

# A zoo of $l_1$ -embeddable polyhedra II

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## Abstract

We complete here the study of  $l_1$ -polyhedra started in our previous paper on this subject, [DeGr97a]. New classes are considered, especially small polyhedra, some operations on Platonic solids and  $k$ -valent polyhedra with only two types of faces.

## 1 Introduction

We use definitions and notation from [DeGr97a]. Call a polyhedron  $P$   $l_1$ -polyhedron (or  $l_1$ -embeddable if its skeleton is embedded isometrically (or with doubled distance) into an  $m$ -cube, and use the notation  $P \rightarrow H_m$  (or  $P \rightarrow \frac{1}{2}H_m$ , respectively).

In this paper we group, in compact form, many results specifying  $l_1$ -polyhedra in the classes defined in the titles of sections. All proofs are obtained by the techniques of [CDGr97], [DeGr97a] and by direct check.

We remind only the following necessary condition (called *5-gonal inequality*) for embedding of a graph which is sufficient for any bipartite and "many" planar graphs:

$$F(x, y; a, b, c) := d_{xy} + (d_{ab} + d_{bc} + d_{ac}) - \sum_{p=x, y; q=a, b, c} d_{pq} \leq 0$$

for any distinct vertices  $x, y, a, b, c$  of the graph.

Moreover: (i)  $F(x, y; a, b, c) \geq 4 - 6 \max_{p=x, y; q=a, b, c} d_{pq}$  with equality, for example, for a graph such that  $x, y$  belong to a triangle  $K_{x, y, z}$  and  $a, b, c$  belong to  $K_{a, b, c, d}$ , and the triangle and  $K_4$  are joined by a path from  $z$  to  $d$ .

(ii)  $F(x, y; a, b, c) \leq d_{xy}$  with equality iff both  $x$  and  $y$  lie on a geodesic between each pair  $(a, b)$ ,  $(a, c)$ ,  $(b, c)$  (for example, as in  $K_{\{x, y\}, \{a, b, c\}}$ ).

(iii)  $F(x, y; a, b, c) \leq (d_{ab} + d_{ac} + d_{bc}) - 2d_{xy}$  with equality iff each of  $a, b, c$  lies on a geodesic between  $x$  and  $y$  (for example, as in (snub  $APrism_n$ )\* for any  $n \geq 5$ , see §5).

## 2 Small polyhedra

$l_1$ -status of all 10 polyhedra with at most 6 faces and their duals is such that they are either  $l_1$ -embeddable or non 5-gonal.

The  $l_1$ -embeddable are

$\rightarrow \frac{1}{2}H_4 : \alpha_3 \simeq \alpha_3^*$ ,  $Pyr_4 \simeq Pyr_4^*$ ,  $BPyr_3 = Prism_3^*$ ,  $\gamma_3^* = \beta_3$ ,

$\rightarrow \frac{1}{2}H_5 : Pyr_5 \simeq Pyr_5^*$ ,  $Prism_3 \simeq BPyr_3^*$ ,  $(2\text{-truncated } \alpha_3)^*$ , a dual with skeleton  $K_6 - P_5$  (# 43 in Proposition 2.1);

$\rightarrow \frac{1}{2}H_6 : \gamma_3$ , 2-truncated  $\alpha_3$ .

Remaining are: self-dual one with the skeleton  $K_6 - P_6$ , one with dual skeleton  $K_{3 \times 2} - e$ , its dual (# 44 in Proposition 2.1 below) and one with dual skeleton  $K_6 - P_5$ .

**Proposition 2.1.** *Among all 34 polyhedra (# # 11-44 on Fig.2) with 7 faces and their duals, we have*

$\rightarrow \frac{1}{2}H_5 : 20^*$ ,  $21^*$ ,  $29^*$ ,  $35$ ,  $36^*$ ,  $38^*$ ,  $39 \simeq 39^*$ ,  $43$ ;

$\rightarrow \frac{1}{2}H_6 : 12^*$ ,  $14^*$ ,  $16^*$ ,  $17^*$ ,  $18$ ,  $19 \simeq 19^*$ ;

$\rightarrow \frac{1}{2}H_7 : 12$ ,  $20$ ,  $23$ ;

*extreme hypermetric:  $13^* = G_4$ ;*

*non 7-gonal:  $32^*$ ,  $37^*$ , and*

*non 5-gonal: all others (incl. self-dual 34, 41 and  $40^* = 42$ ).*

**Remark.** (i) Among above  $l_1$ -polyhedra only  $\gamma_3$  is bipartite and only 3 are not  $l_1$ -rigid:  $\alpha_3 \simeq \alpha_3^* \rightarrow \frac{1}{2}H_3$ ,  $\frac{1}{2}H_4$ ,  $BPyr_3$  and  $39 \simeq 39^* \rightarrow \frac{1}{2}H_5$ ,  $\frac{1}{2}H_6$ . All simple ones between above 10+34 polyhedra with at most 7 faces are:  $\alpha_3$ ,  $Prism_3 = 1\text{-truncated } \alpha_3$ , 2-truncated  $\alpha_3$ ,  $\gamma_3$  and 11, 12 (3-truncated  $\alpha_3$ ), 13, 20 (1-truncated  $\gamma_3$ ),  $23 = Prism_5$ .

(ii) 20 among all 44 polyhedra with at most 7 faces are combinatorially equivalent to a space-filler: all 10, except  $Pyr_5$ , with at most 6 faces (including all 3 non 5-gonal) and # # 12, 15, 18, 20, 22, 23, 28, 30, 36, 43, 44 (between them # # 12, 18, 20, 23, 43 are  $l_1$ -graphs).

**Proposition 2.2.** *Among all 27 cubic graphs on up to 10 vertices, 19 are non 5-gonal while remaining 8 are  $l_1$ -graphs: non-polytopal Petersen graph  $\rightarrow \frac{1}{2}H_6$  and 7 simple polyhedra:  $i$ -truncated  $\alpha_3 \rightarrow \frac{1}{2}H_{4+i}$  for  $0 \leq i \leq 3$ ,  $i$ -truncated  $\gamma_3 \rightarrow \frac{1}{2}H_{6+i}$  for  $i = 0, 1$  and  $Prism_5 \rightarrow \frac{1}{2}H_7$ .*

**Proposition 2.3.** *Among all 14 simple polyhedra with 8 faces (i.e. with 12 vertices) 4 are  $l_1$ -graphs (all are embedded into  $\frac{1}{2}H_8$ ):  $Prism_6$ , dual bisdisphenoid, Dürer's octahedron and  $\gamma_3$  truncated on 2 adjacent vertices. Duals of first two are not 5-gonal and of last two  $\rightarrow \frac{1}{2}H_6$ . 4-truncated  $\alpha_3$  and  $\gamma_3$  truncated on 2 vertices at distance 2 are not 5-gonal; their duals  $\rightarrow \frac{1}{2}H_7$ . 5 polyhedra from Fig.3 are resp. non 5-, 5-, 7-, 7-, 9-gonal; their duals are resp. embeddable into  $\frac{1}{2}H_8$ ,  $\frac{1}{2}H_7$ ,  $\frac{1}{2}H_7$ ,  $\frac{1}{2}H_6$ , non 5-gonal. Remaining 3 polyhedra and their duals are non 5-gonal.*

One of 3 above polyhedra (a 3-truncated  $Prism_3$ ) is the smallest simple polyhedron with  $p = (3, 1, 1, p_6, 0...)$ ; such polyhedra exist (see p.271 in [Grün67]) iff  $p_6$  is odd, other than 1. The Dürer's octahedron above is  $\gamma_3$  truncated on 2 opposite vertices; it is called so, because it appears in Dürer's "Melancholia", 1514, staying on a triangular face.

**Remark.** (i) All simple  $l_1$ -polyhedra with  $f$ ,  $f \leq 8$ , faces (there are 11 of them) embed into  $\frac{1}{2}H_f$ ; all of them, except  $\alpha_3$ , are  $l_1$ -rigid and  $\gamma_3 \rightarrow H_3$ ,  $Prism_6 \rightarrow H_4$ .

(ii) the polyhedron in centre of Fig.3 and one of 3 non 5-gonal simple octahedra, having non 5-gonal duals, are two smallest 3-regular graphs with trivial group of automorphisms.

(iii) Between all 11 simple polyhedra with only  $k$ -gons,  $k \leq 5$ , as faces (duals of all 8 convex deltahedra, 1-truncated  $\alpha_3$ , 1-truncated  $\gamma_3$  and Dürer's octahedron) only 2 (duals of 3-augmented  $Prism_3$  and of 2-capped  $APrism_4$ ) are not  $l_1$ -graphs.

### 3 Truncations and cappings of Platonic solids

Call by *i-truncation* and *i-capping* the (short) truncation on  $i$  vertices of a polytope  $P$  and, respectively, adding a pyramid on  $i$  its faces. A *triakis* (*tetrakis*, *pentakis*, *hexakis*) *i-capping* is an *i-capping* only on 3-faces (4-,5-,6-faces, respectively). Omni-capping is called also *stellation*.

**Proposition 3.1.**

- (i) *i-truncated*  $\alpha_3 \rightarrow \frac{1}{2}H_{4+i}$ ,  $0 \leq i \leq 3$ ; *4-truncated*  $\alpha_3$  is not 5-gonal;  
*dual i-truncated*  $\alpha_3 = i$ -capped  $\alpha_3 \rightarrow \frac{1}{2}H_{3+i}$ ,  $0 \leq i \leq 4$ ;
- (ii) *i-truncated*  $\beta_3 \rightarrow \frac{1}{2}H_4, \frac{1}{2}H_6, H_4, H_6$  for  $i = 0, 1, 2$  (on opposite vertices), 6(=all), non 7-gonal if only 2 vertices of an edge are not truncated; non 5-gonal otherwise.  
*dual i-truncated*  $\beta_3 = i$ -capped  $\gamma_3 \rightarrow \frac{1}{2}H_6$  if  $i \leq 2$  or  $i = 3$  and any 2 capped faces of  $\gamma_3$  are not opposite; otherwise non 5-gonal.
- (iii) *i-truncated*  $\gamma_3 \rightarrow \frac{1}{2}H_{6+i}$  if  $i = 0, 1, 2$  (truncated vertices are at distance 1 or 3), 3 (they form  $P_3$ ), 4 (they form  $C_4$ ), otherwise non 5-gonal;  
*dual i-truncated*  $\gamma_3 = i$ -capped  $\beta_3 \rightarrow \frac{1}{2}H_{4+i}$  for  $0 \leq i \leq 8$ .
- (iv) *i-capped Icosahedron*  $\rightarrow \frac{1}{2}H_{6+i}$  for  $0 \leq i \leq 20$ ;  
*i-capped Dodecahedron*  $\rightarrow \frac{1}{2}H_{10}$  for  $0 \leq i \leq 12$ .

**Remark.** (i) Between above  $l_1$ -graphs only non  $l_1$ -rigid one is  $\alpha_3 \rightarrow \frac{1}{2}H_3, \frac{1}{2}H_4$ .

