between two neighboring zones of 8-gons). Namely, each 4-gon is cut in half and all cutting edges for a fixed sequence are parallel, in one of two, mutually perpendicular, directions. Letters u, v correspond each to one of those two directions.

Nr. 29 (except Nr. 29_c, which is a decoration of (4^4)) comes from (4.8^2) by partitioning it into "mixed zones"of 4-gons alternated by 8-gons. On each 8-gon of each mixed zone, put a diagonal connecting vertices of 4-gons, all diagonals of a fixed mixed zone in the same direction. There are two choices and let them correspond to letters u, v. So, the tiling will be partitioned into decorated (u or v) mixed zones. Then put a diagonal in each 4-gon, so that the valency of each vertex became 4.

Nr. 24: each second vertex of each second column of vertices in (4^4) is truncated and then all resulting 4-gons are capped. Now shift some columns (with truncated vertices) on one step; the letters u, v correspond to the choice "shift or no".

Nr. 26: it is a "complement"of Nr. 24, i.e. it is (4^4), in which are truncated (with capping of resulting 4-gons) exactly those vertices, which were not truncated (with capping of resulting 4-gons) in order to obtain any fixed tiling Nr. 24.

Nr. 25: a tiling is defined by translation of a path of vertices (in king's move on Z_2, i.e. in l_∞-metric on Z_2). The letters u, v correspond to step by a side (of a 4-gon) or by a diagonal; then all vertices of each second (in translation) path are truncated and all resulting 4-gons are capped.

Nr. 33: a tiling is defined by a path of 4-gons in (4^4). The letters u, v correspond to move right or up; then all 4-gons of each second (in translation) path are decorated by diagonals, all in the same direction (the choice of the direction does not affect the combinatorial type; cf. Nr. 15).

The continuums Nrs. 17-19 are described in [**DFSV00**]; see also Figure 3. A tiling Nr. 19 is defined by the translation of a path of pairs of 5-gons. Each such path can be seen as a word in letters u, v, where those letters correspond, respectively, to one of two ways of adjacency of pairs. A tiling Nr. 17 is defined by the translation of a sequence of pairs of 5-gons, alternated by pairs of 7-gons. All edges, separating two 5-gons, of a sequence are supposed to be parallel; the letters u,v correspond, as for Nr. 16, to two, mutually perpendicular, directions.

Remark, that for Nrs. 15, 19, 29, 33 the continuum can be defined over 3 letters; it introduces new symmetries, new metric realizations, but not affects the combinatorial type.

Nr.	k	a,b	t_a,t_b	Nr of all	Nr. of 2-homeohedral	Tilings
1	3	3,7	0,6	1	1	$\frac{1}{6}$-truncated (6^3)
2	3	3,8	0,6	3	1	$\frac{1}{3}$-truncated (6^3)
3	3	3,9	0,6	3	3	$\frac{1}{2}$-truncated (6^3)
4	3	3,10	0,6	3	-	$\frac{2}{3}$-truncated (6^3)
5	3	3,11	0,6	1	-	$\frac{5}{6}$-truncated (6^3)
6	3	3,12	0,6	1	1	trunc.$(6^3) = (3.12^2)$
7	3	4,8	2,6	1	1	4-triakon of Nr.1
8	3	4,10	2,6	3	1	4-triakon of Nr.2
9	3	4,12	2,6	3	1	4-triakon of Nr.3
10	3	4,14	2,6	3	-	4-triakon of Nr.4
11	3	4,16	2,6	1	-	4-triakon of Nr.5
12	3	4,18	2,6	1	1	4-triakon of Nr.6
13	3	4,7	0,5	$1(\infty)+1$	$1+1$	8-halved (4.8^2)
14	3	4,8	0,4	1	1	trunc.$(4^4) = (4.8^2)$
15	3	4,8	1,5	$\infty+1$	$2+1$	4-halved Nr.13
16	3	4,10	1,4	∞	2	4-halved (4.8^2)
17	3	5,7	1,3	∞	2	a 6-halved (6^3)
18	3	5,7	2,4	$\infty+1$	$1+1$	decorated (6^3)
19	3	5,8	2,2	∞	2	a 6-halved (6^3)
20	3	5,8	3,4	1	-	decorated (6^3)
21	3	5,10	3,2	1	-	decorated (6^3)
22	3	5,11	3,1	1	-	decorated (6^3)
23	3	5,12	3,0	1	1	a 5-triakon (6^3)
24	4	3,5	2,4	∞	2	decorated (4^4)
25	4	3,6	2,4	∞	2	decorated (4^4)
26	4	3,7	2,4	∞	-	decorated (4^4)
27	4	3,8	2,4	1	1	4-capped (4.8^2)
28	4	3,5	0,2	1	1	decorated (4^4)
29	4	3,5	1,3	$\infty+1$	$2+1$	decorated (4^4), (4.8^2)
30	4	3,6	0,0	1	1	Archimedean (3.6.3.6)
31	4	3,6	1,2	1	1	decorated (4^4)
32	5	3,4	1,0	1	1	Archimedean $(3^2.4.3.4)$
33	5	3,4	2,2	∞	2	decorated (4^4)

