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Quasi-semi-metrics, Oriented Multi-cuts and Related Polyhedra

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We introduce polyhedral cones and polytopes, associated with quasi-semi-metrics (oriented distances), in particular with oriented multi-cuts, on *n* points. We compute generators and facets of these polyhedra for small values of *n* and study their graphs.

1. INTRODUCTION AND BASIC NOTIONS

The notions of directed distances, quasi-metrics and oriented multi-cuts are generalizations of the notions of distances, metrics and cuts, which are central objects in graph theory and combinatorial optimization.

Define a *quasi-semi-metric* on X as a function d from X^2 to R_+, such that for all $x, y, z \in X$, $d(x, y) \geq 0$, $d(x, x) = 0$, and $d(x, y) \leq d(x, z) + d(z, y)$. If the first inequality is strict for all $x \neq y$, then d is called a *quasi-metric*. If for all $x, y \in X$, $d(x, y) = d(y, x)$, then d is called a *semi-metric* and a *metric*, respectively.

Given a partition $(S_1, \ldots, S_q)$ $(q \geq 2)$ of $X = \{1, \ldots, n\}$, a quasi-semi-metric $\delta'(S_1, \ldots, S_q)$ is called an *oriented multi-cut* if $\delta'(S_1, \ldots, S_q)(i, j) = 1$ for $i \in S_\alpha$, $j \in S_\beta$, $\alpha < \beta$, and $\delta'(S_1, \ldots, S_q)(i, j) = 0$, otherwise.

After short review of general quasi-semi-metrics we consider, for small values of *n*, the cone and the polytope of all quasi-semi-metrics on $X = \{1, \ldots, n\}$ and the cone and the polytope generated by all oriented multi-cuts on X. We list the facets and generators for these polyhedra and tables of their adjacencies and incidences. We study the 1-skeleton graphs and the ridge graphs of these polyhedra: the number of nodes and edges of these graphs, their diameters, adjacency conditions, inclusions among these graphs and their restrictions on some orbits of nodes. Finally, we compare the results obtained for oriented case with similar results for the symmetric case (see [6–8, 10]). All computations were done using the programs *cdd* of [13].

The following notations will be used:

- the *oriented triangle* inequality $T_{ij,k} : x_{ik} + x_{kj} - x_{ij} \geq 0$;
- the *non-negativity* inequality $N_{ij} : x_{ij} \geq 0$;
- the *cone of o-multi-cuts* $OMCUT_n$, generated by all nonzero o-multi-cuts on *n* points;
- the *cone of quasi-semi-metrics* $QMET_n$, defined by all triangle and non-negativity inequalities on *n* points;
- the *o-multi-cut polytope* $OMCUT_n^{\square}$, generated by all o-multi-cuts (i.e., including zero multi-cut) on *n* points;
- the *quasi-semi-metric polytope* $QMET_n^{\square}$, defined by all triangle inequalities, all non-negativity inequalities and the inequalities $G_{ij} : x_{ij} + x_{ji} \leq 2$ on *n* points.

2. QUASI-SEMI-METRICS AND RELATED POLYHEDRA

Given a set X, a mapping $d : X \times X \longrightarrow \mathbb{R}$ is called a *distance* on X if d satisfies:

$$d(i, j) = d(j, i) \quad \text{for all } i, j \in X, \tag{1}$$

$$d(i, j) \geq 0 \quad \text{for all } i, j \in X, \tag{2}$$

$$d(i, i) = 0 \quad \text{for all } i \in X. \tag{3}$$

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If, in addition, d satisfies the triangle inequalities:

$$d(i,j) \le d(i,k) + d(k,j) \qquad \text{for all } i,j,k \in X, \tag{4}$$

then d is called a *semi-metric* on X. Moreover, if

$$d(i,j) = 0 \text{ holds only for } i = j, \tag{5}$$

then d is called a *metric* on X. (For the general theory of metrics see [3, 10].) If we exclude the symmetry condition (1), we obtain the definitions of *oriented distance*, *quasi-semi-metric* and *quasi-metric*, respectively. (X, d) is a *semi-metric, metric, quasi-semi-metric space* provided d has the corresponding property.

It is easy to see that (1) and (2) follow from (3) and (4′), where (4′) is given by

$$d(i,j) \le d(k,i) + d(k,j) \qquad \text{for all } i,j,k \in X. \tag{4′}$$

However, while for the symmetric cases (4) and (3) imply (2), in the oriented case they do not imply (2).

The notion of a semi-metric was first formalized in the classic paper by Frechet [12]. Asymmetric definitions of distance have already been used in [15, p. 145–146], but the first detailed topological analysis of quasi-metrics was given by Wilson [22]. The triangle inequality was first formalized as the central property of distances in [12] and later treated in Hausdorff [15]. The notion of a metric space was also formalized in [12], but the term 'metric' was first proposed in [15, p. 211].

It is known (see Proposition 8 in [17]), that any quasi-metric on n points embeds isometrically into $\mathbb{R}^n$ equipped with some directed norm.

Quasi-metrics are used in the semantics of computation (see, for example, [19]) and are of interest in computational geometry (see, for example, [1]).

Consider now a few examples of quasi-metrics.

EXAMPLE 1. Let (X, D) be a finite metric space and let $X = X_1 \cup \cdots \cup X_n$ be a decomposition of X into the union of pairwise disjoint sets. Then $d'(X_i, X_j) := \min_{x \in X_i} \max_{y \in X_j} D(x, y)$ (for $X_i \neq X_j$) and $d'(X_i, X_i) := 0$ is a quasi-metric on $Y := \{X_1, \ldots, X_n\}$ (compare with the Hausdorff-metric $d(X_i, X_j) = \max_{x \in X_i, y \in X_j} D(x, y)$ on Y).

EXAMPLE 2. If X is the set $\mathbb{R}$ of all real numbers, the mapping

$$d'(x,y) = \begin{cases} \min(1, y - x), & \text{if } x \le y, \\ 1, & \text{otherwise} \end{cases}$$

is a quasi-metric on $\mathbb{R}$ (compare with the ordinary metric $d(x, y) = |x - y|$). It is an example of a non-metrizable quasi-metric space, such that $d(x, y)$, for any fixed x, is a continuous function of y (see [21]).

EXAMPLE 3. For any anti-chain of sets $Z = \{x, y, z, \ldots | x \not\subset y \text{ for all } x \neq y\}$, the function $|x \triangle y|$ is a semi-metric (not a metric), as $|x \triangle y| = |y \triangle x|$, $|x \triangle y| \ge 0$, $|x \triangle x| = 0$, $|x \triangle z| - |x \triangle y| - |y \triangle z| \le -2(y \backslash (x \cup z)) - 2((x \cap z) \backslash y) \le 0$ (here $x \triangle y := (x \backslash y) \cup (y \backslash x)$ is the symmetric difference of sets x and y).

On the other hand, $|x \backslash y| \ge 0$, $|x \backslash x| = 0$, $|x \backslash z| - |x \backslash y| - |y \backslash z| = -|(x \cap z) \backslash y| - |y \backslash (x \cup z)| \le 0$ and the function $|x \backslash y|$ is a quasi-semi-metric (not a quasi-metric).

EXAMPLE 4. A graph $G = (V, E)$ is called *connected* if there is a u–v path between any two of its vertices u and v. The length of the shortest path from u to v in G is called the *distance* $d_G(u, v)$ between vertices u and v. Clearly, the function d_G is a metric on V (path-metric of graph G).

A directed graph $D = (V', E')$ is called *connected*, if there are both directed paths u–v and v–u between any two vertices $u, v \in V'$. The length of the shortest directed path from u to v in D is called the *directed distance* $d'_D(u, v)$ between vertices u and v. The function d'_D is a quasi-metric on V' (the path-metric of the directed graph D); see, for example, [4].

EXAMPLE 5 (CIRCULAR RAILROAD DISTANCE). Consider a circular railroad line, which moves only in a counter-clockwise direction around a circular track, represented by the unit circle $C_1 = \{x \in R^2 | d_e(0, x) = 1\}$ (see, for example, [20]). Let the distance $d_c(x, y)$ be the length of the counter-clockwise circular arc from x to y on C_1. It is easy to see that d_c is not symmetric ($d_c(x, y) + d_c(y, x) = 2\pi$), but it always satisfies (4) and so it is a quasi-metric.