(ii) 4-truncated  $\gamma_3$ , on 4 non-adjacent vertices and on 4 vertices forming 2 opposite edges, are  $Cham(\alpha_3)$  and twisted  $Cham(\alpha_3)$ , respectively. Remaining 4 ways to 4-truncate  $\gamma_3$  are when these 4 vertices induce one of the graphs  $C_4, P_4, P_3 + K_1$  and  $K_{1,3}$ . All 3 ways to 3-truncate  $\gamma_3$  are on  $P_3, P_2 + K_1$  and  $3K_1$ .

Consider now capping of some almost regular  $l_1$ -polyhedra on 3- and 4-gonal faces.

Clearly, any triakis *i-capping* of an  $l_1$ -polyhedron  $P$  embeds into  $\frac{1}{2}H_{m+i}$  if  $P \rightarrow \frac{1}{2}H_m$ .

- 1) *i-capped*  $Pyr_4 \rightarrow \begin{cases} \frac{1}{2}H_{4+i} & \text{if the 4-face is truncated,} \\ \frac{1}{2}H_{3+i} & \text{otherwise;} \end{cases}$

1-truncated  $Pyr_4$  (=dual 1-capped  $Pyr_4$ ) =  $\gamma_3$  if the apex is truncated, or # 43\* otherwise; (so, non 5-gonal).

2) *i-augmented*  $Prism_m$  is  $Prism_m$  capped on  $i$  (4-gonal) faces, which are not 2 base faces. *i-augmented*  $Prism_m$  ( $m = 3, 4$ )  $\rightarrow \frac{1}{2}H_{m+2}$  for  $i = 0, 1, 2$ , non  $l_1$  for  $i \geq 3$ .

3) Snub cube  $\rightarrow \frac{1}{2}H_9$  but any tetrakis *i-capping* of it is non 5-gonal.

4) Rhombicuboctahedron  $\rightarrow \frac{1}{2}H_{10}$ . Only 5-gonal tetrakis *i-capping* of it (which, moreover, embeds into  $\frac{1}{2}H_{10}$ ) is capping on at most 2 or on 3 non-opposite 4-faces, chosen between 6 4-faces adjacent only to 4-faces. Remark similarity with cappings of  $\gamma_3$  (Proposition 3.1 (ii)).

5) Tetrakis omni-capping of truncated  $\beta_3 \rightarrow \frac{1}{2}H_{12}$ .

6) Omni-cappings of  $\alpha_3$  (the  $3_{12}^*(T_d)$ ), of truncated  $\alpha_3$  (the  $3_{36}^*(T_d)$ )  $\rightarrow \frac{1}{2}H_7, \frac{1}{2}H_{11}$ , respectively, but omni-cappings of  $3_{36}(T_d)$  (the  $3_{108}^*(T_d)$ ) and of truncated cube are not  $l_1$ .

### 4 Chamfering of Platonic solids

$Cham P$  denotes the *chamfering* of the polyhedron  $P$ . It is obtained by putting prisms on all faces of  $P$  and deleting original edges; see also §6.

**Proposition 4.1** *Let  $P$  be one of five Platonic solids and  $t \geq 1$ . Then  $\text{Cham}_t P$  or its dual is  $l_1$ -graph only in the following cases:*

- (i)  $(\text{Cham}\alpha_3)^* \rightarrow \frac{1}{2}H_8$ ,
- (ii)  $\text{Cham}\gamma_3 \rightarrow H_7$  (zonohedron, generated by  $e_1, e_2, e_3, e_1 \pm e_2, e_1 \pm e_3$ ),
- (iii)  $\text{Cham}(\text{Dodecahedron}) \rightarrow \frac{1}{2}H_{22}$

**Remark.**

(i)  $\text{Cham}P$  is partially truncated zonohedron in the following cases:

$\text{Cham}\alpha_3 = \gamma_3$ , truncated on 4 non-adjacent vertices;

$\text{Cham}\beta_3, \text{Cham}\gamma_3$  are rhombic dodecahedra truncated on all 3-valent (resp., all 4-valent) vertices,

$\text{Cham}(\text{Icosahedron}), \text{Cham}(\text{Dodecahedron})$  are triacontrahedra truncated on all 3-valent (resp., all 5-valent) vertices.

(ii) Moreover, we have  $\text{Cham}_t\alpha_3$  ( $t = 1, 2$ ),  $\text{Cham}_t\beta_3$  ( $t = 1, 2$ ),  $\text{Cham}_t\gamma_3$  ( $t \geq 2$ ),  $\text{Cham}_2(\text{Icosahedron})$ ,  $(\text{Cham}_t\beta_3)^*$  ( $t \geq 1$ ),  $(\text{Cham}\gamma_3)^*$ ,  $(\text{Cham}(\text{Icosahedron}))^*$ ,  $(\text{Cham}(\text{Dodecahedron}))^*$  are not 5-gonal. Example of proving it:  $\text{Cham}P$  is not 5-gonal if the polyhedron  $P$  contains induced  $K_4 - e$ , because then  $\text{Cham}P$  contains isometric non 5-gonal graph  $G_{11}$  consisting of the cycle  $C_{10}$  with a new point adjacent only to the points 1,5,6 of the cycle  $C_{10}$ .

(iii)  $\text{Cham}(\text{Prism}_6), \text{Cham}(\text{Rhombic Dodecahedron})$  are not 5-gonal.

Iterated chamferings and their duals are used in the image compressing; for example,  $(\text{Cham}_t\gamma_3)^*$  appears there as  $(t + 1)$ -th approximation of some fractal distribution on  $\beta_3$ .

The notion of chamfering can be extended on non-polyhedral graphs. For example,  $\text{Prism}_m$  can be seen as the chamfering of the cycle  $C_m$ . On the other hand, this notion can be extended naturally for a simple  $n$ -polytope  $P$ . Denote by  $\text{Cham } P$  the dual of the convex hull of all vertices of  $P^*$  and of the mid-points of all edges of  $P^*$ . As a generalization of Proposition 4.1(i), we get  $(\text{Cham } \alpha_n)^* \rightarrow \frac{1}{2}H_{2n+2}$ . Remark that  $\alpha_n \rightarrow \frac{1}{2}H_{n+1}$ ,  $\text{ambo } \alpha_n \rightarrow \frac{1}{2}H_{n+1}$  (see §6) and  $(\text{Cham } \alpha_n)^* = \text{conv}(V(\alpha_n) \cup V(\text{ambo } \alpha_n))$ .  $\text{Cham } P$  is (2,0)-leapfrog in terms of [Sah94]. The dual of (3,0)-leapfrog corresponds to putting capped antiprisms on all faces of  $P$  and replacing each original edge (seen as a diagonal of a quadrilateral) by the second diagonal.

## 5 Bifaced polyhedra

Denote by  $(k; a, b; p_a, p_b)$  and call *bifaced* any  $k$ -valent polyhedron whose faces are only  $p_a$   $a$ -gons and  $p_b$   $b$ -gons, where  $3 \leq a < b$  and  $p_a > 0 \leq p_b$ . Any bifaced polyhedron  $(k; a, b; p_a, p_b)$  with  $n$  vertices has  $kn/2 = (ap_a + bp_b)/2$  edges and satisfies the Euler relation  $n - k\frac{n}{2} + (p_a + p_b) = 2$ , i.e.  $n = 2(p_a + p_b - 2)/(k - 2)$  and

$$(*) \quad p_a(2k - a(k - 2)) + p_b(2k - b(k - 2)) = 4k.$$

So,  $a < 2k/(k - 2)$  and only possible  $(k, a)$  are (3,5), (3,4), (3,3), (4,3), (5,3). Now,  $b = 2k/(k - 2)$  only for  $(k, b) = (3, 6), (4, 4)$  (see Sections 7-10 below) and  $b < 2k/(k - 2)$  only for 6 simple polyhedra with  $(a, b) = (3, 4), (3, 5), (4, 5)$ .

The case  $p_b = 0$  of above 5 classes is (combinatorially) Dodecahedron,  $\gamma_3$ ,  $\alpha_3$ ,  $\beta_3$ , Icosahedron, respectively. For  $p_b = 1$  and for  $p_b = 2$ ,  $(k, a) = (3, 3)$ , such polyhedra do

not exist. For  $p_b = 2$  classes  $(k, a) = (3, 4), (4, 3), (3, 5), (5, 3)$  consist of a unique polyhedron each:  $Prism_b$ ,  $APrism_b$ ,  $Barrel_b := (2\text{-capped } APrism_b)^*$  and  $snub APrism_b$  consisting of the following 3 circuits: outer  $(x_1, \dots, x_b)$ , middle  $(y_1, y'_1, \dots, y_b, y'_b)$ , and inner  $(z_1, \dots, z_b)$ , where  $x_i \sim y_i, y'_i, y_{i+1}$  and  $z_i \sim y'_{i-1}, y_i, y'_i$  for all  $i = 1, \dots, b$  ordered cyclically. The last polyhedron is called  $snub APrism_b$ , since for  $b = 4$  it is (combinatorially) well known regular-faced  $snub APrism_4$  which is 5-gonal, but not 7-gonal. Remark that  $snub APrism_3$  is sometimes called  $snub tetrahedron$ ; it has the same skeleton as Icosahedron, but its symmetry is  $T$ .  $Barrel_b$  is an analog of  $Prism_b$ , where the layer of 4-gons is replaced by two layers of 5-gons; it has  $4b$  vertices. We have

**Proposition 5.1** (i)  $Prism_b \rightarrow \frac{1}{2}H_{b+2}$  (moreover,  $\rightarrow \frac{1}{2}H_{(b+2)/2}$  for even  $b$ ),

$Prism_b^* \rightarrow \frac{1}{2}H_4, \frac{1}{2}H_4$ , non 5-gonal for  $b = 3, 4 \geq 5$ , respectively;

(ii)  $APrism_b \rightarrow \frac{1}{2}H_{b+1}$ ,

$APrism_b^* \rightarrow H_3$ , non 5-gonal for  $b = 3, \geq 4$ ;

(iii)  $Barrel_3 = \text{Dürer's octahedron} \rightarrow \frac{1}{2}H_8$ ,  $Barrel_3^* \rightarrow \frac{1}{2}H_6$ ,

$Barrel_5 = \text{Dodecahedron} \rightarrow \frac{1}{2}H_{10}$ ,  $Barrel_5^* \rightarrow \frac{1}{2}H_6$ ,

$Barrel_4$  is non 5-gonal,  $Barrel_4^*$  (a convex deltahedron) is an extreme hypermetric.

For  $b > 5$  both  $Barrel_b$  and its dual are non 5-gonal;

(iv)  $snub APrism_3 = \text{Icosahedron} \rightarrow \frac{1}{2}H_6$ , its dual  $\rightarrow \frac{1}{2}H_{10}$ ;

$snub APrism_4$  is not 7-gonal, its dual is not 5-gonal.

For  $b \geq 5$  both  $snub APrism_b$  and its dual are not 5-gonal.

From now on we consider mainly bifaced polyhedra with  $p_b \geq 3$ .

Consider first *simple* bifaced polyhedra, i.e. the case of  $k = 3$ . For  $b < 6$  there are only 6 such polyhedra: the Dürer's octahedron ( $\gamma_3$  truncated on two opposite vertices) and 5 dual deltahedra ( $Prism_3$  and duals of  $BPy_5$ , of bisdisphenoid, of 3-augmented  $Prism_3$ , of 2-capped  $APrism_4$ ). Moreover, there are only 11 simple polyhedra with  $\sum_{i \geq 3} p_i = p_3 + p_4 + p_5$ : 6 above polyhedra, remaining 3 dual deltahedra ( $\alpha_3, \gamma_3, \text{Dodecahedron}$ ) and 1-truncated  $\gamma_3$ , 2-truncated  $\alpha_3$ , having  $(p_3, p_4, p_5) = (1, 3, 3), (2, 2, 2)$ , respectively. The  $l_1$ -status of all these polyhedra and their duals is known (see §2 above).