Table 2: All face-regular bifaced tilings

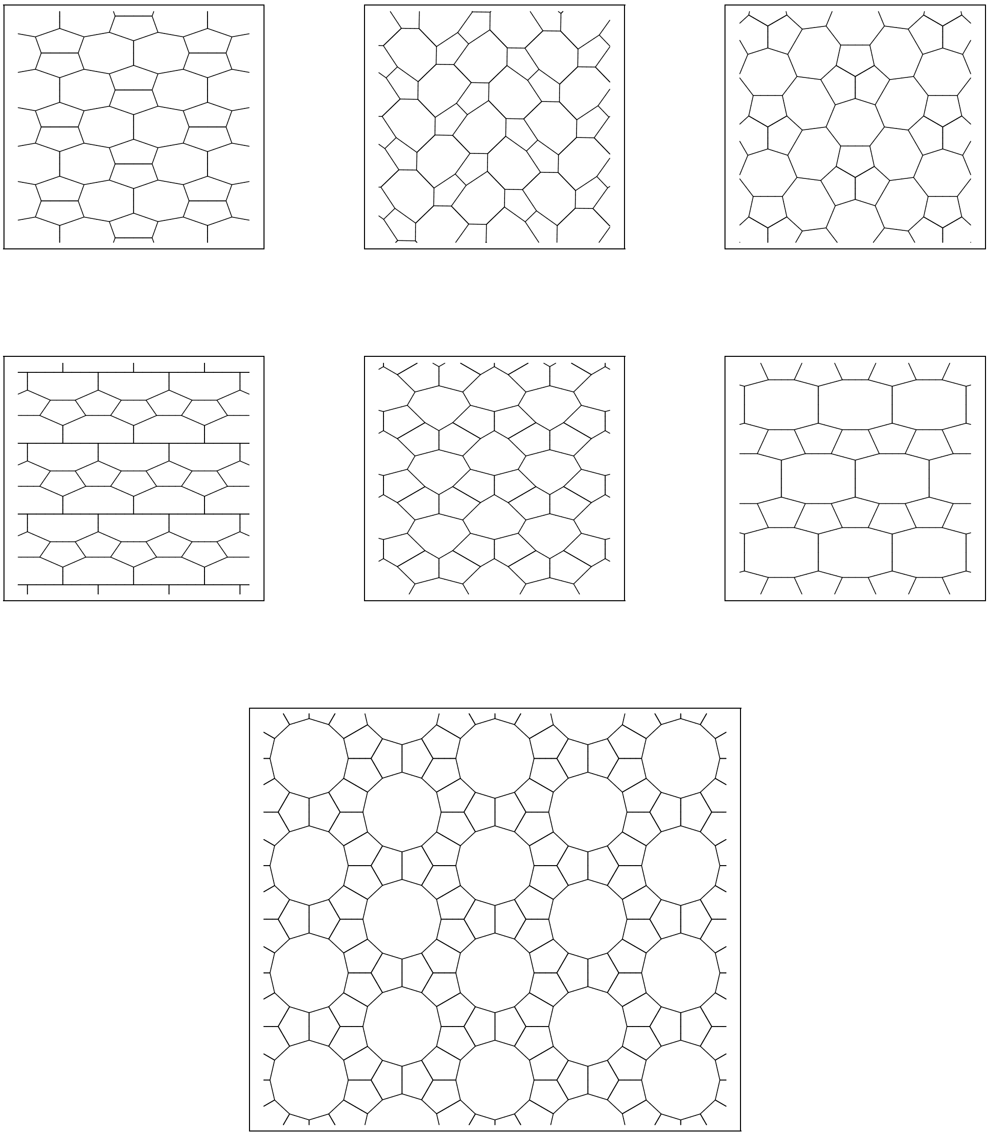
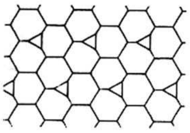
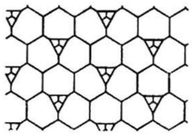
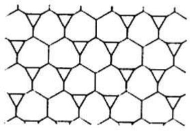
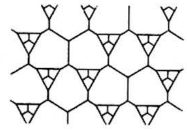