Note that Examples 4 and 5 represent the much wider class of 'one-way path' distances, which commonly occur in practice. For example, the presence of one-way streets in a city produces exactly the same type of distances (such as shortest travel-time distance), which satisfy the triangle inequality, but fail to be symmetric.

Set $V_n := \{1, \dots, n\}$, $E_n := |\{ij | i, j \in V_n, i \neq j\}|$, where ij denotes the unordered pair of the integers i, j, and $I_n := |\{\langle i, j\rangle | i, j \in V_n, i \neq j\}|$, where $\langle i, j\rangle$ denotes the ordered pair of the integers i, j. Let d be a semi-metric on the set V_n. Due to the symmetry (1) and since $d(i, i) = 0$ for $i \in V_n$, we can view the semi-metric d as a vector $(d_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{E_n}$, where $E_n = \frac{n(n-1)}{2}$. In the same way, we can view a quasi-semi-metric d' on the set V_n as a vector $(d'_{ij})_{i \neq j} \in \mathbb{R}^{I_n}$, where $I_n = n(n-1)$. Hence, a semi-metric (a quasi-semi-metric) on V_n can be viewed alternatively as a function on $V_n \times V_n$ or as a vector in $\mathbb{R}^{E_n}$ (in $\mathbb{R}^{I_n}$). We will use both these representations. Moreover, we will use both symbols $d(i, j)$ ($d'(i, j)$) and d_{ij} (d'_{ij}) for the values of the semi-metric (quasi-semi-metric) between points i and j. Clearly, one can also view a semi-metric (a quasi-semi-metric) as an $n \times n$ matrix with $d_{ii} = 0$ on the main diagonal (and with $d_{ij} = d_{ji}$ in the first case).

Denote by MET_n the set of all semi-metrics on n points, and by $QMET_n$ the set of all quasi-semi-metrics on n points. MET_n is a full-dimensional cone in $\mathbb{R}^{E_n}$, defined by the $\frac{n(n-1)(n-2)}{2}$ triangle inequalities (4). $QMET_n$ is a full-dimensional cone in $\mathbb{R}^{I_n}$, defined by the $n(n-1)(n-2)$ triangle inequalities (4) and the $n(n-1)$ inequalities (2). (In the symmetric case, (2) follows from (4) and (3), see the remark above.)

Note, that without condition (2) we have in the cone $QMET_n$ the subspace of all mappings d^* satisfying $d^*(i, j) = -d^*(j, i)$ and $d^*(i, j) + d^*(j, n) = d^*(i, n)$ for all $1 \le i, j \le n$. The dimension of this subspace is $n(n-1) - \left(\binom{n}{2} + \binom{n-1}{2}\right) = n - 1$.

Denote by $MET_n^{\square}$ the polytope of all semi-metrics on n points, defined by the $\frac{n(n-1)(n-2)}{2}$ triangle inequalities (4) and by the $\frac{n(n-1)(n-2)}{6}$ inequalities

$$d(i, j) + d(i, k) + d(j, k) \le 2 \qquad \text{for all } x, y, z \in X \tag{4''}$$

(the non-homogeneous triangle inequalities, see [10, p. 421]).

Denote by $QMET_n^{\square}$ the polytope of all quasi-semi-metrics on n points, defined by the $n(n-1)(n-2)$ triangle inequalities (4) and by the $n(n-1)$ inequalities

$$d'(i, j) + d'(j, i) \le 2 \qquad \text{for all } x, y \in X \tag{4'''}$$

('oriented analogue' of non-homogeneous triangle inequalities, see Section 5).

3. Oriented Multi-cuts and Related Polyhedra

We start with the notion of a cut semi-metric. Given a subset S of $V_n = \{1, \dots, n\}$, let $\delta(S)$ denote the vector in R^{E_n}, defined by $\delta(S)_{ij} = 1$, if $|S \cap \{i, j\}| = 1$, and $\delta(S)_{ij} = 0$, otherwise, for $1 \le i < j \le n$. Obviously, $\delta(S)$ defines a semi-metric on V_n, and for this reason $\delta(S)$ is called a *cut semi-metric* (or a *cut vector*, or simply a *cut*).

In the same way, given a subset S of V_n, let $\delta'(S)$ denote the vector in R^{I_n}, defined by $\delta'(S)_{ij} = 1$, if $i \in S$, $j \notin S$, and $\delta'(S)_{ij} = 0$, otherwise, for $1 \le i \ne j \le n$. Clearly, $\delta'(S)$ defines a quasi-semi-metric on V_n, called an *oriented cut* (or *o-cut vector*, or *o-cut*).

Consider now the notion of a multi-cut semi-metric. Let $q \ge 2$ be an integer and let $S_1, \dots, S_q$ be a partition of V_n. Then the *multi-cut semi-metric* $\delta(S_1, \dots, S_q)$ is the vector in $\mathbb{R}^{E_n}$, defined by $\delta(S_1, \dots, S_q)_{ij} = 0$, if $i, j \in S_h$ for some h, $1 \le h \le q$, and $\delta(S_1, \dots, S_q)_{ij} = 1$, otherwise, for $1 \le i < j \le n$. In the same way, given an ordered partition $S_1, \dots, S_q$ of V_n, let $\delta'(S_1, \dots, S_q)$ denote the vector in $\mathbb{R}^{I_n}$, defined by $\delta'(S_1, \dots, S_q)_{ij} = 1$, if $i \in S_\alpha$, $j \in S_\beta$, where $\alpha < \beta$, and $\delta'(S_1, \dots, S_q)_{ij} = 0$, otherwise. It may be verified that $\delta'(S_1, \dots, S_q)$ defines a quasi-semi-metric on V_n, which is called an *oriented multi-cut* (or *o-multi-cut vector*, or *o-multi-cut*). (This notion was considered, for example, in [18].)

Note, that the number of all oriented cuts on n points is 2^n, and the number $p'(n)$ of all oriented multi-cuts on n points is the number of all ordered partitions of n. In fact, $p'(n) = \frac{1}{2}A_n(2)$, where $A_n(x)$ is the Euler's polynomial

$$A_n(x) = \sum_{\pi \in S_n} x^{1+d(\pi)},$$

with $d(\pi) := |\{i \le n | a_i > a_{i+1}\}|$ for the permutation

$$\pi := \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

Table 1.
The number of o-multi-cuts for small values of n.

n	3	4	5	6	7
$p'(n)$	13	75	541	4683	41338

Note, that the notion of a cut semi-metric is connected with the notion of symmetric difference of sets, and the notion of an o-cut quasi-semi-metric is connected with the notion of asymmetric difference. For example, the cut $\delta(\{1\}) = (1, 1, 0)$ can be defined by the symmetric difference of sets $\{1\}$, $\{\emptyset\}$, $\{\emptyset\}$: $\delta(\{1\}) = (|\{1\} \triangle \{\emptyset\}|, |\{1\} \triangle \{\emptyset\}|, |\{\emptyset\} \triangle \{\emptyset\}|)$, and o-cut $\delta'(\{1\}) = (1, 1, 0, 0, 0, 0)$ can be defined by the asymmetric difference of sets $\{1\}$, $\{\emptyset\}$, $\{\emptyset\}$: $\delta'(\{1\}) = (|\{1\} \setminus \{\emptyset\}|, |\{1\} \setminus \{\emptyset\}|, |\{\emptyset\} \setminus \{1\}|, |\{\emptyset\} \setminus \{\emptyset\}|, |\{\emptyset\} \setminus \{1\}|, |\{\emptyset\} \setminus \{\emptyset\}|)$.

The full-dimensional cone in $\mathbb{R}^{E_n}$, generated by all non-zero cut semi-metrics $\delta(S)$ for $S \subseteq V_n$, is called the *cut cone* and denoted by CUT_n. The full-dimensional cone in $\mathbb{R}^{E_n}$, generated by all non-zero multi-cut semi-metrics $\delta(S_1, \dots, S_q)$ on V_n, is called *the multi-cut cone* and denoted by $MCUT_n$. The polytope in $\mathbb{R}^{E_n}$, which is defined as the convex hull of all cut semi-metrics (multi-cut semi-metrics) on V_n, is called the *cut polytope* (*multi-cut polytope*) and is denoted by $CUT_n^\square$ ($MCUT_n^\square$).