Simple bifaced  $n$ -vertex polyhedra with  $b = 6$  will be denoted as  $3_n, 4_n, 5_n$  if  $a = 3, 4, 5$ , respectively. They will be considered in the next sections as well as interesting case of  $(4; 3, 4; p_3, p_4)$ , having constant  $p_3 = 8$ ; denote such a polyhedron by  $(3, 4)_n$ .

The polyhedra dual to 3-valent polyhedra without  $i$ -gonal faces,  $i > 6$ , are studied in exceptionally deep work [Thur98] they are called there non-negatively curved triangulations. (Actually, the reference [Sah94] is an application of a preliminary version of [Thur98].) The main Theorem 0.1 there describes them as the elements of  $L_+/G$ , where  $L$  is a lattice in complex Lorenz space  $C^{(1,9)}$ ,  $G$  is a group of automorphisms and  $L_+$  is the set of lattice points of positive square-norm. The square of the norm of a lattice point is the number of triangles in the triangulation and the number of such triangulations with up to  $2n$  triangles is  $O(n^{10})$ .

Simple bifaced polyhedra with  $b > 6$  were considered in [Malk70]. Besides Euler's relation  $(6 - a)p_a = 12 + (b - 6)p_b$ , we have

if  $a = 3$ , then  $b \in \{7, 8, 9, 10\}$ ;

if  $a = 4$ ,  $b \equiv 0 \pmod{8}$ , then  $p_b$  is even;

if  $a = 5$ ,  $b \equiv 0 \pmod{10}$ , then  $p_b$  is even.

[Malk70] asserts that above necessary conditions (for existence of such a polyhedron) are sufficient, with only a finite number of exceptions. Polyhedra  $(3; 3, b; p_3, p_b)$  with  $b < 6$  are only  $Prism_3$  and Dürer's octahedron; both are  $l_1$ -embeddable into  $\frac{1}{2}H_5, \frac{1}{2}H_8$ , respectively. The polyhedra  $(3; 3, b; p_3, p_b)$  with  $b = 6, 8, 10$  are non 5-gonal, since they contain (triangles are isolated) isometric non 5-gonal subgraph consisting of a vertex surrounded by a triangle and two even-gons. Examples of  $l_1$ -graphs between their duals are 3 omni-capped Platonic solids  $\alpha_3, \beta_3$ , Icosahedron.

Simple bifaced polyhedra were studied in order to find non-Hamiltonian members between them; for example, they exist (J.Zaks) for  $(a, b) = (5, 8)$ , but do not exist ([Good75], [Good77]) for  $(a, b) = (3, 6), (4, 6)$  or  $6 > b$ . The case  $(5, 6)$  of fullerenes is undecided but their duals are Hamiltonian.

Consider now *non-simple* bifaced polyhedra with  $b \leq 6$ . All possible  $(k; a, b)$  are  $(4; 3, 6), (4; 3, 5), (4; 3, 4), (5; 3, 6), (5; 3, 5), (5; 3, 4)$ . In each of those cases there is an infinity of such polyhedra. Namely, we have

[Fisch75]:  $(5; 3, 4; p_3, p_4)$  exists for any  $p_4 > 1$ ,

[Grün67], p.282:  $(4; 3, 4; p_3, p_4)$  exists for any  $p_4 > 1$ ,

[Grün96]: an infinity of examples is constructed for each of 4 remaining cases (all of them, except tetrakis truncated octahedron  $(4; 3, 6; 24, 8)$ , are not  $l_1$ ).

Remark that cases (ii), (iii),  $(i_3)$  of *ambo*  $P$  construction of §6, produce also an infinity of bifaced polyhedra for  $(k = 4; a = 3, b)$  with  $b = 4, 5, 6$ .

There are only two classes of non-simple bifaced polyhedra, namely,  $(4; 3, b; p_3, p_b)$  and  $(5; 3, b; p_3, p_b)$ . Examples of  $l_1$ -polyhedra between them are Rhombicuboctahedron  $\rightarrow \frac{1}{2}H_{10}$  ( $(4; 3, 4; 8, 18)$ ), tetrakis truncated  $\beta_3 \rightarrow \frac{1}{2}H_{12}$  ( $(4; 3, 6; 24, 8)$ ), *snub*  $\gamma_3 \rightarrow \frac{1}{2}H_9$  ( $(5; 3, 4; 32, 6)$ ), *snub* dodecahedron  $\rightarrow \frac{1}{2}H_{15}$  ( $(5; 3, 5; 80, 12)$ ), and, between their duals, both Catalan zonohedra. Duals of above six  $l_1$ -polyhedra are non 5-gonal. Only 3 Archimedean polyhedra are not bifaced: Rhombicosidodecahedron  $\rightarrow \frac{1}{2}H_9$  and two large zonohedra.

The list of known  $l_1$ -polyhedra between bifaced polyhedra and their duals is given in Tables 2 and 3.

**Remark** (i) For all  $l_1$ -polyhedra of the tables below (except  $Prism_3$ , Dürer's octahedron and 2-capped  $C_4 \times P_{m+1}$ ) the dual polyhedra are not 5-gonal.

(ii) For majority of  $l_1$ -polyhedra  $P$  from both Tables 2 and 3,  $P \rightarrow \frac{1}{2}H_{2d}$ , where  $d$  is the diameter of  $P$ . But, for example, the diameter of dual truncated  $\alpha_3$  is 2 and of dual truncated dodecahedron is 4.

(iii) Between triangulations ## 1-12 of Table 3 (and denoting by " $<$ " isometric subgraph) we have: Icosahedron  $<$  #4  $<$  #12;  $\beta_3 <$  # 2, # 3  $<$  # 11; # 6  $<$  # 5.

*Self-dual* polyhedron with  $p = (p_a, p_b)$ ,  $3 \leq a < b$ , exists, applying [Juco70] iff  $a = 3$ , all  $p_i = v_i$  and  $p_a = p_3 = n - p_b$ ,  $p_b = \frac{n-4}{b-3}$ . So, it is  $\alpha_3, Pyr_b$  for  $p_b = 0, 1$ . For  $b = 4$ , the subcase  $n \equiv 1 \pmod{4}$  is realized by  $k$ -elongated  $Pyr_4$  having  $p = (p_3 = 4, p_4 = 4k + 1)$ . All above polyhedra are embedded into a half-cube. But the polyhedra with  $b = 4$  and  $n = 6, 7, 8$  are not 5-gonal (the skeleton is  $K_6 - P_6$  for  $n = 6$ ). Also the gyrobifastigium (a regular-faced polyhedron) has  $p = (p_3, p_4) = v = (v_3, v_4) = (4, 4)$ , but it is not self-dual; it and its dual are non 5-gonal.

Other nice example of a bifaced polyhedron with two vertex-valencies is the following chimera of Cube and Dodecahedron. It has  $p = (p_4 = 6, p_5 = 12)$  and  $v = (v_3 = 20, v_4 = 6)$ ; it is the fundamental polyhedron of the smallest known closed hyperbolic space

(see [Week85]). Neither it, nor its dual  $K_6 - K_3 - K_2$  are 5-gonal.

**Table 1.** All  $k$ -valent polyhedra with only  $a$ -gonal and  $b$ -gonal faces,  $3 \leq a < b \leq 6$ ,  $p_a > 0 < p_b$ .

$(a, b) \setminus k$	3	4	5
(5, 6)	$5_n$ (fullerene) exists iff $p_6 \geq 2$	--	--
(4, 6)	$4_n$ exists iff $p_6 \geq 2$	--	--
(3, 6)	$3_n$ exists iff $p_6 \geq 4$ , even	$p_3 = 8 + 2p_6,$ $n = 6 + 3p_6$	$p_3 = 20 + 8p_6,$ $n = 12 + 6p_6$
(4, 5)	only 4 dual deltahedra	--	--
(3, 5)	only Dürer's octahedron	$p_3 = 8 + p_5,$ $n = 6 + 2p_5$	$p_3 = 20 + 5p_5,$ $n = 12 + 4p_5$
(3, 4)	only $Prism_3$	$p_3 = 8,$ $n = 6 + p_4$ exists iff $p_4 \geq 2$	$p_3 = 20 + 2p_4,$ $n = 12 + 2p_4$ exists iff $p_4 \geq 2$

**Table 2.** Known  $l_1$ -polyhedra between bifaced polyhedra

$k$	$a, b$	$p_a, p_b$	polyhedron	embeds into
3	4, 6	6, 8	truncated $\beta_3$	$H_6$
3	4, 6	6, 12	chamfered $\gamma_3$	$H_7$
3	4, 6	6, 12	twisted chamfered $\gamma_3$	$H_7$
3	5, 6	12, 3	fullerene $5_{26}(D_{3h})$	$\frac{1}{2}H_{12}$
3	5, 6	12, 12	fullerene $5_{44}(T_d)$	$\frac{1}{2}H_{16}$
3	5, 6	12, 30	chamfered Dodecahedron	$\frac{1}{2}H_{22}$
3	3, 5	2, 6	Dürer's octahedron	$\frac{1}{2}H_8$
3	4, 5	4, 4	dual bisdisphenoid	$\frac{1}{2}H_8$
3	4, $b$	$b, 2$	$Prism_b$ (incl. $b = 3$ )	$\frac{1}{2}H_{b+2}$
4	3, $b$	$2b, 2$	$APrism_b$	$\frac{1}{2}H_{b+1}$
4	3, 4	$8, 4m$	2-capped ( $C_4 \times P_{m+1}$ )	$\frac{1}{2}H_{2m+4}$
4	3, 4	8, 3	ambo $Prism_3$	$\frac{1}{2}H_6$
4	3, 4	8, 18	Rhombicuboctahedron	$\frac{1}{2}H_{10}$
4	3, 6	24, 8	tetrakis truncated $\beta_3$	$\frac{1}{2}H_{12}$
5	3, 4	32, 6	snub $\gamma_3$	$\frac{1}{2}H_9$
5	3, 5	80, 12	snub dodecahedron	$\frac{1}{2}H_{15}$



**Table 3. Known  $l_1$ -polyhedra between dual bifaced polyhedra**

$k$	$a, b$	$p_a, p_b$	polyhedron	embeds into
3	3, 6	4, 4	truncated $\alpha_3$	$\frac{1}{2}H_7$
3	3, 6	4, 6	chamfered $\alpha_3$	$\frac{1}{2}H_8$
3	3, 6	4, 6	twisted chamfered $\alpha_3$	$\frac{1}{2}H_8$
3	3, 6	4, 12	4-truncated Dodecahedron	$\frac{1}{2}H_{10}$
3	3, 6	4, 18	truncated omnicapped $\alpha_3$	$\frac{1}{2}H_{11}$
3	5, 6	12, 4	fullerene $5_{28}(T_d)$	$\frac{1}{2}H_7$
3	5, 6	12, 8	fullerene $5_{36}(D_{6h})$	$\frac{1}{2}H_8$
3	5, 6	12, 20	$5_{60} =$ truncated Icosahedron	$\frac{1}{2}H_{10}$
3	3, 4	2, 4	Prism <sub>3</sub>	$\frac{1}{2}H_4$
3	3, 5	2, 6	Dürer's octahedron	$\frac{1}{2}H_6$
3	3, 8	8, 6	truncated $\gamma_3$	$\frac{1}{2}H_{12}$
3	3, 10	20, 12	truncated Dodecahedron	$\frac{1}{2}H_{26}$
4	3, 4	8, 6	Cuboctahedron	$H_4$
4	3, 5	20, 12	Icosidodecahedron	$H_6$
4	3, 4	$8, 4m$	$2-(C_4 \times P_{m+1}) = (C_4 \times P_{m+2})^*$	$H_{m+3}$

## 6 Constructions of bifaced polyhedra

Consider some operations on a bifaced polyhedron  $P$  with parameters  $(k; a, b; p_a, p_b)$ . If  $b = 2k$ , then the dual of omnicapped  $P$ , called *leapfrog* of  $P$  in [Fowl93], has parameters  $(3; a, b; p_a, p_b + |V(P)|)$  and  $k|V(P)|$  vertices. If  $b = 2k = 6$ , then the *chamfering* of  $P$  (replace all edges by hexagons) has parameters  $(3; a, b; p_a, p_b + |E(P)|)$  and  $4|V(P)|$  vertices.