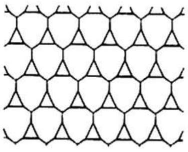
Figure 3: The 3-valent 2-homeohedral types of tilings by 5- and b-gons: Nrs. $17_A, 17_B, 18_A, 18_c, 19_B, 19_A, 23$

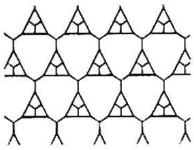

Nr. 1

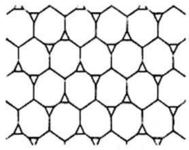

Nr. 7

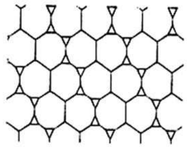

Nr. 2'

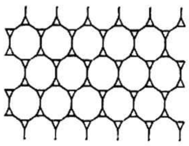

Nr. 8'

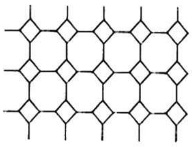

Nr. 3'

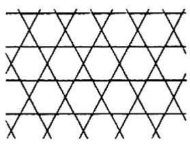

Nr. 9'

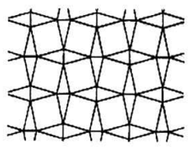

Nr. 3''

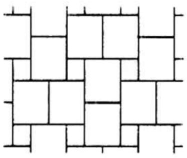

Nr. 3'''