In the same way, denote by $OCUT_n$ ($OMCUT_n$) the full-dimensional cone in $\mathbb{R}^{I_n}$, which is generated by all non-zero o-cut semi-metrics (o-multi-cut semi-metrics) on V_n. Denote by $OCUT_n^\square$ ($OMCUT_n^\square$) the polytope in $\mathbb{R}^{I_n}$, which is the convex hull of all o-cut semi-metrics (o-multi-cut semi-metrics) on V_n.

For example, $OCUT_3$ is the simplicial cone in $\mathbb{R}^6$ generated by the six oriented cuts $\delta'(\{1\}) = (1, 1, 0, 0, 0, 0)$, $\delta'(\{2\}) = (0, 0, 1, 1, 0, 0)$, $\delta'(\{3\}) = (0, 0, 0, 0, 1, 1)$, $\delta'(\{1, 2\}) = (0, 1, 0, 1, 0, 0)$, $\delta'(\{1, 3\}) = (1, 0, 0, 0, 0, 1)$ and $\delta'(\{2, 3\}) = (0, 0, 1, 0, 1, 0)$.

4. Facets, Extreme Rays, Vertices and their Orbits in Polyhedra

Let C be a polyhedral cone in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, the inequality $v^T x \leq 0$ is said to be *valid* for C if it holds for all $x \in C$. Then the set $\{x \in C | v^T x = 0\}$ is called the *face* of C, induced by the valid inequality $v^T x \leq 0$. A face of dimension $\dim(C) - 1$ is called a *facet* of C; a face of dimension 1 is called an *extreme ray* of C. Let P be a polytope in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$ and $v_0 \in \mathbb{R}$, the inequality $v^T x \leq v_0$ is said to be *valid* for P if it holds for all $x \in P$. Then the set $\{x \in P | v^T x = v_0\}$ is called a *face* of P, *induced* by the inequality $v^T x \leq v_0$. A face of dimension $\dim(P) - 1$ is called a *facet* of P. A face of dimension 1 (0) is called an *edge* (a *vertex*) of P.

Two vertices x, y of P are said to be *adjacent* on P if the segment $\{\alpha x + (1-\alpha) y | 0 \leq \alpha \leq 1\}$ is an edge of P. Two facets of P (or C) are said to be *adjacent* if their intersection has codimension 2.

The 1-*skeleton* graph of P (or C) is the graph G_P (or G_C) whose node set is the set of vertices of P (or extreme rays of C) and with an edge between two nodes if they are adjacent on P (or C). The *ridge graph* of P (or C) is the graph G^*_P (or G^*_C) whose node set is the set of facets of P (or C) and with an edge between two facets if they are adjacent on P (or C). Thus, the ridge graph of a polyhedron is the 1-skeleton of its dual.

An isometry $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a *symmetry* of P (or C), if it is an isometry, satisfying $f(P) = P$ (or $f(C) = C$). (*An isometry* of $\mathbb{R}^n$ is a linear mapping preserving the Euclidean distance.) Given a face F, the *orbit* $\Omega(F)$ of F consists of all faces that can be obtained from F by a symmetry.

Clearly, all faces of $CUT_n^\square$ and CUT_n are preserved by any permutation of the nodes. For a vector $v \in \mathbb{R}^{E_n}$ and a cut vector $\delta(A)$, let $v^{\delta(A)}$ be defined by $v_{ij}^{\delta(A)} = -v_{ij}$ if $\delta(A)_{ij} = 1$ and $v_{ij}^{\delta(A)} = v_{ij}$ if $\delta(A)_{ij} = 0$. Consider the mapping $r_{\delta(A)} : \mathbb{R}^{E_n} \longrightarrow \mathbb{R}^{E_n}$ defined by $r_{\delta(A)}(v) = v^{\delta(A)} + \delta(A)$. The mapping $r_{\delta(A)}$ is an affine bijection of the space $\mathbb{R}^{E_n}$, called a *switching* mapping. The facets of $CUT_n^\square$ are preserved under a switching operation too (see [10, pp. 403–409]): a consequence of the simple fact that the symmetric difference of two cuts is again a cut. Moreover, it is shown in [9] that for $n \neq 4$, switchings and permutations are the only symmetries of CUT_n and $CUT_n^\square$. For $n = 4$ there are some additional symmetries. It is shown in [16] that the semi-metric polytope $MET_n^\square$ has the same group of symmetries as $CUT_n^\square$; that is, $Is(MET_n^\square) = Is(CUT_n^\square)$.

In the oriented case all orbits of faces of quasi-semi-metric polyhedra on V_n are preserved under any permutation of the set $V_n = \{1, \ldots, n\}$, but a switching is not a symmetry of $OMCUT_n$ and $OMCUT_n^\square$, because the set of o-multi-cuts is not closed under the symmetric difference. However, the orbits of faces of $OMCUT_n$ are preserved under the so-called *reversal* operation. For an o-multi-cut $\delta'(S_1, \ldots, S_q)$ on V_n define the reversal of $\delta'(S_1, \ldots, S_q)$ as the o-multi-cut $\delta'(S_q, \ldots, S_1)$ (in the symmetric case the reversal of a multi-cut is the same multi-cut). We conjecture that the symmetry group of $OMCUT_n$ and $OMCUT_n^\square$ consists only of permutations and reversals, i.e., it is the group $Z_2 \times Sym(n)$ of signed permutations, and the symmetry group of $QMET_n$ and $QMET_n^\square$ is $Sym(n)$.

5. Some Connections Between Semi-metric and Quasi-semi-metric Polyhedra

As every cut is a multi-cut and as every multi-cut is a semi-metric, we have

$$CUT_n \subseteq MCUT_n \subseteq MET_n \subseteq \mathbb{R}_+^{E_n}.$$

In the same way, an oriented cut is an oriented multi-cut and every oriented multi-cut is a quasi-semi-metric; so, we have

$$OCUT_n \subseteq OMCUT_n \subseteq QMET_n \subseteq \mathbb{R}_+^{I_n}.$$

It is easy to check that $\delta(S_1, \dots, S_q) = \frac{1}{2}\sum_{1 \le i \le q} \delta(S_i)$. Thus, $MCUT_n = CUT_n$. A similar property fails for oriented multi-cuts. For example, $\delta'(\{1, 2, 3\}) \notin OCUT_3$. Hence, $OCUT_n \subset OMCUT_n$ (strictly) for any $n \geq 3$.

Among the facets of $CUT_n^\square$ the most simple ones are the triangle facets, i.e., those defined by the triangle inequalities (4) and (4$''$). Hence,

$$CUT_n^\square \subseteq MCUT_n^\square \subseteq MET_n^\square \subseteq [0, 1]^{E_n}.$$

Among the facets of $OMCUT_n^\square$ the most simple ones are the triangle facets, induced by inequalities (4), and the facets, induced by inequalities (4$'''$). Hence,

$$OCUT_n^\square \subseteq OMCUT_n^\square \subseteq QMET_n^\square \subseteq [0, 1]^{I_n}.$$

Compare now some semi-metric and quasi-semi-metric polyhedra on n points for small n. The triangle inequalities are sufficient for describing the cut polyhedra for $n \leq 4$, but $CUT_n \subset MET_n$ and $CUT_n^\square \subset MET_n^\square$ (strictly) for $n \geq 5$. The complete description of all the facets of the cut polyhedra CUT_n and $CUT_n^\square$ is known for $n \leq 8$, the complete description of the semi-metric polyhedra MET_n and $MET_n^\square$ is known for $n \leq 7$ (see, for example, the linear description of MET_7 in [14]). Here the 'combinatorial explosion' starts from $n = 8$ (for example, CUT_8 has 49 604 520 facets).

In the oriented case, $OCUT_3 \subset QMET_3$ and $OCUT_3^\square \subset QMET_3^\square$, while $OMCUT_n = QMET_n$ and $OMCUT_n^\square = QMET_n^\square$ for $n = 3$ only. We computed all facets, extreme rays (vertices) and their adjacencies and incidences of $OMCUT_n$ ($OMCUT_n^\square$) and $QMET_n$ ($QMET_n^\square$) for $n = 3, 4$ only. In fact, the 'combinatorial explosion' starts in the oriented case from $n = 5$ (for instance, $QMET_5$ has 43 590 extreme rays). The amount of computation and memory is much bigger in the oriented case, because the quasi-semi-metrics are not symmetric (so, the dimension of the quasi-semi-metric polyhedra is doubled) and the o-multi-cuts do not lie in the cone of o-cuts.

