EMBEDDING OF SKELETONS OF VORONOI AND DELONE PARTITIONS INTO CUBIC LATTICES

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The famous Deuxième mémoire of Voronoi (1908, 1909) in Crelle Journal contains, between other things, deep study of two dual partitions of $\mathbf{R}^n$ related to an n-dimensional lattice Λ. In modern terms, they are called *Voronoi partition* and *Delone partition* (Voronoi himself called the second one *L-partition*). Both partitions coincide for the cubic lattice; we denote by Z_n the skeleton of the cubic n-dimensional lattice. Denote by $Vo(\Lambda)$, $De(\Lambda)$ skeletons of Voronoi and Delone partitions for lattice Λ. So, edges of these graphs are edges of the Voronoi parallelotope and of the Delone polytopes of Λ; any minimal vector of Λ is an edge of $De(\Lambda)$ but not vice versa, in general.

We are interested whether infinite graph G, where $G = Vo(\Lambda)$ or $De(\Lambda)$, either is embedded isometrically (or with doubled distances) into a Z_m for some $m \geq n$, or not; we use notation $G \to Z_m$ or $G \to \frac{1}{2}Z_m$ in the first two cases.

In this note we report what we got, in this direction, for irreducible root lattices, for two generalizations of the diamond bilattice and for 3-dimensional case.

The validity of the following *5-gonal* inequality for distances is known [Dez60] to be necessary for embedding of any graph (in fact, of any metric space) into some Z_m: for any vertices a, b, x, y, z we have

$$d(a,b) + \{d(x,y) + d(x,z)) + d(y,z)\} \leq$$

$$\leq \{d(a,x) + d(a,y) + d(a,z)\} + \{d(b,x) + d(b,y) + d(b,z)\}.$$

It turns out that cases of non-embedding given in this note, came out by violation of this 5-gonal inequality.

Let us start with irreducible root lattices, i.e. A_n, D_n, E_n.

For small dimension n, we have: $De(A_2) = (3^6) \to \frac{1}{2}Z_3$, $Vo(A_2) = (6^3) \to Z_3$ ([As81]); $D_2 = Z_2$, $A_2^* = A_2$, $D_3 = A_3$.

Theorem 1.

(i) *$De(E_n)$ is not 5-gonal for $n = 6, 7, 8$;*

(ii) *for $n \geq 3$, we have: $De(A_n) \to \frac{1}{2}Z_{n+1}$, $Vo(A_n) \to Z_{n+1}$, $Vo(A_n^*) \to Z_m$ (where $m = \binom{n+1}{2}$), $De(A_n^*)$ is not 5-gonal;*

(iii) *for* $n \geq 4$, *we have:* $De(D_n)$, $Vo(D_n)$, $De(D_n^*)$ *are not 5-gonal.*

Remark that $De(D_4)$ is not embedded, contrary to 2b) of [As81]; take 5 points

$$a = (0,0,0,0),\ b = (1,1,0,0),$$

$$x = (1,0,1,0),\ y = (1,0,-1,0),\ z = (0,1,0,1)$$

forming non 5-gonal graph $K_5 - K_3$. For example, (x,y) is not an edge, since the middle point of the segment $[x,y]$ is the center of the square $(a,x,c = (2,0,0,0),y)$ with edges (a,x), (x,c), (c,y), (y,a) from the graph $De(D_4)$. Apropos, $De(D_4)$ is a metric subspace of $De(D_n)$ for $n \geq 5$.

Remark also, that we have isometric embedding of Z_n into $De(D_{2n})$ and Z_2 into $De(A_3)$.

Now we consider 5 types (depending on their Voronoi polyhedron) of 3-dimensional lattices, obtained by Fedorov [Fe1885]. Besides Z_3, A_3=*f.c.c.* and A_3^*=*b.c.c.*, there are two other types of lattices having 6-prism and elongated dodecahedron as the Voronoi polyhedron. Let us take $A_2 \times Z_1$ and, say, Λ' as representatives of the lattices of these two types. Remark that $De(A_3)$, $De(\Lambda')$ coincide as graphs, but the *partitions* of $\mathbf{R}^3$ (for which they are skeletons) are different.