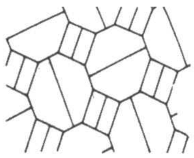

Nr. 6

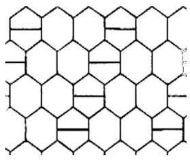

Nr. 14

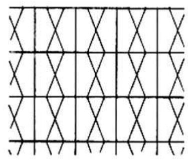

Nr. 30

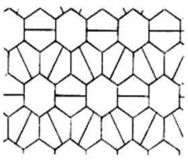

Nr. 32

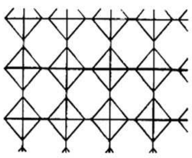

Nr. 13_c

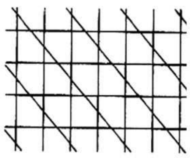

Nr. 15_c


Nr. 18_c


Nr. 29_c


Nr. 23


Nr. 27


Nr. 28

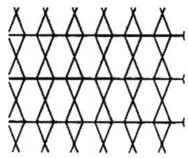

Nr. 31

Figure 4: 20 sporadic 2-homeohedral types of tilings

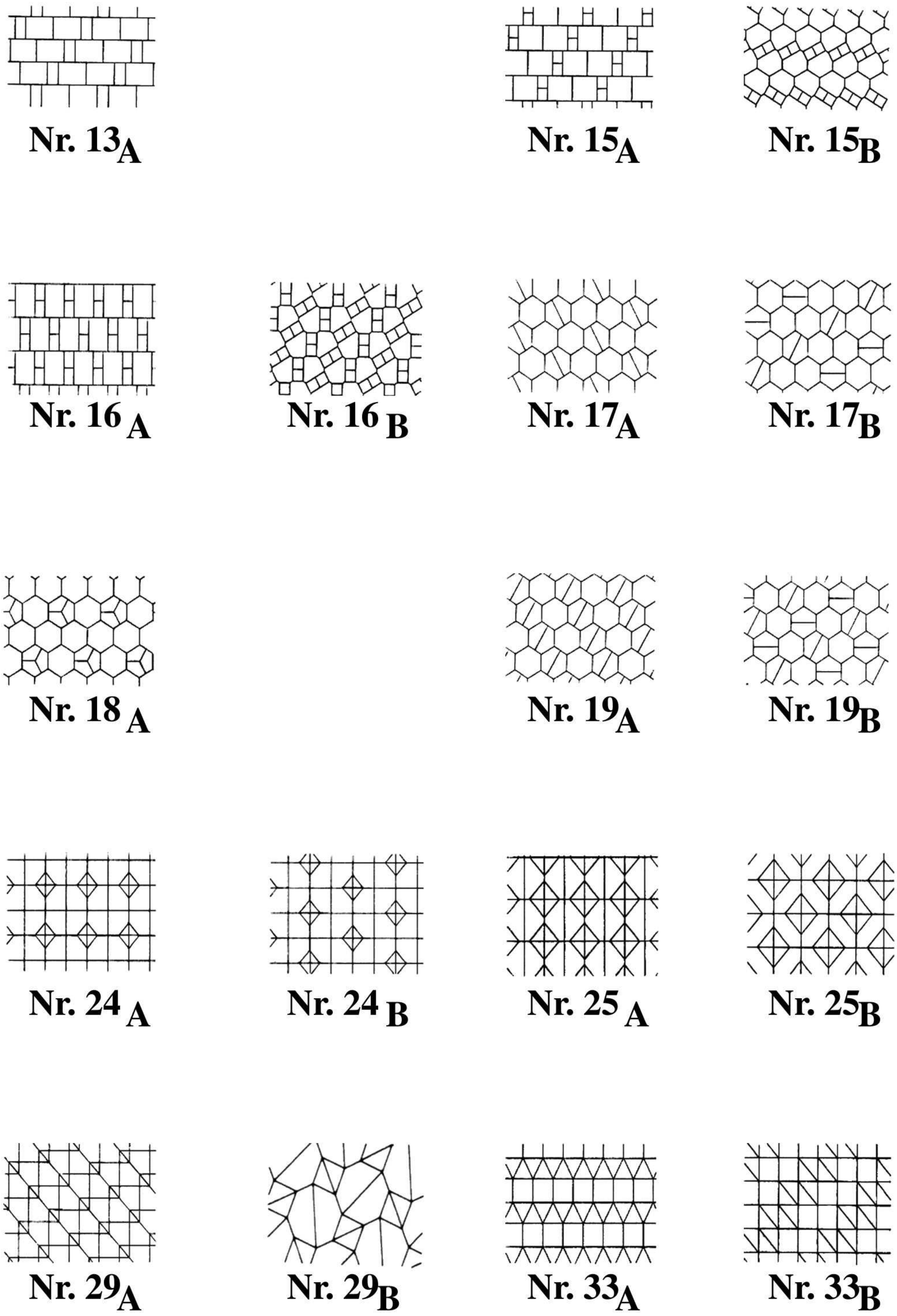


Figure 5: 18 2-homeohedral types from continuums

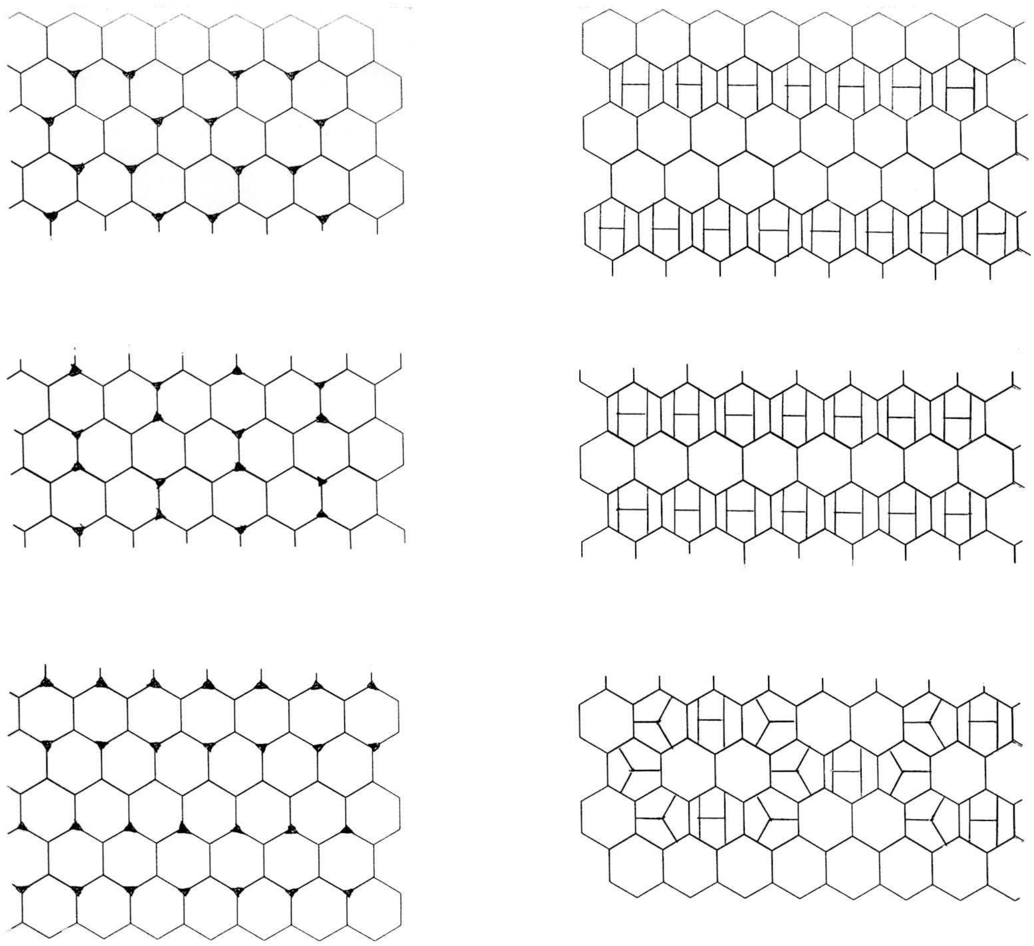


Figure 6: Face-regular tilings Nrs. 2', 2'', 2''' and 20, 21, 22 (only Nr. 2' is 2-homeohedral)

Remarks:

(i) The classification for 11 continuums of Table 2 is just an extension of ideas used in [**DFSV00**] to classify continuums Nrs. 17-19. Moreover, [**DFSV00**] gives, for tilings Nr. 17, all possible groups of symmetry - strip groups p111, p112, p1m1 and plane groups cmm (only for $Nr.17_A$), pgg (including $Nr.17_B$), pg, p2, p1 - and minimal polyhedral torus or Klein bottle for tilings Nr. 17 of each of those 7 symmetry groups. (See [**L68**], pp.61-72, on the symmetry groups of 2-isohedral tilings.)

(ii) Amongst face-regular bifaced tilings, only Nrs. $6, 14, 30, 32, 33_A$ - Archimedean plane tilings (3.12^2), (4.8^2), $(3.6.3.6)$, $(3^2.4.3.4)$, $(3^3.4^2)$ (with symmetry $p6m$, $p4m$, $p6m$, $p4g$, cmm, respectively; unique remaining bifaced Archimedean tiling $(3^4.6)$ is not face-regular) are *mosaics*, i.e. tilings of Euclidean plane by regular polygons; see [**DS99**] for the list of 58 mosaics T with embeddable skeletons of T or T^*. All five are 2-isohedral. Dual (3.12^2), (4.8^2), dual $(3.6.3.6)$, $(3^2.4.3.4)$, $(3^3.4^2)$ embed isometrically (or isometrically up to scale 2, that is indicated by "$\frac{1}{2}$") into $\frac{1}{2}Z_\infty$, Z_4, Z_3, $\frac{1}{2}Z_4$, $\frac{1}{2}Z_3$, respectively.

(iii) Tedious check show that *all* embeddable ones amongst the face-regular bifaced tilings are: Nr. 3' (the 2-homohedral one), Nr. 13_c, Nr. 14, all Nr. 29 (except of Nr. 29_c), Nr. 32, Nr. 33_A and all others Nr. 33; they embeds into $\frac{1}{2}Z_3$, $\frac{1}{2}Z_8$, Z_4, $\frac{1}{2}Z_5$, $\frac{1}{2}Z_4$, $\frac{1}{2}Z_3$, $\frac{1}{2}Z_4$, respectively.

(iv) Tilings Nrs. 1-6 are truncations of the regular tiling (6^3) (Archimedean Nr. 6 being its full truncation); Nrs. 7-12, respectively, are their 4-triakon decorations. Nrs. 17-23 are decorated (6^3). Archimedean Nr. 14 is the full truncation of the regular tiling (4^4); Nrs. 13-16, 27 and all Nr. 29, except of Nr. 29_c, are decorations of it. All other face-regular bifaced tilings (including Archimedean Nrs. 30, 32, 33_A) are decorated (4^4).

(v) Only one of 33 sets of parameters of face-regular bifaced tilings (3 tilings Nr. 8, with $(k;\, a,b;\, t_a,t_b) = (3;\, 4,10;\, 2,6)$), is one of those of 71 polyhedra (Nr. 38); all are 4-triakon.

8. Aknowlegement

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