6. The Case of Three Points

We present here the complete linear description for the case $n = 3$.

Clearly, $OCUT_3 \subset QMET_3$ strictly. However, for $n = 3$ the triangle inequalities (4) with the non-negativity inequalities (2) describe $OMCUT_3$.

There are 12 non-zero o-multi-cuts, including six o-cuts (see Table 2 below) on V_3, which form two orbits: the orbit O_1 of o-cuts and the orbit O_2 of other o-multi-cuts.

Note, that all o-cuts above can be obtained from $\delta'(\{1\})$ by a permutation ($\delta'(\{2\})$ and $\delta'(\{3\})$) or by a reversal and a permutation ($\delta'(\{1, 2\})$, $\delta'(\{1, 3\})$ and $\delta'(\{2, 3\})$); all o-multi-cuts above can be obtained from $\delta'(\{1\}, \{2\}, \{3\})$ by some permutation.

TABLE 2.
Non-zero o-multi-cuts on three points.

o-multi-cut	$(v_{12}, v_{13}, v_{21}, v_{23}, v_{31}, v_{32})$	Orbit number
$\delta'(\{1\})$	(1, 1, 0, 0, 0, 0)	O_1
$\delta'(\{2\})$	(0, 0, 1, 1, 0, 0)	O_1
$\delta'(\{3\})$	(0, 0, 0, 0, 1, 1)	O_1
$\delta'(\{1, 2\})$	(0, 1, 0, 1, 0, 0)	O_1
$\delta'(\{1, 3\})$	(1, 0, 0, 0, 0, 1)	O_1
$\delta'(\{2, 3\})$	(0, 1, 0, 0, 1, 0)	O_1
$\delta'(\{1\}, \{2\}, \{3\})$	(1, 1, 0, 1, 0, 0)	O_2
$\delta'(\{1\}, \{3\}, \{2\})$	(1, 1, 0, 0, 0, 1)	O_2
$\delta'(\{2\}, \{1\}, \{3\})$	(0, 1, 1, 1, 0, 0)	O_2
$\delta'(\{2\}, \{3\}, \{1\})$	(0, 0, 1, 1, 1, 0)	O_2
$\delta'(\{3\}, \{1\}, \{2\})$	(1, 0, 0, 0, 1, 1)	O_2
$\delta'(\{3\}, \{2\}, \{1\})$	(0, 0, 1, 0, 1, 1)	O_2

TABLE 3.
The adjacencies of facets in $OMCUT_3$.

Orbit	Representative	F_1	F_2	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	3	5	8	6
F_2	N_{12}	5	2	7	6

The only facet-defining inequalities of $OMCUT_3$ are the six triangle inequalities

$$T_{ij,k} : x_{ij} - x_{ik} - x_{kj} \leq 0$$

and six non-negativity inequalities

$$N_{ij} : x_{ij} \geq 0,$$

which form two orbits F_1 and F_2, respectively. (Reversals coincide here with some permutations.)

Adjacencies of facets (extreme rays) of $OMCUT_3$ are shown in Tables 3 and 4. For each orbit a representative and a number of adjacent ones from other orbits are given, as well as the total number of adjacent ones and the cardinality of orbits.

The 1-skeleton graph G_{OMCUT_3} has 12 nodes and $45 = \frac{1}{2}(9 \times 6 + 6 \times 6)$ edges. Figure 1 shows the complement $\bar{G}_{OMCUT_3}$ of this graph (here $a_1 := \delta'(\{2, 3\})$, $a_2 := \delta'(\{1, 3\})$, $a_3 := \delta'(\{1, 2\})$, $a_1^* := \delta'(\{1\})$, $a_2^* := \delta'(\{2\})$, $a_3^* := \delta'(\{3\})$, $b_1 := \delta'(\{2\}, \{3\}, \{1\})$, $b_2 := \delta'(\{1\}, \{3\}, \{2\})$, $b_3 := \delta'(\{1\}, \{2\}, \{3\})$, $b_1^* := \delta'(\{3\}, \{2\}, \{1\})$, $b_2^* := \delta'(\{3\}, \{1\}, \{2\})$, and $b_3^* := \delta'(\{2\}, \{1\}, \{3\})$).

As any two nodes of G_{OMCUT_3} have at least three common neighbors, we obtain

PROPOSITION 1. *The diameter of the* 1*-skeleton graph* G_{OMCUT_3} *is* 2.

The graph $G^*_{OMCUT_3}$ also has 12 nodes and 45 edges. Figure 2 shows its complement.

As any two nodes of $G^*_{OMCUT_3}$ have at least three common neighbors, we obtain

PROPOSITION 2. *The diameter of the ridge graph* $G^*_{OMCUT_3}$ *is* 2.

It is easy to see that, in $G^*_{OMCUT_3}$, a triangle facet is adjacent to some other facet if and only if they are non-conflicting. Two vectors from $\{0, 1, -1\}^n$ are said to be *conflicting* if there exists a pair ij such that the two vectors have non-zero coordinates of distinct signs at the position ij. More exactly, we obtain the following result.

TABLE 4.
The adjacencies of extreme rays in $OMCUT_3$.

Orbit	Representative	O_1	O_2	Total adjacency	$\|O_i\|$
O_1	$\delta'\{1\}$	5	4	9	6
O_2	$\delta'(\{1\},\{2\},\{3\})$	4	2	6	6

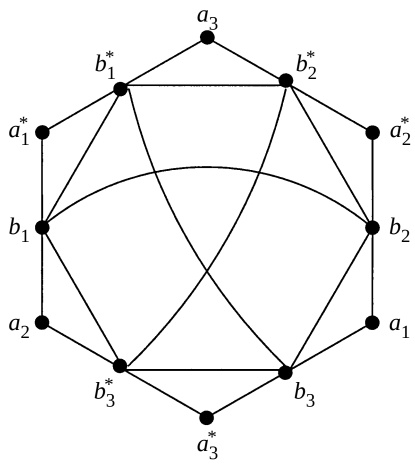

FIGURE 1. Graph $\bar{G}_{OMCUT_3}$.

PROPOSITION 3. *For the ridge graph $G^*_{QMET_3}$ it holds that:*

(i) the triangle facet $T_{ij,k}$ is adjacent to a facet if and only if they are non-conflicting;
(ii) the non-negativity facet N_{ij} is adjacent also to the facets N_{im}, N_{kj} ($m \neq j, k \neq i$).

The incidences of facets and extreme rays for $OMCUT_3$ are shown in Table 5. Namely, for each orbit F_i we give the number of extreme rays from orbits O_j, which belong to a representative of F_i and the total number of extreme rays, which belong to it. Note, that, in general, the table of incidences of extreme rays and facets can be obtained from the table of incidences of facets and extreme rays by the formulae

$$F_{ij} \times |O_j| = O_{ji} \times |F_i|, \tag{6}$$

where $|O_j|$ and $|F_i|$ are the orbit sizes, F_{ij} is the number of elements from orbit O_j, which are incident to a representative of F_i, and O_{ji} is the number of elements from orbit F_i, to which is incident a representative from O_j.

Consider now the polytope $OMCUT_3^{\square}$—the convex hull of all 13 o-multi-cuts on V_3.

This polytope has 13 vertices, which form three orbits (the orbit O_1 of o-cuts, the orbit O_2 of other o-multi-cuts and the new orbit O_1^p, consisting of only $\delta'(\emptyset)$). $OMCUT_3^{\square}$ has 15 facets: six facets of type $T_{ij,k}$ (orbit F_1), six facets of type N_{ij} (orbit F_2) and three new facets (orbit F_1^p), which are induced by the inequalities

$$G_{ij} : x_{ij} + x_{ji} \leq 2.$$

As the cone $OMCUT_3$ coincides with the cone $QMET_3$, we define $QMET_3^{\square}$ by all inequalities of types $T_{ij,k}$ (triangle inequalities), N_{ij} (non-negativity inequalities) and G_{ij} (oriented analogue of non-homogeneous triangle inequalities). Hence, $OMCUT_3^{\square} = QMET_3^{\square}$.