All  $5_n, 4_n, 3_n$  with icosahedral, octahedral, tetrahedral symmetry, respectively, are characterized in [Gold37]. They have  $n = 20(a^2 + ab + b^2), 8(a^2 + ab + b^2), 4(a^2 + ab + b^2)$ , resp. for  $a \geq b \geq 0$  (in Coxeter notation they are  $\{m+, 3\}_{a,b}$  for  $m = 5, 4, 3$ ) and cases  $b = 0$  and  $a = b$  correspond to full such symmetry  $I_h, O_h, T_d$ . The leapfrog (and chamfering) of Dodecahedron,  $\gamma_3, \alpha_3$  correspond to the case  $a = b = 1$  ( $a = 2, b = 0$ , resp.); above leapfrogs are truncated Icosahedron,  $\beta_3, \alpha_3$ , respectively. Clearly, each of classes of all  $5_n, 4_n, 3_n$  is closed under operations of leapfrog and chamfering.

Let  $P, P'$  be two Platonic polyhedra having  $(k, l, n, f)$  and  $(k' = k, l', n', f')$  resp. as size of faces, valency, the number of vertices and the number of faces. Consider the convex polyhedron  $P + fP'$  obtained by adjoining a copy of  $P'$  on each face of  $P$ . Then  $(P + fP')^*$  is a bifaced polyhedron with parameters  $(k; l', l(l' - 1); f(n' - k), n)$ . The case  $P' = \alpha_3$  corresponds to omnicapping of simplicial  $P$ . While, clearly,  $\alpha_3 + i\alpha_3, \beta_3 + i\alpha_3$  are  $l_1$ -embeddable for  $0 \leq i \leq f, \alpha_3 + i\beta_3$  for  $2 \leq i \leq 4$  and  $\beta_3 + i\beta_3$  for  $1 \leq i \leq 8$  are not  $l_1$ .

Given a polyhedron  $P$ , let  $P + Pyr_3$  ( $P + Prism_q$ , respectively) be the polyhedron defined by join of new vertex to all vertices of a triangular face (respectively, join of  $Prism_q$  to a  $q$ -face). It is easy to check the following

- Proposition 6.1** (i)  $P + Pyr_3$  is  $l_1$ -embeddable iff  $P$  is  $l_1$ -embeddable;  
(ii)  $P + Prism_q$  is  $l_1$ -embeddable iff  $P$  is  $l_1$ -embeddable.

In particular,  $P \rightarrow \frac{1}{2}H_m$  implies  $P + Pyr_3 \rightarrow \frac{1}{2}H_{m+1}$ , but, for example, for  $P = Prism_3 \rightarrow \frac{1}{2}H_5$ , we have  $P + Pyr_3 \rightarrow \frac{1}{2}H_6$  and  $\frac{1}{2}H_5$ , also. While  $\alpha_3 + iPyr_3 \rightarrow \frac{1}{2}H_{3+i}$  ( $1 \leq i \leq 4$ ) and  $Prism_3$  are  $l_1$ -rigid, we have  $\alpha_3 + iPrism_3 \rightarrow \frac{1}{2}H_{3+2i}, \frac{1}{2}H_{4+2i}$  ( $0 \leq i \leq 4$ ). Now,  $P \rightarrow \frac{1}{2}H_m$  implies  $P + Prism_m \rightarrow \frac{1}{2}H_{m+2}$ . The case (i) above implies that for a planar triangulation  $T$ , the subgraph  $T_1$  induced by all vertices of degree  $\geq 4$  is an isometric triangulation and  $T$  is embeddable iff  $T_1$  is embeddable. Remark that  $APrism_n \rightarrow \frac{1}{2}H_{n+1}$  but  $APrism_n + APrism_n$  is not 5-gonal. The *golden dodecahedron* of [HiPe89], i.e. Dodecahedron+12 $APrism_5$  is not 5-gonal. The property (ii) above implies the following embeddings. Let  $P$  be a polyhedron with a vertex  $v$  such that  $P - v \rightarrow \frac{1}{2}H_m$ . Then  $P$ , truncated on the vertex  $v$ , is embedded into  $\frac{1}{2}H_{m+2}$ . For example, this truncation of  $P$  on two opposite caps is embedded into:  $\frac{1}{2}H_{10}$  if  $P$  is the icosahedron (i.e. 2-capped  $APrism_5$ );  $\frac{1}{2}H_9$  if  $P$  is 2-capped  $APrism_4$  (which is an extreme hypermetric);  $H_4$  if  $P$  is  $\beta_3 = APrism_3$ ;  $\frac{1}{2}H_{11}$  if  $P$  is  $5_{24}^*$  (i.e. 2-capped  $APrism_6$ , which is not 5-gonal).

Among 92 regular-faced polyhedra, 19 are *elongated P* (i.e. obtained from P by adjoining or inserting a prism), 27 are *i-augmented P* i.e. i-cappings of P. For example, pentakis of  $APrism_5$  and of the Dodecahedron are embeddable, pentakis of 5-Pyramid and of  $Prism_5$  are not 5-gonal, tetrakis of  $Prism_3$  and of  $APrism_4$  are extreme hypermetrics.

The following constructions produce bifaced polyhedra from columns of prisms and antiprisms. Denote by  $Prism_n^t, APrism_n^t$  the column of  $t$   $n$ -prisms (resp. of  $t$   $n$ -antiprisms) piled up on their  $n$ -gonal faces. Denote by  $2-Prism_n^t, 2-APrism_n^t$  their cappings on both external  $n$ -gonal faces. Remark that  $2-Prism_n^t = (Prism_n^{t+1})^*$ , that  $APrism_n^t$  can be seen as a helix and that identifying of both external faces in  $Prism_n^t, APrism_n^t$  gives a realization of maps  $(4^4), (3^6)$  on the torus. Now  $(APrism_3^t)^*, (2-APrism_5^t)^*, (2-APrism_6^t)^*$  are nanotubes  $4_{6t+2}, 5_{10t+10}, 5_{12t+12}$  (all are non 5-gonal for  $t > 1$ ) and  $(Prism_4^t)^*$  is a  $(3,4)_{4t+2} \rightarrow \frac{1}{2}H_{2t+2}$  (its dual  $\rightarrow H_{t+2}$ ). (An example of use of such columns of prisms and antiprisms in Chemistry is that  $APrism_3^t$  ( $t = 1, 2, 3, \infty$ ) and  $Prism_3^t$  ( $t = 1, 2$ ),  $2-Prism_5^2$  are metal cluster polyhedra for molybdenum sulfides  $Mo_6S_8^{4-}, Mo_9S_{11}^{4-}, Mo_{12}S_{14}^{6-}, (Mo_6S_6)_\infty^{2-}$  and for platinum carbonils  $Pt_6(CO)_{12}^{2-}, Pt_9(CO)_{18}^{2-}, Pt_{19}(CO)_{22}^{4-}$ , respectively; see [King87].)

The next construction of bifaced polyhedra is based on the notion of an ambo-polytope. For given polytope  $P$ , denote by ambo  $P$  the convex hull of the midpoints of all edges of  $P$ ; i.e. ambo  $P$  is the mid-edge truncation of  $P$ . Sometimes, moreover, the skeleton of ambo  $P$  is the line graph of the skeleton of  $P$ ; in such a case we write ambo  $P = L(P)$ . For example, ambo  $\alpha_n = L(\alpha_n) \rightarrow \frac{1}{2}H_{n+1}$ , ambo  $\alpha_3 = \beta_3$ , ambo  $\beta_4 = 24$ -cell, ambo  $\gamma_3 = L(\gamma_3) = \text{Cuboctahedron} = \text{ambo } \beta_3$ , ambo  $Dodecahedron = L(Dodecahedron) = Icosidodecahedron = \text{ambo } Icosahedron, L(Cuboctahedron) = Rhombicuboctahedron$ .

Clearly, ambo  $P$ , if  $P$  is a polyhedron  $(k, a, b; p_a, p_b)$  with  $n$  vertices, is a 4-valent polyhedron with  $p_a$   $a$ -faces,  $p_b$   $b$ -faces and, in addition,  $n$   $k$ -faces. So, ambo  $P$  is bifaced iff  $k \in \{a, b\}$ ; all possible cases for its parameters are:

- (i)  $k = a = 3, 4, 5$ ,
- (ii)  $(3, 4)_{2n}$  if  $P = (3, 4)_n$ ,
- (iii)  $(4; 3, 5; 20 + 5p_5, 12 + 5p_5)$  if  $P$  has parameters  $(5; 3, 5; p_3, p_5)$ .

If we consider only  $P$  with  $b \leq 6$ , then (see Table 1) we have, in addition to (ii), (iii):

- (i<sub>1</sub>)  $(3, 4)_9$  if  $P = Prism_3$ ,
- (i<sub>2</sub>)  $(4; 3, 5; 14, 6)$  if  $P$  is the Dürer's octahedron,

(i<sub>3</sub>)  $(4; 3, 6; n + 4, \frac{n}{2} - 2)$  if  $P = 3_n$ .

Finally, we give two constructions (by O. Delgado and M. Deza) generalizing two smallest *fulleroids* (it will mean here simple polyhedra with  $p = (p_5, p_7)$  and of the icosahedral symmetry) the existence and unicity of which is proved in [DrBr96]. Clearly,  $p_7 = p_5 - 12$  and divisible by 60; a fulleroid has  $v = 4(p_7 + 5)$  vertices. For any positive integer  $m$ , we get a fulleroid with  $v = 120m(m + 1) + 20$  from the  $5_{20(2m+1)^2}(I_h)$  by the *triacon* decoration of some  $\frac{p_6}{4} = 10m(m + 1)$  of its hexagonal faces with no pentagonal neighbors. (The triacon means adding a vertex connected to 3 alternated mid-edges of the hexagon.) Also we get simple polyhedra of the icosahedral symmetry and with  $p = (p_5 = 72, p_7 = 60)$ ,  $(p_5 = 72, p_8 = 30)$ ,  $(p_5 = 72, p_9 = 20)$ , from the icosahedral fullerenes  $5_{140}, 5_{80}, 5_{60}$ , respectively, by following decoration of each of 12 pentagonal faces. Add the 5-cycle of new vertices  $u_1, \dots, u_5$  to the 5-cycle of vertices  $v_1, \dots, v_5$  of above face; then connect each  $u_i$  to the mid-point of corresponding edge  $(v_i, v_{i+1})$ . Those polyhedra have, respectively, 260, 200, 180 vertices.