Theorem 2.
(i) $De(A_2 \times Z_1) \rightarrow \frac{1}{2}Z_4$, $Vo(A_2 \times Z_1) \rightarrow Z_4$;
(ii) $De(\Lambda') \rightarrow \frac{1}{2}Z_4$, $Vo(\Lambda') \rightarrow Z_5$.

So, Delone partition of unique general lattice A_3^* is only non-embeddable $De(\Lambda)$, $Vo(\Lambda)$ for 5 types of 3-dimensional lattices.

In $\mathbf{R}^3$, the combinatorial type of a parallelohedron P determines the combinatorial type of the corresponding tiling by P; also the type of the dual partition is determined by the type of its *star* (i.e. the configuration around a vertex). For normal partitions we have 5 types of parallelohedra and 5 dual types of partition (whose skeletons are of 4 types, it was already in [Fe1885]), their embeddings are described in the theorem 2 above. [Sh80] found all 3 types of convex parallelohedra for essentially non-normal (i.e. non-normalizable) partitions of $\mathbf{R}^3$. Denote them by S_1, S_2, S_3; denote by $P(S_i)$, $P^*(S_i)$ the tiling by S_i and dual partition for i=1, 2, 3. All S_i, i=1, 2, 3, are centrally-symmetric 10-hedrons obtained by a decoration of the parallelepiped; their p-vectors are (p_4=10), (p_4=6, p_6=4), (p_4=4, p_6=4, p_8=2) respectively. S_1 is (combinatorially) β_3 truncated in 2 opposite vertices; S_2, S_3 have 2-valent vertices [Sh80]. All $P^*(S_i)$ have the same combinatorial type of skeleton; they are partitions of $\mathbf{R}^3$ by non-convex bodies.

Theorem 3.
For i*=1, 2, 3 we have* $S_i \rightarrow H_{3+i}$, $P(S_i) \rightarrow Z_{2+i}$ *and* $P^*(S_i)$ *is not 5-gonal.*

Two most interesting *lattice complexes* in 3-space are bilattices *J-complex* and *D-complex* (D-complex called also *diamond* or *tetrahedral packing* and denoted by D_3^+).

Theorem 4.

(i) *De(D-complex)* $\to \frac{1}{2}Z_5$*, but Vo(D-complex), De(J-complex), Vo(J-complex) are not 5-gonal;*

(ii) *any Kelvin packing* K *by* α_3 *and* β_3 *(except* A_3*, but including the bilattice h.c.p., i.e. hexagonal close packing) has non 5-gonal* $De(K)$, $Vo(K)$.

Between packings of 3-space, considered above, $De(Z_3)$, $De(A_2 \times Z_1)$, $Vo(A_2 \times Z_1)$, $De(A_3)$, $Vo(A_3^*)$, *De(J-complex)*, *De(h.c.p.)* are *uniform* partitions of $\mathbf{R}^3$ by regular and semiregular polyhedra. These polyhedra are, respectively: cubes γ_3, truncated octahedra β_3, 3-prisms, 6-prisms, tetrahedra α_3 with β_3, β_3 with cuboctahedra, α_3 with β_3. The list of all such partitions and their embedding will be considered in [DGS97].