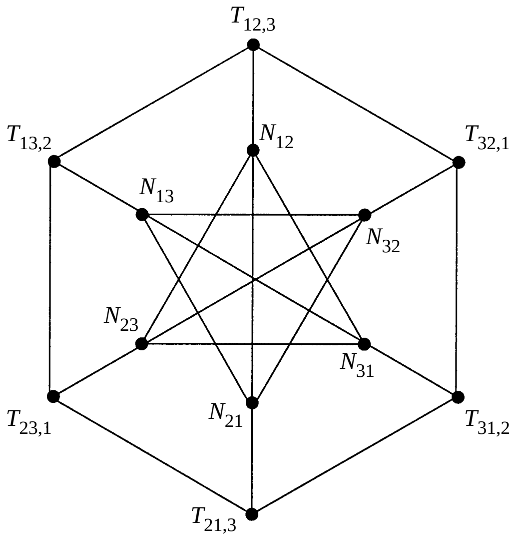

FIGURE 2. Graph $\bar{G}^*_{OMCUT_3}$.

TABLE 5.
The incidences of facets and extreme rays in $OMCUT_3$.

Orbit	O_1	O_2	Total incidence
F_1	4	3	7
F_2	4	3	7

Connections between facets and vertices of the o-multi-cut polytope $OMCUT_3^\square$ are shown in Tables 6–8, which are constructed in the same way as Tables 3–5.

$G_{OMCUT_3^\square}$ has 13 nodes and 57 edges. Figure 3 shows the complement of it. Here the points $a_1, \ldots, b_3^*$ are the same as in Figure 1, and $a_0 := \delta'(\emptyset)$.

Since any two nodes of $G_{OMCUT_3^\square}$ have $\delta'(\emptyset)$ as a common neighbor, we obtain

PROPOSITION 4. *The diameter of the* 1*-skeleton graph* $G_{OMCUT_3^\square}$ *is* 2.

$G^*_{OMCUT_3^\square}$ has 15 nodes and 72 edges; Figure 4 shows the complement of it.

As any two nodes of $G^*_{OMCUT_3^\square}$ have at least three common neighbors, we obtain

PROPOSITION 5. *The diameter of ridge graph* $G^*_{OMCUT_3^\square}$ *is* 2.

It is easy to check the following proposition.

PROPOSITION 6. *For the ridge graph* $G^*_{QMET_3^\square}$ *it holds that:*

(i) the triangle facet $T_{ij,k}$ is adjacent to a facet if and only if they are non-conflicting;
(ii) the facet N_{ij} is adjacent also to N_{im}, N_{kj} ($m \neq j$, $k \neq i$) and to all G_{mk}; and
(iii) the facet G_{ij} is adjacent also to all non-triangle facets.

It turns out that the 1-skeleton graph G_{OMCUT_3} and the ridge graph $G^*_{OMCUT_3}$ are induced subgraphs of $G_{OMCUT_3^\square}$ and $G^*_{OMCUT_3^\square}$, respectively.

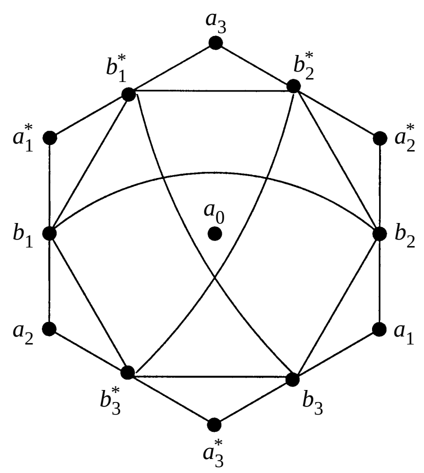

FIGURE 3. Graph $\bar{G}_{OMCUT_3^\square}$.

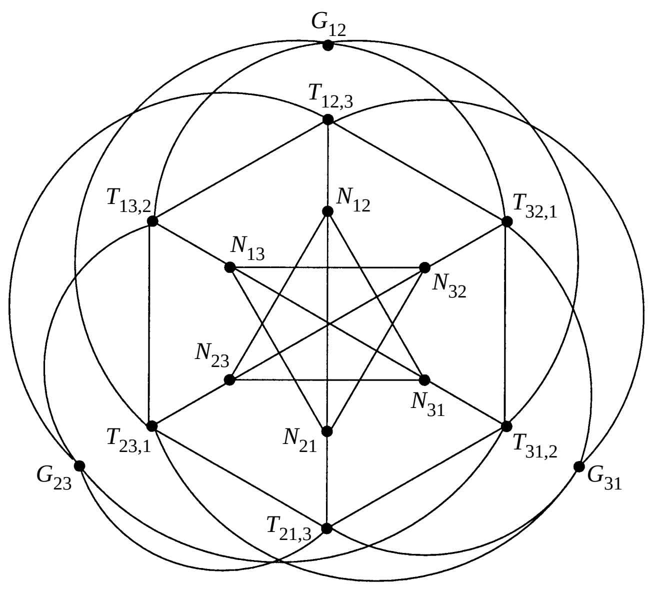

FIGURE 4. Graph $\bar{G}^*_{OMCUT_3^\square}$.

TABLE 6.
The adjacencies of facets in $OMCUT_3^{\square}$.

Orbit	Representative	F_1	F_2	F_1^p	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	3	5	1	9	6
F_2	N_{12}	5	2	3	10	6
F_1^p	G_{12}	2	6	2	10	3

TABLE 7.
The adjacencies of vertices in $OMCUT_3^{\square}$.

Orbit	Representative	O_1	O_2	O_1^p	Total adjacency	$\|O_i\|$
O_1	$\delta'(\{1\})$	5	4	1	10	6
O_2	$\delta'(\{1\},\{2\},\{3\})$	4	2	1	7	6
O_1^p	$\delta'(\emptyset)$	6	6	0	12	1

Recall, that in the symmetric case $CUT_3 = MET_3$ ($CUT_3^{\square} = MET_3^{\square}$) and, hence, the only facet-defining inequalities for CUT_3 and $CUT_3^{\square}$ are the triangle inequalities, three inequalities (from one orbit, obtained by permutations) for CUT_3 and four inequalities (from one orbit, obtained by permutations and switchings) for $CUT_3^{\square}$.

7. THE CASE OF FOUR POINTS

We present here the complete linear description of $OMCUT_4$, $QMET_4$, $OMCUT_4^{\square}$ and $QMET_4^{\square}$.

$OMCUT_4$ has 74 extreme rays (all non-zero o-multi-cuts on V_4), which form five orbits with the representatives $\delta'(\{1\})$ (orbit O_1), $\delta'(\{1,2\})$ (orbit O_2), $\delta'(\{1\},\{2\},\{3,4\})$ (orbit O_3), $\delta'(\{1\},\{2,3\},\{4\})$ (orbit O_4) and $\delta'(\{1\},\{2\},\{3\},\{4\})$ (orbit O_5). $OMCUT_4$ has 72 facets from four orbits, which are induced by 24 triangle inequalities (orbit F_1)

$$T_{ij,k} : x_{ij} - x_{ik} - x_{kj} \leq 0,$$

12 non-negativity inequalities (orbit F_2)

$$N_{ij} : x_{ij} \geq 0,$$

12 inequalities (orbit F_3)

$$L_{i,j,k,m} : x_{ij} + x_{ji} + x_{km} \leq x_{im} + x_{jm} + x_{ki} + x_{kj} + x_{mk}$$

and 24 inequalities (orbit F_4)

$$Q_{i,j,k,m} : x_{ij} + x_{ji} + x_{km} \leq x_{ik} + x_{im} + x_{jk} + x_{jm} + x_{ki} + x_{kj},$$
$$Q^r_{i,j,k,m} : x_{ij} + x_{ji} + x_{mk} \leq x_{ik} + x_{jk} + x_{ki} + x_{kj} + x_{mi} + x_{mj}.$$

Note, that for the orbits F_1, F_2, F_3, but not for F_4, the reversal operation coincides with some permutation; the reversal of $Q_{i,j,k,m}$ is $Q^r_{i,j,k,m}$.