## 7 Polyhedra $3_n$

Theorem 2 of [GrMo63] gives that  $3_n$  exists for any  $n \equiv 0 \pmod{4}$ , except  $n = 8$ , and provides complete description of their skeletons. [Good77] showed that all  $3_n$  are Hamiltonian; their duals are also Hamiltonian.

$3_4 = \alpha_3 \rightarrow \frac{1}{2}H_3$ ,  $\frac{1}{2}H_4$  is not  $l_1$ -rigid. The "would-be"  $3_8$  is not 3-connected and non 5-gonal.

There are exactly  $N_3(n)$  polyhedra  $3_n$  ([Dill96]) for  $1 \leq \frac{n}{4} \leq 7$ , where  $N_3(n)$  is given below

$\frac{n}{4}$	1	2	3	4	5	6	7
$N_3(n)$	1	0	1	2	1	2	2

Th.3.4 in [Sah94] implies that  $N_3(n) = O(n)$ . Clearly,  $\alpha_3$  is unique  $3_n$  with abutting triangles. So, any  $3_n$ ,  $n > 4$ , is not 5-gonal since it contains an isometric subgraph  $G_{11}$  given in §4. We know six  $l_1$ -polyhedra  $3_n^*$  (no other exists for  $n \leq 28$ ):  $3_4^* \rightarrow \frac{1}{2}H_3$  (and  $\rightarrow \frac{1}{2}H_4$ ),  $3_{12}^* \rightarrow \frac{1}{2}H_7$ , both  $3_{16}^* \rightarrow \frac{1}{2}H_8$ , a  $3_{28}^* \rightarrow \frac{1}{2}H_{10}$ , a  $3_{36}^* \rightarrow \frac{1}{2}H_{11}$ ; last 5 are 4-capped  $\alpha_3, \beta_3, \beta_3, 5_{20}^*$  (on disjoint faces),  $5_{28}^*(T_d)$  = hexakis (truncated  $\alpha_3$ ), respectively.

Except  $\alpha_3$  and unique  $3_{12}$  = truncated  $\alpha_3$ , any  $3_n$  is a 4-vertex truncation of a simple polyhedron with  $\sum_{i \geq 3} p_i = p_4 + p_5 + p_6$  (so  $p_5 = 12 - p_4$  from the well-known equality  $\sum_{i \geq 3} (6 - i)p_i = 12$ ) and  $n - 8$  vertices (so  $n = 28 + 2p_6 - 2p_4$ ). In particular,  $3_n$  is 4-vertex truncation of a fullerene  $5_{n-8}$  iff this  $5_{n-8}$  has 4 vertices of type  $(5, 5, 5)$  at pairwise distance at least 3; so any pair of triangles is separated by more than one hexagon. Such  $5_{n-8}$  has either 4 isolated triples of pentagons, or two isolated clusters of 6 pentagons.  $3_n^* \rightarrow \frac{1}{2}H_m$  iff  $5_{n-8}^* \rightarrow \frac{1}{2}H_{m-4}$ .

For small values of  $n$  we have (see Fig. 4; marked 4 vertices indicate truncation):

1) exactly seven of  $3_n$  come as 4-vertex truncation of dual deltahedra: unique  $3_{12}$  from  $\alpha_3$ , both  $3_{16}$  (chamfered and twisted chamfered  $\alpha_3$ ) from  $\gamma_3$ , unique  $3_{20}$  from dual bisdisphenoid, both  $3_{24}$  from dual 2-capped  $APrism_4$  and one (of two)  $3_{28}$  from the (pentagonal) dodecahedron. The second one  $3_{28}$  comes from a simple dodecahedron with  $p = (p_4, p_5, p_6) = (4, 4, 4)$ .

2) exactly six  $3_n$  come as 4-vertex truncation of fullerenes  $5_m$ ,  $20 \leq m \leq 36$ : one  $3_{28}$  from  $5_{20}$ , a  $3_{32}$  from  $5_{24}$ , a  $3_{36}$  from  $5_{28}(D_2)$ , another  $3_{36}$  from  $5_{28}(T_d)$ , a  $3_{40}$  from  $5_{32}(D_2)$ , a  $3_{44}$  from  $5_{36}(D_2)$ ; each of these 6 fullerenes has a unique, up to a symmetry, set of 4 vertices at pairwise distance  $\geq 3$ .

The class of  $5_{n-8}$  with 4 isolated triples of pentagons contains all tetrahedral  $5_{n-8}$ ; those have (using [Gold37])  $n = 4(a^2 + ab + b^2)$  with  $a \geq b \geq 0$ ,  $a \geq 2$ . In particular,  $Cham_t\alpha_3$ ,  $t \geq 2$ , comes from  $5_{4t+1-8}(T_d)$ . (But already  $5_{56}^*(T_d)$  is not  $l_1$ ; so this  $3_{64}^*$  is not  $l_1$  implying that  $(Cham_t\alpha_3)^*$  is  $l_1$  iff  $t = 1$ .) But such fullerenes  $5_{n-8}$  exist also for  $n - 8 = 40, 44, 56$  (unique for each), 68. On the other hand, for  $n = 40, 44, 48, 52, 68$  there are  $5_{n-8}$  with two isolated groups of 6 pentagons (3 such fullerenes for  $n = 48$ , unique for others) such that a 4-vertex truncation of them is a  $3_n$  (see, for example, last 3 on Fig. 4).

## 8 Polyhedra $4_n$

Theorem 1 of [GrMo63] gives that  $4_n$  exists for any even  $n \geq 8$  except  $n = 10$ . [Good75] showed that all  $4_n$  are Hamiltonian; their duals are also Hamiltonian. Clearly,  $4_n$  is bipartite, and there is an infinity of centrally symmetric  $4_n$ . Hence it either  $\rightarrow H_m$  or is non 5-gonal. For  $4 \leq \frac{n}{2} \leq 22$ , there are  $N_4(n)$  polyhedra  $4_n$  ([Dill96]), where  $N_4(n)$  is given in the table below.

$\frac{n}{2}$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$N_4(n)$	1	0	1	1	1	1	3	1	3	3	3	2	8	3	7	7	7	5	14

Th.3.4 in [Sah94] implies that  $N_4(n) = O(n^3)$ . Unique  $l_1$ -polyhedron between  $4_n^*$  is  $4_8^* = \beta_3 \rightarrow \frac{1}{2}H_4$ , since other  $4_n^*$  contain the following non 5-gonal isometric subgraph  $G_7$ :  $Cyc_{1,\dots,6}$  with chords (1,3), (4,6) plus a new vertex connected with vertices 1,3,4,6. Clearly  $4_8 = \gamma_3$ ,  $4_{12} = Prism_6$ , truncated  $\beta_3$  is  $4_{24}$  (one of 3), two  $4_{32}$  are  $Cham\gamma_3$  and twisted  $Cham\gamma_3$  (see Fig. 5). Those 5 polyhedra are all known  $l_1$ - $4_n$ ; all 17 other  $4_n$ ,  $n < 32$ , are not  $l_1$ . They are embedded into  $H_3, H_4, H_6, H_7, H_7$ , respectively. The first 3 are Voronoi polyhedra,  $Cham\gamma_3$  is non space-filling zonohedron, twisted one is not centrally symmetric.

$Cham_t\gamma_3$  is  $l_1$  iff  $t = 1$ . For  $n \equiv 2 \pmod{6}$  the dual of column of  $\frac{n-2}{6}$  octahedra  $\beta_3 = APrism_3$  gives a  $4_n$ ; for  $n > 8$ , it and its dual are not 5-gonal. Examples of  $4_n$  without abutting pairs of 4-gons are  $Cham_t\gamma_3$ ,  $t \geq 1$ ) and duals of tetrakis cube, Cuboctahedron, triangular ortobicupola ("anticuboctahedron"), giroelongated triangular bicupola and snub cube, having, respectively  $8a^2$  ( $a \geq 2$ ), 24, 32, 32, 44, 56 vertices. Last two are non 5-gonal. "Dual tetrakis" above, means a truncation on six 4-valent vertices of the dual polyhedra.

On the other hand, any  $4_n$  with each pair of 4-gons separated by at least 3 edges (for example,  $Cham_t\gamma_3$ ,  $t \geq 2$ ) comes as 6-edge truncation (put 4-gons instead of edges) of a fullerene  $5_{n-12}$ . Hence such a  $4_n$  has no abutting pair of 4-gons and, moreover, the corresponding  $5_{n-12}$  has 6 pairs of pentagons separated by only one edge. Many  $4_n$  come as 6-edge truncation of  $5_n$  under weaker conditions:  $Cham\gamma_3$  from  $5_{20}$ , truncated  $\beta_3$  from  $\alpha_3$ . Also truncated  $\beta_3$  is 6-(disjoint) edges truncation of  $4_{12} = Prism_6$ ,  $Prism_6$  is

2-(disjoint) edges truncation of  $4_8 = \gamma_3$ ,  $\gamma_3$  is 2-(disjoint) edges truncation of  $\alpha_3$ . Suitable 6-(disjoint) edges truncation of  $5_{24}$  (dual 2-capped  $APrism_6$ ) gives a  $4_{36}$  and so on (see Fig. 5).

## 9 Polyhedra $5_n$

Theorem 1 of [GrMo63] gives that  $5_n$  exists for any even  $n \geq 20$  except  $n = 22$ . The polyhedra  $5_n$ , i.e. the bifaced polyhedra  $(3; 5, 6; 12, p_6)$ , are called *fullerenes* in Chemistry; see, for example, [Fowl93], [DDGr96] for a sample of vast literature on them. In fact, all  $5_n$  are the cases  $f = 2 + \frac{n}{2} \geq 12$  of *medial* polyhedra, introduced in [Gold35] as putative best (isoperimetrically) approximation of a sphere within the class of polyhedra having given number  $f$  of faces. A medial polyhedron with  $f$  faces is a bifaced polyhedron  $(3; a = \lfloor 6 - \frac{12}{f} \rfloor, b = a + 1; p_a, p_b = f - p_a)$ ; it exists for any  $f \geq 4$ , except  $f = 11, 13$ . For  $f = 4, \dots, 10, 12$  they are exactly 8 dual convex deltahedra:  $\alpha_3$ ,  $Prism_3$ ,  $\gamma_3$ ,  $Prism_5$ , dual bisdisphenoid, dual 3-augmented  $Prism_3$ , dual 2-capped  $APrism_4$  and the Dodecahedron (see the case  $k = 3$  of Table 1 above).