All embeddings into Z_m (i.e. isometric ones) of skeletons of Voronoi partitions considered here, except the Theorem 3, were related to Voronoi tilings by a zonotope with m zones; for example, by the permutahedron for $Vo(A_n^*)$. But the Voronoi partition corresponding to A_3, elongated by layers of 3-prisms, is embedded into Z_4. It is a tiling of $\mathbf{R}^3$ by a half of the rhombic dodecahedron (i.e. 6-prism with new vertex connected to 3 alternated vertices of a hexagonal face), which is *not* centrally-symmetric. (An example of non-zonotopal plane tiling is given by $[3^6; 3^2.6^2] \to Z_\infty$. Apropos, the simplest graph, embeddable only into Z_∞, is the caterpillar with vertices $a_i = (i, 0)$, $b_i = (i, 1)$ for $i \in N$ and edges (a_i, a_{i+1}), (a_i, b_i).) It will be interesting to find some non-zonotopal, but embeddable into Z_m, tiling of $\mathbf{R}^3$ by *centrally-symmetric* polyhedrons. It will be an infinite analog of non representable oriented matroid. Example of *non-normal* such tiling is $P(S_1)$, given in [Sh80]. It is a tiling of $\mathbf{R}^3$ by centrally-symmetric convex parallelohedrons $\gamma_3 + \gamma_3$; the skeleton of this tiling is Z_3, see Theorem 3 above.

Consider now following two bilattices generalizing D-complex

$$D_n^+ := D_n \cup (d + D_n),$$

where the new point d is the center of greatest Delone polytope, and

$$A_n^+ := A_n \cup (a + A_n),$$

where the new point a is the center of regular n-simplex, Delone polytope of A_n, see [CS88].

D_n^+ is a lattice if and only if n is even; $D_2^+ = Z_2$, $D_4^+ = Z_4$, $D_8^+ = E_8$; A_n^+ is always bilattice. It obtained from A_n by the centering of its smallest Delone polytope, the n-simplex α_n; the centering of *all* Delone polytopes of A_n will give A_n^*. Remind that $De(A_2^+) = Vo(A_2) = (6^3) \to Z_3$, $Vo(A_2^+) = (3^6) \to \frac{1}{2}Z_3$, $A_3^+ = D_3^+$.

Theorem 5.

(i) $De(D_3^+) \to \frac{1}{2}Z_5$, $Vo(D_3^+)$ *is non 5-gonal,* $De(D_n^+)$ *is non 5-gonal for* $n \geq 5$;

(ii) $De(A_n^+) \to \frac{1}{2}Z_{n+2}$ *for* $n \geq 3$.

All above results are obtained by techniques given in [CDG97] and [DS96]. For example, non 5-gonality of $De(A_n^*)$, $n \geq 3$, given by 5 points:

$$a = (0, 0, 0, 0, ..., 0), b = (1, 1, 1, 0, ..., 0),$$

$$x=(1,0,0,0,...0), y=(0,1,0,0,...,0), z=(0,0,1,0,...,0)$$

in Selling-reduced basis of the lattice; the points *a, x, x+y, b* are vertices of a face of a Delone simplex [Del37].

Another example of non-embedding (also generalizing the remark after Theorem 2) is following:

Theorem 6.

A closed simplicial n-manifold M_n*,* $n \geq 3$*, is not embedded if it has (n–2)-face which belongs to at least five n-simplices.*

In fact, in the conditions of Theorem 6, the skeleton of M_n contains isometric subgraph $K_7 - C_5$ (i.e. the skeleton of 4-polytope $Pyr(Pyr_5)$) which is 5-gonal but is not embedded. For example, the regular 4-polytope 600-cell, which is a closed simplicial 3-manifold, has five tetrahedra on each edge. If, moreover, (n-2)-face from Theorem 6 belongs to at least *six* n-simplices (and so to at least six (n–1)-simplices), then M_n is not 5-gonal, since it contains isometric subgraph $K_5 - K_3$. For example, $De(A_3^*)$ is not 5-gonal, because it has six tetrahedra on some edges.

See [DGS97], [DS96], [DS97], [DG97] for embedding of other classes of polyhedral graphs. See [DL97] for general theory of isometric embedding into space l_1^m and, up to scale, into hypercubes. See [RB79] and [CS88] for notions of lattice theory.

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