Tables 9–11 show connections between the facets and the extreme rays of $OMCUT_4$.

The 1-skeleton graph G_{OMCUT_4} of $OMCUT_4$ has 74 nodes and 1479 edges. As 14 o-cuts (orbits O_1 and O_2 together) form a dominating clique, we obtain

PROPOSITION 7. *The diameter of the* 1*-skeleton graph* G_{OMCUT_4} *is* 2 *or* 3.

TABLE 8.
The incidences of facets and vertices in $OMCUT_3^{\square}$.

Orbit	O_1	O_2	O_1^p	Total incidence
F_1	4	3	1	8
F_2	4	3	1	8
F_1^p	4	6	0	10

TABLE 9.
The adjacencies of facets in $OMCUT_4$.

Orbit	Representative	F_1	F_2	F_3	F_4	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	17	11	5	8	41	24
F_2	N_{12}	22	6	12	8	48	12
F_3	$L_{1,2,3,4}$	10	12	0	2	24	12
F_4	$Q_{1,2,3,4}$	8	4	1	3	16	24

The ridge graph $G^*_{OMCUT_4}$ has 72 nodes and 1404 edges.

The quasi-semi-metric cone $QMET_4$ has 36 facets, distributed in two orbits: 24 triangle facets (orbit F_1) and 12 non-negativity facets (orbit F_2). There are 164 extreme rays in $QMET_4$, which form 10 orbits: orbits O_1–O_5 with the same representatives as in $OMCUT_4$ and five other orbits with the representatives

$v_6(\{1\}, \{2\}, \{3\}, \{4\}) = (1, 1, 2, 0, 1, 1, 0, 0, 1, 0, 0, 0)$ (orbit O_6),
$v_7(\{1\}, \{2\}, \{3\}, \{4\}) = (1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1)$ (orbit O_7),
$v_8(\{1\}, \{2\}, \{3\}, \{4\}) = (1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0)$ (orbit O_8),
$v_9(\{1\}, \{2\}, \{3\}, \{4\}) = (1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1)$ (orbit O_9),
$v_{10}(\{1\}, \{2\}, \{3\}, \{4\}) = (1, 1, 2, 1, 1, 1, 0, 0, 1, 0, 0, 1)$ (orbit O_{10}).

The adjacencies and incidences of the facets and extreme rays of $QMET_4$ are given in Tables 12–14.

The 1-skeleton graph G_{QMET_4} has 164 nodes and 2647 edges. As 14 o-cuts (orbits O_1 and O_2 together) form a dominating clique, and as there are two representatives from orbit O_{10}, which do not have common neighbors, we obtain

PROPOSITION 8. *The diameter of the* 1*-skeleton graph* G_{QMET_4} *is* 3*.*

Note, that the complement of the graph of neighbors for a representative of O_{10} is $4K_1 + P_3 + P_2 + K_{1,3}$.

The ridge graph $G^*_{QMET_4}$ has 36 nodes and 504 edges. One can check (case by case)

PROPOSITION 9. *For the ridge graph* $G^*_{QMET_4}$ *it holds that:*

(i) *the triangle facet* $T_{ij,k}$ *is adjacent to a facet if and only if they are non-conflicting;*
(ii) *the non-negativity facet* N_{ij} *is adjacent also to the facets* N_{im}, N_{kj}, N_{km} *(*$m \neq j$, $k \neq i$*); and*
(iii) *the diameter of this graph is 2.*

Note, that in $QMET_4$ the adjacencies of facets $T_{ij,k}$ and N_{ij} are the same as in $OMCUT_4$ (see Tables 9 and 12); hence, $G^*_{QMET_4}$ is an induced subgraph of $G^*_{OMCUT_4}$. However, the adjacencies of o-multi-cuts from orbits O_3, O_4 and O_5 are decreased in the cone $QMET_4$ (see Tables 10 and 13); hence, G_{OMCUT_4} is not an induced subgraph of G_{QMET_4}.

TABLE 10.
The adjacencies of extreme rays in $OMCUT_4$.

Orbit	Representative	O_1	O_2	O_3	O_4	O_5	Total adjacency	$\lvert O_i \rvert$
O_1	$\delta'\{1\}$	7	6	21	9	18	61	8
O_2	$\delta'(\{1,2\})$	8	5	20	12	8	53	6
O_3	$\delta'(\{1\},\{2\},\{3,4\})$	7	5	15	7	10	44	24
O_4	$\delta'(\{1\},\{2,3\},\{4\})$	6	6	14	6	8	40	12
O_5	$\delta'(\{1\},\{2\},\{3\},\{4\})$	6	2	10	4	12	34	24

TABLE 11.
The incidences of facets and extreme rays in $OMCUT_4$.

Orbit	O_1	O_2	O_3	O_4	O_5	Total incidence
F_1	6	4	14	7	12	43
F_2	6	4	14	7	12	43
F_3	4	4	8	4	8	28
F_4	3	4	4	4	2	17

Consider now $OMCUT_4^{\square}$—the convex hull of all 75 o-multi-cuts on V_4 (with $\delta'(\emptyset)$). It has 75 vertices, which belong to six orbits (orbits O_1–O_5 with the same representatives as in $OMCUT_4$ and new orbit O_1^p, which has only $\delta'(\emptyset)$). $OMCUT_4^{\square}$ has 106 facets from seven orbits: 72 facets from the orbits F_1–F_4, induced by the inequalities of the types $T_{ij,k}$, N_{ij}, $L_{i,j,k,m}$ and $Q_{i,j,k,m}$, respectively, six facets (orbit F_1^p), induced by the inequalities

$$G_{ij} : x_{ij} + x_{ji} \le 2,$$

four facets (orbit F_2^p), induced by the inequalities

$$M_{i,j,k,m} : x_{ik} + x_{im} + x_{ki} + x_{km} + x_{mi} + x_{mk} - x_{ij} - x_{ji} - x_{jk} - x_{jm} - x_{kj} - x_{mj} \le 1,$$

and 24 facets (orbit F_3^p), induced by the inequalities

$$R_{i,j,k,m} : x_{jk} + x_{jm} + x_{km} + x_{mk} - x_{ik} - x_{im} - x_{ji} - x_{ki} - x_{mi} \le 1.$$

Tables 15–17 give connections between the facets and the vertices of this polytope.

The 1-skeleton graph $G_{OMCUT_4^{\square}}$ has 75 nodes and 1604 edges. As $\delta'(\emptyset)$ is adjacent to all other vertices, we obtain

PROPOSITION 10. *The diameter of the* 1*-skeleton graph* $G_{OMCUT_4^{\square}}$ *is* 2.

The ridge graph of $G^*_{OMCUT_4^{\square}}$ has 72 nodes and 1683 edges.

It turns out that $G^*_{OMCUT_4}$ is an induced subgraph of $G^*_{OMCUT_4^{\square}}$ (see Tables 9 and 15), and $G^*_{QMET_4}$ is the induced subgraph of $G^*_{OMCUT_4^{\square}}$ (see Tables 12 and 18), but G_{OMCUT_4} is not an induced subgraph of $G_{OMCUT_4^{\square}}$ (see Tables 10 and 16).

Similarly to $QMET_3^{\square}$, we define $QMET_4^{\square}$ by the inequalities of the types $T_{ij,k}$, N_{ij} and G_{ij}. Hence, this polytope has 42 facets from three orbits: 24 triangle facets (orbit F_1), 12 non-negativity facets (orbit F_2) and six facets, induced by inequalities G_{ij} (orbit F_3). $QMET_4^{\square}$ has 221 vertices, which belong to 14 orbits: 10 orbits O_1–O_{10} with the same representatives as in $QMET_4$, orbit O_1^p (see $OMCUT_4^{\square}$) and three new orbits O_2^p–O_4^p with the representatives $v_2^p(\{1\},\{2\},\{3\},\{4\}) = (1,1,1,1,2,2,0,0,1,0,0,1)$ (orbit O_2^p), $v_3^p(\{1\},\{2\},\{3\},\{4\}) = (1,1,2,1,1,2,0,0,1,0,0,0)$ (orbit O_3^p) and $v_4^p(\{1\},\{2\},\{3\},\{4\})$

TABLE 12.
The adjacencies of facets in $QMET_4$.