The number  $N_5(n)$  of polyhedra  $5_n$  is  $O(n^9)$  from Th.3.4 in [Sah94]. All known  $l_1$ - $5_n$  are the Dodecahedron  $5_{20} \rightarrow \frac{1}{2}H_{10}$ ,  $5_{26} \rightarrow \frac{1}{2}H_{12}$ ,  $5_{44}(T) \rightarrow \frac{1}{2}H_{16}$  and the chamfered Dodecahedron  $5_{80}(I_h) \rightarrow \frac{1}{2}H_{22}$ . The last one is the dual pentakis Icosidodecahedron; its twisted version is not  $l_1$ . All known  $l_1$ - $5_n^*$  are the Icosahedron  $5_{20}^* \rightarrow \frac{1}{2}H_6$ , hexakis (truncated  $\alpha_3$ ) =  $5_{28}(T_d) \rightarrow \frac{1}{2}H_7$ , hexakis( $APrism_6^2$ ) =  $5_{36}^*(D_{6h}) \rightarrow \frac{1}{2}H_8$  and pentakis Dodecahedron  $5_{60}^*(I_h) \rightarrow \frac{1}{2}H_{10}$ . In fact, no other  $l_1$ - $5_n$ ,  $l_1$ - $5_n^*$  exist for  $n < 60$  (see [DDGr96]) and they are not expected for other  $n$ . All 7 above fullerenes, such that it or its dual is  $l_1$ , have the highest symmetry for its  $n$  and all, except the dual of  $5_{28}(T_d)$  of diameter 3, have the corresponding embedding into the half- $2d$ -cube, where  $d$  is the diameter of the skeleton. The  $5_{28}(T_d)$  is also unique not antipodal among them. Many parameters of those 7 fullerenes, related to their distance matrix  $D$ , can be found in [Bala95]. In particular, the *distance polynomials*  $\det(xI - D)$  for Dodecahedron  $5_{20}$  and its chamfering  $5_{80}(I_h)$  are  $x^9(x+2)^4(x-50)(x^2+14x+4)^3$  and  $x^{63}(x+8)^4(x-440)(x^4+136x^3+2824x^2+16800x+1312)^3$ , respectively. All integer eigenvalues of  $D$  are exactly 1st, 2nd, 4th for both; they are (with multiplicities)  $50^1, 0^9, -2^4$  and  $440^1, 0^6, 3, -8^4$ , respectively. It hints to special properties of the spectrum of the distance matrix of chamfering; compare with preservaton of closed shell (in terms of the spectrum of the adjacency matrix) known (P. Fowler) for leapfrog of fullerenes.

Some interesting classes of  $5_n$  were mentioned above:  $5_{10(t+1)} = (2-APrism_5^t)^*$ ,  $5_{12(t+1)} = (2-APrism_6^t)^*$  and those (with 4 isolated triples of pentagons) coming from collapsing of 4 triangles in some  $3_n$ . Let us consider a generalization of the last 2 classes.

We say that a fullerene  $5_n$  is  $PR_i$  (*pentagon-regular of degree i*) if each of 12 pentagons is adjacent to exactly  $i$  other pentagons. Clearly, the dodecahedron  $5_{20}$  and the hexagonal barrel  $Barrel_6 = 5_{24}$  are unique  $PR_5$  and  $PR_4$ , respectively.  $RP_0$ 's are fullerenes with isolated pentagons; chemists call them *preferable* or IP fullerenes. The smallest ones among them are the  $5_{60}(I_h)$  and a  $5_{70}(D_{5h})$ .  $PR_1$ 's are those having 6 isolated pairs of abutting pentagons; all 130 of such  $5_n$  with  $n \leq 84$  are listed in [Fowl93], the 4 smallest ones are a  $5_{50}(D_3)$ , the  $5_{52}(T)$ , a  $5_{54}(C_2)$ , a  $5_{54}(D_3)$ . The only  $PR_3$  are the  $5_{28}(T_d)$  and

the  $5_{32}(D_{3h})$ . It will be interesting to characterize all fullerenes  $PR_2$ . Their 12 pentagons form isolated  $k$ -cycles ( $k = 3, 6, 9, 12$ ): four 3-cycles (including a  $5_{48}(D_2)$  and at least one tetrahedral  $5_n$  for any  $n = 4(a^2 + ab + b^2) - 8$ , starting with the  $5_{40}(T_d)$ , the  $5_{44}(T)$ , the  $5_{56}(T_d)$ , a  $5_{58}(T)$  and so on); two 3-cycles and one 6-cycle; one 3-cycle and one 9-cycle (including the  $5_{38}(C_{3v})$ ); two 6-cycles (including any  $(t - 2)$ -elongated hexagonal barrel  $5_{12t} = (2-APrism_6^{t-1})^*$  with  $t \geq 3$ , it has symmetry  $D_{6h}$  for odd  $t$  and  $D_{6d}$  for even  $t$ ); one 12-cycle. The number of vertices of type  $(5^3)$ ,  $(5^2.6)$ ,  $(5.6^2)$  for above 5 classes (by faces surrounding a vertex) are  $(4, 12, 24)$ ,  $(2, 18, 18)$ ,  $(1, 21, 15)$ ,  $(0, 24, 12)$ ,  $(0, 24, 12)$ , respectively. We not found  $PR_2$  fullerenes of second class; they are also absent among all 1812 fullerenes  $5_{60}$ . All  $PR_i$  with  $n \leq 50$  are 15 mentioned above: one for  $i = 5, 4, 1$ , two for  $i = 3$  and ten for  $i = 2$ . We find that in the third and in the fifth class there are exactly 5 fullerenes of Fig. 6.

**Proposition 9.1** *All fullerenes with 12-cycle of pentagons are those four given on Fig.6.*

In fact, let  $C_{12}$  and  $C'_{12}$  be the inner and the outer cycles of a belt of 12 adjacent squares. We have to set 12 new vertices on edges of  $C_{12}$  and  $C'_{12}$  such that the 12 squares of the ring are transformed into 12 pentagons. Let  $a$  and  $a' = 12 - a$  be the numbers of the new vertices on  $C_{12}$  and  $C'_{12}$ , respectively. Now we connect the  $a$  vertices of  $C_{12}$  by lines such that the inner domain of the ring is partitioned into hexagons. Similarly we partition the outer domain. The Euler relation implies that  $a = a' = 6$ . If  $k$  is the number of new vertices in the inner part (their degree is 3), then the number of hexagons is  $\frac{k}{2} + 4$ . Hence  $k$  is even. Consider a configuration of the six new vertices on  $C_{12}$ . Let  $q$  be the maximal number of vertices of  $C_{12}$  lying between two consecutive new vertices. Then  $q \leq 4$ , since two consecutive new vertices and  $q$  vertices of  $C_{12}$  between them belong to a hexagon. It is easy to see that  $q$  takes only 3 values: 4, 3 and 2. Not very complicated enumeration of configurations shows the following.

If  $q = 4$ , we have a  $5_{36}(D_{2d})$  with  $k = 0$  (one of two  $5_{36}$  with symmetry  $D_{2d}$ ).

If  $q = 3$ , we obtain two nonisomorphic fullerenes  $5_{44}$  with  $k = 4$  (with symmetry  $D_2$ ,  $D_{3d}$ ).

If  $q = 2$ , we have a homogeneous configuration of 6 new vertices on  $C_{12}$ . In this case  $k = 6$ , and we obtain a fullerene  $5_{48}(D_{6d})$ , (unique, except  $(2-APrism_6^3)^*$ ,  $5_{48}(D_{6d})$ ).

The partitions of the inner and outer domains into hexagons are similar.

**Proposition 9.2** *There is a unique fullerene with one 3-cycle and one 9-cycle of pentagons, namely the fullerene  $5_{38}(C_{3v})$ .*

**Proof.** Let  $C_9$  and  $C'_9$  be the inner and the outer cycles of a ring of 9 adjacent squares. We have to set 9 new vertices on edges of  $C_9$  and  $C'_9$  such that the 9 squares of the ring are transformed into 9 pentagons. Let  $a$  and  $a' = 9 - a$  be the numbers of the new vertices on  $C_9$  and  $C'_9$ , respectively. Let the 3-cycle of pentagons lies in the inner region of  $C_9$ . Hence the outer region of  $C'_9$  has no pentagons. We connect the  $a'$  vertices of  $C'_9$  by lines such that the outer region of  $C'_9$  is partitioned into hexagons. The Euler relation implies that  $a' = 3$ . Hence  $a = 9 - 3 = 6$ . The simple enumeration shows that the 3 new vertices of  $C'_9$  lie uniformly, i.e. there are exactly 3 vertices of  $C'_9$  between two consecutive new vertices. Besides, there is only one other vertex of degree 3. Hence the outer region of  $C'_9$  is partitioned into 3 hexagons.

Consider the inner region of  $C_9$ . We have to connect the 6 vertices of degree 2 of the

3-cycle of pentagons with the 6 new vertices of the cycle  $C_9$  by lines. It is easy to see that if there is one vertex of the 3-cycle and one new vertex of  $C_9$  connected by an edge, then the partition of the inner region of  $C_9$  into hexagons is unique. In this case each new vertex of  $C_9$  is connected by an edge with a vertex of degree 2 of the 3-cycle. The 6 new vertices of  $C_9$  lie uniformly, i.e. there are two and three vertices of  $C_9$  between a new vertex and its left and right neighbours. We obtain the fullerene  $5_{38}(C_{3v})$ .

If there is no pair of vertices of the 3-cycle and  $C_9$  connected by an edge, then the 3-cycle is circumscribed by a ring of 6 hexagons. The outer cycle  $C_{15}$ , including these 6 hexagons, contains 9 vertices of degree 2. It is not difficult to show that it is not possible to connect these 9 vertices of  $C_{15}$  with 6 vertices of  $C_9$  and obtain a partition into hexagons.

Similarly, define  $HR_j$  (*hexagon j-regular*) be any fullerene such that any hexagon is adjacent to exactly  $j$  hexagons. Then all  $HR_0$  are the  $5_{24}$ , the  $5_{26}$ , the  $5_{28}(T_d)$  and no  $HR_6$  exists. Examples of  $HR_1$  are the  $5_{28}(D_2)$ , the  $5_{32}(D_3)$ ; of  $HR_2$  - the  $5_{30}(D_{5h})$ , the  $5_{32}(D_{3d})$ , the  $5_{32}(D_2)$ , the  $5_{32}(D_{3h})$ ,  $a5_{36}(D_{2d})$ ; of  $HR_3$  - a  $5_{36}(D_2)$ , the  $5_{44}(T)$ , the  $5_{52}(T)$ , the  $5_{60}(I_h)$ ; of  $HR_4$  - a  $5_{40}(D_{5d}) = (2 - AP_{5}^3)^*$ , the  $5_{56}(T_d)$ , the  $5_{68}(T_d)$ , the  $5_{80}(I_h)$ , a  $5_{80}(D_{5h})$ ; of  $HR_5$  - the  $5_{140}(I)$ . For  $n \leq 36$  no other  $HR_j$  (than above ten) exists. The only possible  $(j, n)$  for preferable fullerenes  $HR_j$  are  $(3, 60), (4, 80), (5, 140)$ , since they have  $p_6 = \frac{60}{6-j}$ ; there are only above two for  $n = 80$ . Those four  $HR_j$  and  $PR_0$  fullerenes have (see Section 4.4 in [FoMa95] minimal *steric strain* among  $PR_0$  fullerenes.  $HR_j$  have (see page 79 in [FoMa95] maximal steric strain among fullerenes. Any  $5_n$  which is  $HR_j$  has exactly  $30 - \frac{p_6(6-j)}{2}$  edges separating two pentagons; so  $20 + \frac{10}{6-j} \leq n \leq 20 + \frac{120}{6-j} \leq 140$ .