Orbit	Representative	F_1	F_2	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	17	11	28	24
F_2	N_{12}	22	6	28	12

TABLE 13.
The adjacencies of extreme rays in $QMET_4$.

Or.	Representative	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	Ad.	$\|O_i\|$
O_1	$\delta'(\{1\})$	7	6	21	9	18	6	9	6	3	6	91	8
O_2	$\delta'(\{1,2\})$	8	5	20	12	8	12	16	4	4	8	97	6
O_3	$\delta'(\{1\},\{2\},\{3,4\})$	7	5	7	5	10	4	4	2	0	2	46	24
O_4	$\delta'(\{1\},\{2,3\},\{4\})$	6	6	10	2	8	4	4	0	2	4	46	12
O_5	$\delta'(\{1\},\{2\},\{3\},\{4\})$	6	2	10	4	3	4	2	1	0	1	33	24
O_6	$v_6(\{1\},\{2\},\{3\},\{4\})$	2	3	4	2	4	0	2	1	0	0	18	24
O_7	$v_7(\{1\},\{2\},\{3\},\{4\})$	3	4	4	2	2	2	0	1	1	2	21	24
O_8	$v_8(\{1\},\{2\},\{3\},\{4\})$	4	2	4	0	2	2	2	0	0	0	16	12
O_9	$v_9(\{1\},\{2\},\{3\},\{4\})$	4	4	0	4	0	0	4	0	0	4	20	6
O_{10}	$v_{10}(\{1\},\{2\},\{3\},\{4\})$	2	2	2	2	1	0	2	0	1	0	12	24

$= (1,1,1,0,1,1,0,1,1,0,1,1)$ (orbit O_4^p). Connections of the facets and the vertices of $QMET_4^\square$ are given in Tables 18–20.

The 1-skeleton graph $G_{QMET_4^\square}$ has 221 nodes and 3534 edges. As the orbits O_1 and O_2 together form a dominating clique, and as there are two representatives from the orbit O_4^p, which have no common neighbors, we obtain

PROPOSITION 11. *The diameter of the* 1*-skeleton graph* $G_{QMET_4^\square}$ *is* 3.

The ridge graph $G^*_{QMET_4^\square}$ has 80 nodes and 686 edges. It is easy to check the following proposition.

PROPOSITION 12. *For the ridge graph* $G^*_{QMET_4^\square}$ *it holds that:*

(i) the triangle facet $T_{ij,k}$ is adjacent to a facet if and only if they are non-conflicting;
(ii) the facet N_{ij} is adjacent also to N_{im}, N_{kj}, N_{km} ($m \neq j, k \neq i$) and to all G_{mk}; and
(iii) the facet G_{ij} is adjacent also to all non-triangle facets.

Figures 5 and 6 show some subgraphs of $G_{QMET_4^\square}$. The restriction of $G_{QMET_4^\square}$ on the union of orbits O_1 and O_4^p consists of two disjoint cube graphs; see Figure 6: here the black (or white) points are the elements from O_1 (or O_4^p).

Note, that in $QMET_4^\square$ the adjacencies of facets $T_{ij,k}$ and N_{ij} are the same as in $QMET_4$ (see Tables 12 and 18) and the ridge graph $G^*_{QMET_4}$ is an induced subgraph of $G^*_{QMET_4^\square}$. Similarly (see Tables 15 and 18), the graph $G^*_{QMET_4^\square}$ is an induced subgraph of $G^*_{OMCUT_4^\square}$. $G_{OMCUT_4^\square}$ is the induced subgraph of $G_{QMET_4^\square}$ (see Tables 16 and 19), but G_{OMCUT_4} is not an induced subgraph of $G_{QMET_4^\square}$ (see Tables 10 and 19).

Recall, that in the symmetric case $CUT_4 = MET_4$ and $CUT_4^\square = MET_4^\square$. The only facet-defining inequalities for CUT_4 and $CUT_4^\square$ are the triangle inequalities, 12 inequalities (from one orbit, obtained by permutations) for CUT_4 and 16 inequalities (from one orbit, obtained by permutations and switchings) for $CUT_4^\square$.

TABLE 14.
The incidences of extreme rays and facets in $QMET_4$.

Orbit	F_1	F_2	Total incidence
O_1	18	9	27
O_2	16	8	24
O_3	14	7	21
O_4	14	7	21
O_5	12	6	18
O_6	10	6	16
O_7	10	5	15
O_8	10	5	15
O_9	8	4	12
O_{10}	8	4	12

TABLE 15.
The adjacencies of facets in $OMCUT_4^\square$.

Orbit	Representative	F_1	F_2	F_3	F_4	F_1^p	F_2^p	F_3^p	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	17	11	5	8	4	1	4	50	24
F_2	N_{12}	22	6	12	8	6	4	8	66	12
F_3	$L_{1,2,3,4}$	10	12	0	2	1	0	0	25	12
F_4	$Q_{1,2,3,4}$	8	4	1	3	4	0	1	21	24
F_1^p	G_{12}	16	12	2	6	5	2	12	65	6
F_2^p	$M_{1,2,3,4}$	6	12	0	0	3	0	6	27	4
F_3^p	$R_{1,2,3,4}$	4	4	0	1	3	1	0	13	24

TABLE 16.
The adjacencies of vertices in $OMCUT_4^\square$.

Orbit	Representative	O_1	O_2	O_3	O_4	O_5	O_1^p	Total adjacency	$\|O_i\|$
O_1	$\delta'(\{1\})$	7	6	21	9	18	1	62	8
O_2	$\delta'(\{1,2\})$	8	5	20	12	8	1	54	6
O_3	$\delta'(\{1\},\{2\},\{3,4\})$	7	5	16	8	10	1	47	24
O_4	$\delta'(\{1\},\{2,3\},\{4\})$	6	6	16	6	10	1	45	12
O_5	$\delta'(\{1\},\{2\},\{3\},\{4\})$	6	2	10	5	12	1	36	24
O_1^p	$\delta'(\emptyset)$	8	6	24	12	24	0	74	1

TABLE 17.
The incidences of facets and extreme rays in $OMCUT_4^\square$.

Orbit	O_1	O_2	O_3	O_4	O_5	O_1^p	Total incidence
F_1	6	4	14	7	12	1	34
F_2	6	4	14	7	12	1	34
F_3	4	4	8	4	8	1	29
F_4	3	4	4	4	2	1	18
F_1^p	4	2	4	2	0	1	13
F_2^p	6	0	12	6	0	0	24
F_3^p	1	0	8	1	3	0	13

TABLE 18.
The adjacencies of facets in $QMET_4^\square$.

Orbit	Representative	F_1	F_2	F_1^p	Total adjacency	$\|F_i\|$
F_1	$T_{12,3}$	17	11	4	32	24
F_2	N_{12}	22	6	6	34	12
F_1^p	G_{12}	16	12	5	33	6

TABLE 19.
The adjacencies of vertices in $QMET_4^{\square}$.

Or.	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	O_1^p	O_2^p	O_3^p	O_4^p	Ad.	$\|O_i\|$
O_1	7	6	21	9	18	6	6	6	3	6	1	9	9	3	110	8
O_2	8	5	20	12	8	12	16	4	4	8	1	0	0	0	98	6
O_3	7	5	16	8	10	4	4	2	0	2	1	4	6	2	71	24
O_4	6	6	16	6	10	4	4	0	2	6	1	4	4	0	69	12
O_5	6	2	10	5	12	4	2	1	0	1	1	2	0	0	46	24
O_6	2	3	4	2	4	0	2	1	0	0	1	0	0	0	19	24
O_7	2	4	4	2	2	2	0	1	1	2	1	1	0	0	22	24
O_8	4	2	4	0	2	2	2	0	0	0	1	0	0	0	17	12
O_9	4	4	0	4	0	0	4	0	0	4	1	0	0	0	21	6
O_{10}	2	2	2	3	1	0	2	0	1	0	1	2	0	0	16	24
O_1^p	8	6	24	12	24	24	24	12	6	24	0	0	0	0	164	1
O_2^p	3	0	4	2	2	0	1	0	0	2	0	0	1	0	15	24
O_3^p	3	0	6	2	0	0	0	0	0	0	0	1	2	1	15	24
O_4^p	3	0	6	0	0	0	0	0	0	0	0	0	3	0	12	8

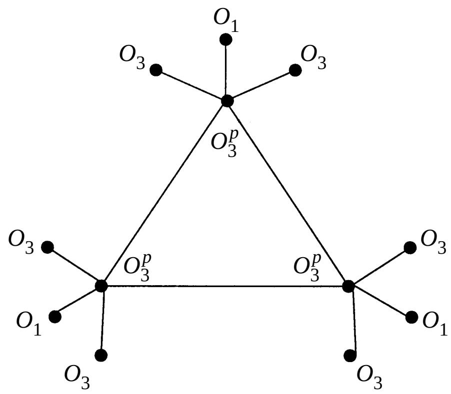

FIGURE 5. The complement of the graph of neighbors for a representative of O_4^p.