The fullerenes  $5_n$  which are both  $PR_i$  and  $HR_j$  have  $n = 20 + 24\frac{(5-i)}{(6-j)}$ . Using it one can show that the only possible  $(i, j)$  are  $(5,0), (4,0), (3,0), (3,2), (2,2), (2,3), (1,3), (2,4), (0,3), (1,4), (0,4), (0,5)$  with  $n=20, 24, 28, 32, 38, 44, 52, 56, 60, 68, 80, 140$  respectively. Each of above 12 cases is realized by (unique with its symmetry) fullerene  $I_h, D_{6d}, T_d, D_{3h}, C_{3v}, T, T, T_d, I_h, T_d, I_h, I$  respectively. For  $n = 80$  there is exactly one other fullerene: a  $5_{80}(D_{5h})$ =twisted chamfered Dodecahedron. In other 11 cases (except unchecked  $n = 56, 68$ ) above fullerenes are unique. Among all 7 known  $5_n$ , such that either it, or its dual is  $\ell_1, 5_{26}$  is not  $PR_i$  and  $5_{36}(D_{6h})$  is not  $HR_j$ ; all others are both  $PR_i$  and  $HR_j$ .

Any  $3_n$ , except Tetrahedron, have isolated all 4 triangular faces. There are only 4  $3_n$  which are  $HR_j$ : unique  $3_{12}$  with  $j = 3$ , both  $3_{16}$  with  $j = 4$  and a  $3_{28}$ = 4-truncated Dodecahedron with  $j = 5$  (see Fig 4).

Any  $4_n$  which is  $HR_j$  has  $8 + 8/(6-j) \leq n \leq 8 + 48/(6-j)$ . So the  $4_{12}$  is unique  $HR_0$ , there is no a  $HR_1$ , only  $HR_2$  are unique  $4_{14}$  and  $4_{16}$ . Only  $HR_3$  are 2 (among all 3)  $4_{20}$  and 1 (among all 3)  $4_{24}$ . Remaining (the tetrahedral)  $4_{20}$  and at least 2 (among all 8)  $4_{32}$  (chamfered cube and its twist) are  $HR_4$ ; any other  $HR_4$  should have  $n = 26, 28, 30, 32$ . Any  $HR_5$  should have  $n = 26, \dots, 56$ ; an example is a  $4_{56}$  =dual tetrakis snub cube.

So all polyhedra  $3_n, 4_n, 5_n$  with isolated non-hexagons are:

- 1) if  $HR_3$ , then truncated, respectively, Tetrahedron, Octahedron, Icosahedron, i.e. the leapfrog of their duals,
- 2) if  $HR_4$ , then chamfering and its twist of, respectively, Tetrahedron, Cube and Dodecahedron,
- 3) if  $HR_5$ , then, respectively, dual 4-triakis snub tetrahedron, dual tetrakis snub cube

and dual pentakis snub dodecahedron.

In terms of generalized leapfrog (see, for example, [Sah94]) the leapfrog, the chamfering and the capping of snubs as in 3) above, correspond to the cases  $(a, b) = (1, 1), (2, 0)$  and  $(2, 1)$ , respectively. All polyhedra  $3_n, 4_n, 5_n$  such that they are  $HR_j$  with  $j = i$  and each non-hexagon is adjacent to exactly  $i$  others are, respectively, "would be"  $3_8$  with  $i = 1$ , the  $4_{14}$  with  $i = 2$  and the  $5_{38}(C_{3v})$  with  $i = 2$ . If the condition  $j = i$  above is replaced by  $p_6 =$  the number of non-hexagonal faces, then all such polyhedra are, respectively, the  $3_{12}$ , two  $4_{20}$  and the  $5_{44}(T)$  with  $(i, j, p_6) = (0, 3, 4), (2, 4, 6), (1, 3, 6), (2, 3, 12)$ .

Table 1 above gives all 6, other than  $5_n, 4_n$  and  $3_n$ , simple polyhedra with only  $a$ - and  $b$ -faces,  $3 \leq a < b \leq 6$ . All 6 polyhedra have  $i$   $a$ -faces adjacent to each  $a$ -face and  $j$   $b$ -faces adjacent to each  $b$ -face with  $(i, j) = (0, 2), (0, 4)$  and  $(2, 0), (1, 1), (0, 3), (0, 4)$  for  $Prism_3$ , Dürer's octahedron and 4 dual deltahedra (duals of 5-bipyramide, bisdisphenoid, 3-augmented  $Prism_3$ , 2- $APrism_4$ ), respectively. In general, any bifaced polyhedron  $(k; a, b; p_a, p_b)$  with  $j$   $b$ -neighbours for each  $b$ -face has  $p_b \leq p_a \frac{a}{b-j}$ . If, moreover, any  $a$ -face has  $i$   $a$ -neighbors, then  $p_b = p_a \frac{a-i}{b-j}$  (by double counting of  $a$ - $b$  edges). The classification of the face-regular bifaced polyhedra will be addressed in [DeGr97b].

The fullerenes  $5_n$  are the case:

1. "at least 20 vertices" of the medial polyhedra [Gold35],
2. " $a=5$ " of simple bifaced polyhedra with  $p = (p_a, p_6)$ ; see Sections 7,8 above,
3. " $b=6$ " of simple bifaced polyhedra with  $p = (p_5, p_b)$ . (For general  $b$  we have  $p_5 + p_b(6 - b) = 12$  and it has  $2\frac{b-5}{b-6}p_5 - \frac{4b}{b-6}$  vertices; see remark on fullerenes in the end of Section 6. For  $b = 3, 4$  it is the Dürer's octahedron,  $Prism_5$ , respectively.)

3-valent polyhedra with  $p = (p_4 = 2, p_5 = 8, p_6)$  with high symmetry were proposed in Chemistry (see [GaHe93]); some are chemically better (i.e. they have smaller number of pairs of adjacent pentagons) than fullerenes with the same number of vertices. 3-valent maps with  $p = (p_5, p_6, p_7)$  have  $p_5 = p_7 + 12(1 - g), p_6$  any. Many species of plancton (for example, famous Aulonia hexagona of E.Haeckel, 1887) have, as their rigid skeleton, such polyhedral maps. In the chemically important case  $p_5 = p_7$ , they are realizable on the torus (having the genus  $g = 1$ ) and they have  $4p_5 + 2p_6$  vertices. They are called *toroidal polyhexes* if  $p_5 = p_7 = 0$  and *azulenoids* if  $p_5 = p_7 > 0$ . In many applications 5- and 7-gons come by joined pairs, i.e. (in chemical terms) by *azulene* units. Toroidal polyhexes with  $p_6 = 3, 7, 8, 12$  are not 5-gonal, but for  $p_6 = 4$ , it realizes the cube  $H_3$ . Toroidal realizations of 3-valent maps with  $p = (p_5, p_6, p_8)$  also used in Chemistry; they have  $p_5 = 2p_8, p_6$  any and  $3p_5 + 2p_6$  vertices.

Fowler and co-workers (see [FMSZ97] and references there) considered chemical relevance of 3-valent polyhedra with  $p = (p_5, p_6, p_t)$  and  $t = 4, 7, 8$ . The smallest such polyhedra with  $p_t = 1$  have  $(t, n) = (3, 22), (4, 22), (7, 30), (7, 32), (7, 32), (8, 34), (8, 36), (8, 36)$ ; all of them and their duals are not  $l_1$ -embeddable.

3-valent maps with  $p = (p_6, p_7, p_8)$  have  $p_7 + 2p_8 = 12(g - 1), p_6$  any. They are realized on some minimal surface of negative curvature and are called *schwarzites*. [King96] gives 4 examples, with the genus  $g = 3$ , of chemical relevance having  $p = (p_7 = 24)$  (Klein map  $\{7^3\}$ ) on the D surface),  $p = (p_6 = 56, p_7 = 24)$  (the leapfrog of the previous one, it is an analog of  $5_{60}(I_h)$ ),  $p = (p_6 = 80, p_8 = 12)$  and  $p = (p_6 = 80, p_7 = 24)$ . They have 56, 168, 192, 216 vertices, respectively, and last two are realized on the the surface P.



## 10 Polyhedra $(3, 4)_n$

Those are 4-valent bifaced polyhedra with parameters  $(4; 3, 4; p_3, p_4)$ ; so  $p_3 = 8$  and  $n = 6 + p_4$ . Besides  $3_n, 4_n, 5_n$  it is the only case of bifaced polyhedra for which  $p_a$  is fixed for given  $(k, b)$ . [Grün67], p.282, gives the existence of  $(3, 4)_n$  for any  $n \geq 6$ , except 7.

From §6 above we have ambo  $(3, 4)_n = (3, 4)_{2n}$ . Another operation, namely, inserting a ring of  $m$  4-gons into some  $(3, 4)_n$ , produces a  $(3, 4)_{n+m}$ ; let us call it *m-elongation*. For example,  $2\text{-Prism}_4^t = (3, 4)_{4t+6} \rightarrow \frac{1}{2}H_{2t+4}$  is,  $m$  times iterated, 4-elongation of  $2\text{-Prism}_4$ . Already iterated 3-elongations of  $\beta_3$ , 4-elongations of  $A\text{Prism}_4$  and of  $2\text{-Prism}_4$  give  $(3, 4)_n$  for  $n = 6 + 3m, 8 + 4m, 10 + 4m$  for any  $m$ .

First examples of  $(3, 4)_n$  are:  $\beta_3 = (3, 4)_6 \rightarrow \frac{1}{2}H_4$ ,  $A\text{Prism}_4 = (3, 4)_8 \rightarrow \frac{1}{2}H_5$ , ambo  $\text{Prism}_3 = 3\text{-elongated } \beta_3 = (3, 4)_9 \rightarrow \frac{1}{2}H_6$ ,  $2\text{-Prism}_4 = (3, 4)_{10} \rightarrow \frac{1}{2}H_6$ . The six polyhedra on Fig.7 are not 5-gonal:  $(3, 4)_{11}$  and 4 polyhedra  $(3, 4)_{12}$ , (Cuboctahedron = ambo  $\beta_3 = 4\text{-elongated } A\text{Prism}_4$ , anticuboctahedron, another 2 times 3-elongation of  $\beta_3$  and another  $(3, 4)_{12}$  and ambo  $A\text{Prism}_4 = (3, 4)_{16}$ . Now, Rhombicuboctahedron  $(3, 4)_{24} \rightarrow \frac{1}{2}H_{10}$  but its twisted version ("14th Archimedean solid") is not 5-gonal; they are 8-elongations of twisted ambo  $A\text{Prism}_4$  which is a  $(3, 4)_{16} \rightarrow \frac{1}{2}H_8$ , and of ambo  $A\text{Prism}_4$  which is not 5-gonal. Exactly eleven  $(3, 4)_n$  are regular-faced: above 4,  $\beta_3, A\text{Prism}_4, 2\text{-Prism}_4$ , Cuboctahedron, anticuboctahedron and 6-elongations of last two, which are non 5-gonal  $(3, 4)_{18}$ .

Between the duals of above  $(3, 4)_n$ , all embeddable are three zonohedra:  $\beta_3^* = \gamma_3$ , (Cuboctahedron)\*  $\rightarrow H_4$  and  $(2\text{-Prism}_4^t)^* = \text{Prism}_4^{t+1} \rightarrow H_{t+3}$ . Apropos, (ambo  $\text{Prism}_3 = (3, 4)_9$ )\* is the smallest convex polyhedron with odd number of faces, all of which are quadrilaterals; it is not 5-gonal.

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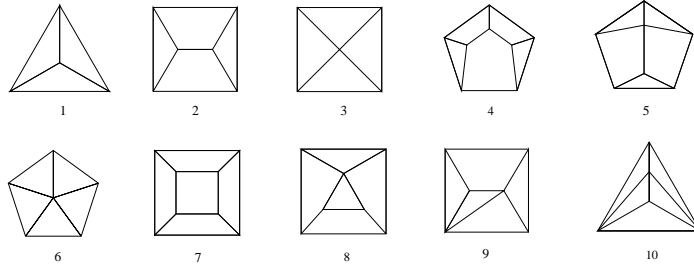


Figure 1: All polyhedra with at most six faces

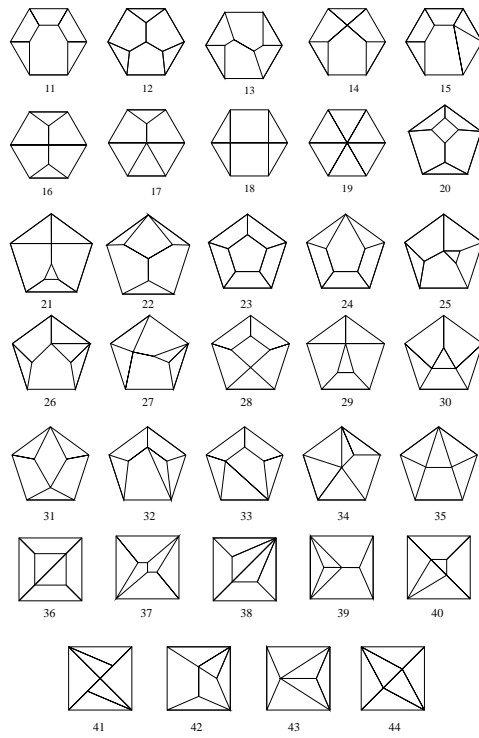


Figure 2: All polyhedra with seven faces

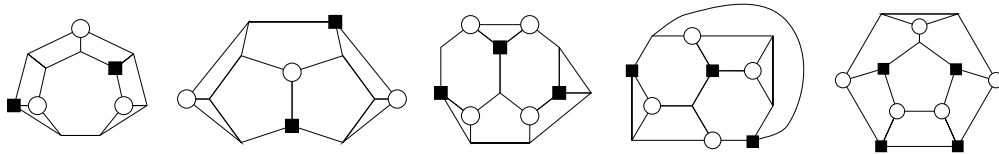


Figure 3: Some simple octahedra

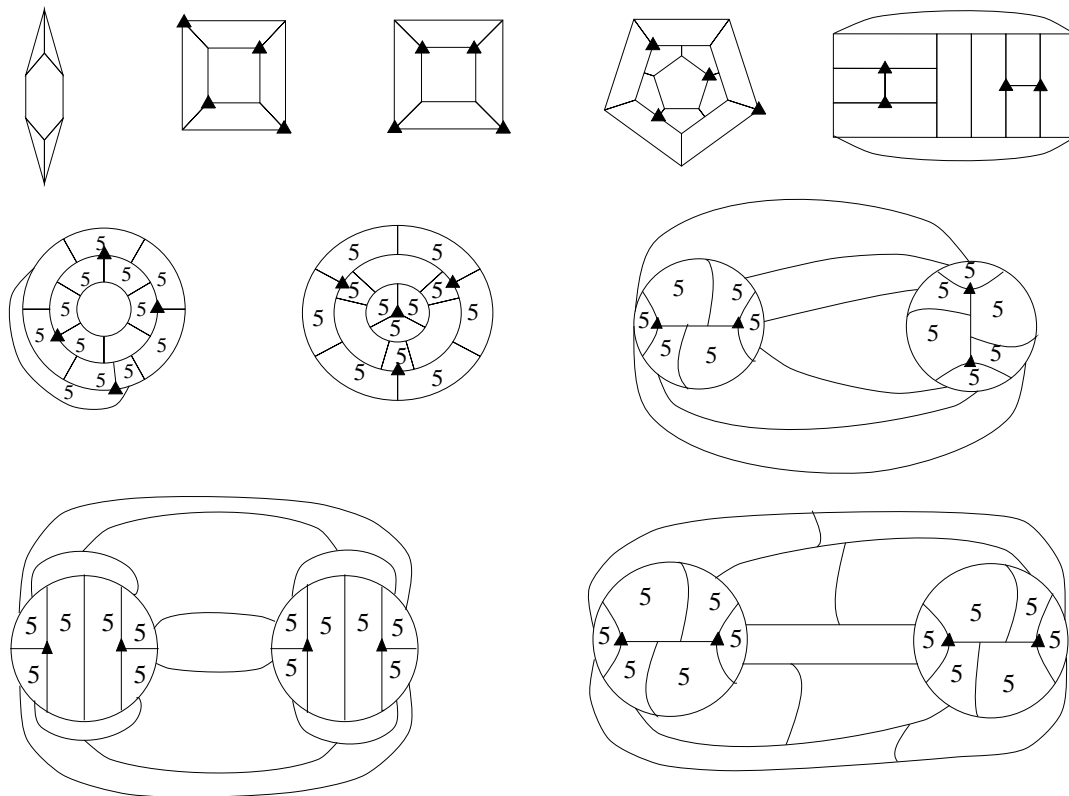


Figure 4: Some  $3_n$ : "would-be"  $3_8$ , both  $3_{16}$ , both  $3_{28}$ ; 4-truncations of both  $5_{28}$ , of the  $5_{32}(D_2)$ , of two  $5_{40}$

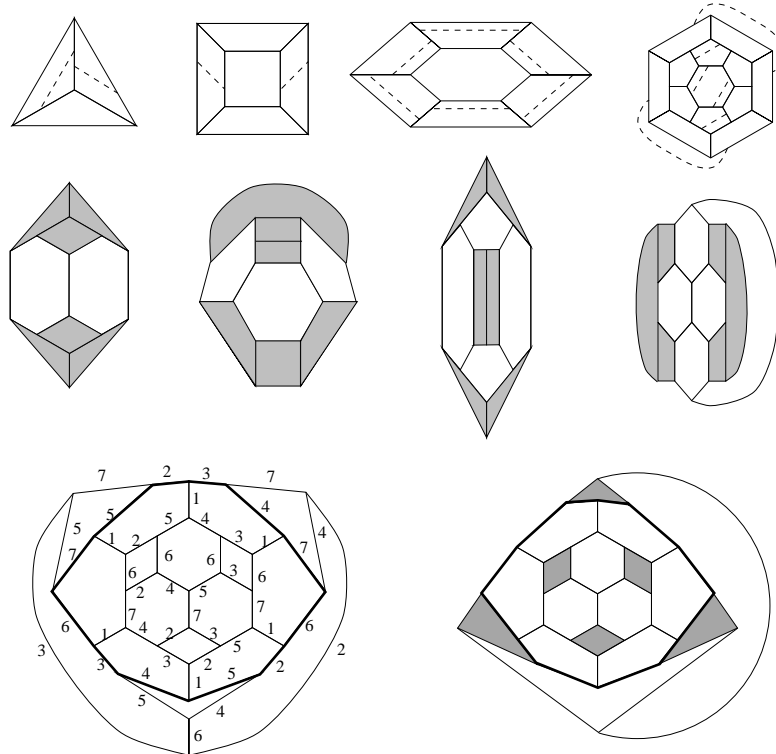


Figure 5: Some  $4_n$ :  $\gamma_3$ ,  $Prism_6$ , truncated  $\beta_3$ , a  $4_{36}$  (as edge-truncations); unique  $4_{14}$ ,  $4_{16}$ ,  $4_{18}$ ; a  $4_{20}$ , twisted Cham  $\gamma_3$ , Cham  $\gamma_3$

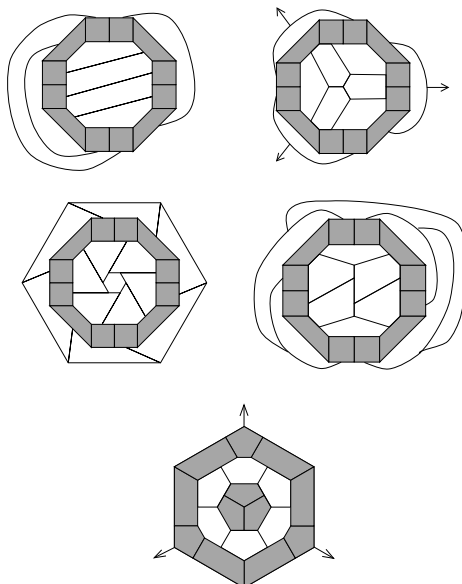


Figure 6: All pentagon-regular fullerenes with 12- or 9-belt of pentagons: a  $5_{36}(D_{2d})$ , a  $5_{44}(D_{3d})$ , a  $5_{48}(D_{6d})$ , a  $5_{44}(D_2)$ , the  $5_{38}(C_{3v})$

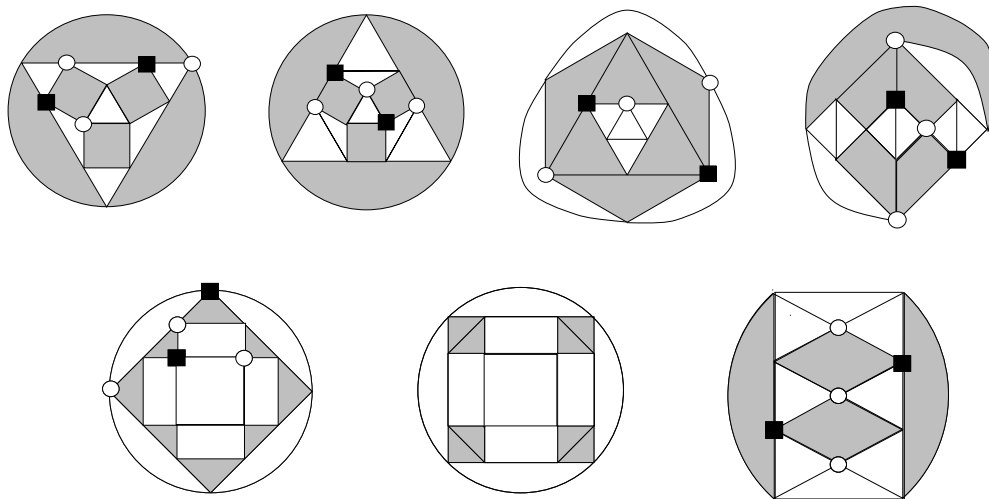


Figure 7: Some  $(4,3)_n$ : Cuboctahedron, its twist, two other  $(3,4)_{12}$ , ambo  $APrism_4$ , its twist, the  $(3,4)_{11}$