8. THE CASE OF FIVE POINTS

$QMET_5$ has 80 facets from two orbits (60 triangle facets and 20 non-negativity facets) and 43 590 extreme rays from more than 58 orbits. For instance, together with 540 o-multi-cuts (10 orbits), there are orbits with the representatives

(0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 2, 0, 0, 0, 0) (the adjacency 404, the incidence 41),
(0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 1) (the adjacency 58, the incidence 30),
(0, 0, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1, 1, 0, 0, 1, 1, 1, 1) (the adjacency 20, the incidence 20).

$QMET_5^{\square}$ has 90 facets (three orbits) and 79391 vertices and more than 113 orbits of extreme rays.

$OMCUT_5$ has 540 generators (all non-zero o-multi-cuts on V_5), which form 10 orbits, and more than 1200000 facets. $OMCUT_5^{\square}$ has 541 vertices (11 orbits) and more than 128 orbits of facets. Besides the extensions of all facets of $OMCUT_4$, the cone $OMCUT_5$ has, for example, three facets A, B, C, given below. Here $S_{a\ldots x}$ denotes the sum of distances along the oriented cycle $a \ldots x$.

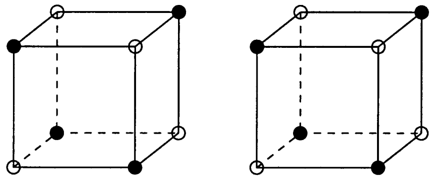

FIGURE 6. The restriction of $G_{QMET_4^{\square}}$ on $O_1 \cup O_4^p$.

TABLE 20.
The incidences of vertices and facets in $QMET_4^{\square}$.

Orbit	F_1	F_2	F_1^p	Total incidence
O_1	18	9	3	30
O_2	16	8	4	28
O_3	14	7	5	26
O_4	14	7	5	26
O_5	12	6	6	24
O_6	10	6	1	17
O_7	10	5	1	16
O_8	10	5	1	16
O_9	8	4	2	14
O_{10}	8	4	3	15
O_1^p	24	12	0	36
O_2^p	6	4	4	14
O_3^p	6	5	3	14
O_4^p	6	3	3	12

$$A : S_{axb} + S_{ayb} + S_{azb} \geq S_{ab} + S_{xyz},$$
$$B : S_{abx} + S_{zay} + S_{zby} = S_{xayzabyzb} \geq S_{ab} + (S_{xz} + S_{yz}),$$
$$C : S_{axb} + S_{ayx} + S_{byz} + S_{az} \geq S_{ab} + (S_{xyz} + S_{xzy}).$$

An example of a quasi-metric violating inequality A is provided by the oriented graph on $\{a, b, x, y, z\}$, having ai, ib, ba, xy, zy (for $i = x, y, z$) as the set of arcs. If in A we replace all (oriented) cycles by non-oriented ones, then it will became a *pentagonal* inequality; 10 pentagonal inequalities, together with 30 triangle inequalities, give all facets of the cone CUT_5.

Recall, that for the symmetric case $CUT_5 \subset MET_5$ and $CUT_5^{\square} \subset MET_5^{\square}$. While CUT_5 has 40 facets from two orbits, the cone $CUT_5^{\square}$ has 56 facets from two orbits. The extreme rays of MET_5 and the vertices of $MET_5^{\square}$ are also known; namely, besides the cut vectors, all of them arise by a switching of the vector $(2/3, \dots, 2/3)$. So, MET_5 has 25 extreme rays from three orbits and $MET_5^{\square}$ has 32 vertices from two orbits.

The complete linear description of the semi-metric polyhedra is known for $n \leq 7$. The cut cone CUT_6 (respectively, $CUT_6^{\square}$) has 210 facets from four orbits (respectively, 368 facets from three orbits); CUT_7 (respectively, $CUT_7^{\square}$) has 38 780 facets from 36 orbits (respectively, 116 764 facets from 11 orbits); CUT_8 (respectively, $CUT_8^{\square}$) has 49 604 520 facets (respectively, 217 093 472 facets from 147 orbits) (see [5]). The semi-metric cone MET_6 (respectively, $MET_6^{\square}$) has 296 extreme rays from seven orbits (respectively, 544 vertices from three orbits); MET_7 (respectively, $MET_7^{\square}$)has 55 226 extreme rays from 41 orbits (respectively, 275 840 vertices from 13 orbits).

See [2] and [11] for a study of similar small cones and polyhedra.

9. Conjectures for General n

Four conjectures about the graphs of quasi-semi-metric polyhedra on n points, proposed below, have been verified for $n \leq 4$; we also computer-checked Conjectures 3 and 4 for $n = 5$.

CONJECTURE 1. The ridge graphs $G^*_{QMET_n}$ and $G^*_{QMET_n^\square}$ are induced subgraphs of $G^*_{OMCUT_n}$ and $G^*_{OMCUT_n^\square}$, respectively.

CONJECTURE 2. For $OMCUT_n$, $QMET_n$, $OMCUT_n^\square$ and $QMET_n^\square$ it holds that:

— every o-cut is adjacent to all other o-cuts;
— every extreme ray (vertex) is adjacent to some o-cut;
— the diameter of the 1-skeleton graph is equal to 2 or 3.

CONJECTURE 3. For the ridge graph $G^*_{QMET_n}$ it holds that:

(i) a triangle facet $T_{ij,k}$ is adjacent to a facet if and only if they are non-conflicting; and
(ii) a non-negativity facet N_{ij} is adjacent to $N_{i'j'}$ if and only if neither $i' = j$ nor $j' = i$.

CONJECTURE 4. For the ridge graph $G^*_{QMET_n^\square}$ it holds that:

(i) a triangle facet $T_{ij,k}$ is adjacent to some facet if and only if they are non-conflicting;
(ii) a non-negativity facet N_{ij} is adjacent to $N_{i'j'}$ if and only if neither $i' = j$ nor $j' = i$; it is adjacent also to all facets G_{mk}; and
(iii) a facet G_{ij} is adjacent also to all non-triangle facets.

The above conjecture implies that if we take a triangle facet $T_{ij,k}$ in $QMET_n^\square$, then the ‘conflicting’ graph (the graph of ‘non-neighbors’) of $T_{ij,k}$ has $4(n-2)+1$ nodes (facets N_{ij}, G_{jk}, G_{ik}, $T_{ik,j}$, $T_{kj,i}$, $T_{il,j}$, $T_{ik,l}$, $T_{li,j}$, $T_{kj,l}$, where $l \neq i, j, k$) and $2(n-2)$ edges (between facets G_{jk} and $T_{ik,j}$; G_{ik} and $T_{kj,i}$, $T_{il,j}$ and $T_{ik,l}$; $T_{lj,i}$ and $T_{kj,l}$). Thus, it is the graph $2(n-2)K_2 + K_1$.

Conjectures 3 and 4 would imply that the ridge graphs of $QMET_n$ and $QMET_n^\square$ have diameter 2. Let us see it for $QMET_n$. The ‘non-conflicting’ graph, restricted on the orbit of all triangle facets, has diameter 2. This follows (case by case check) from the fact that the complement of its local graph is $2K_1 + (2n-6)K_2$. For example, $T_{ij,k}$ conflicts with $4n$–10 facets: $T_{ik,j}$; $T_{kj,i}$; $T_{ik,l}$; $T_{il,j}$; $T_{kj,l}$; $T_{lj,i}$ for all l different of i, j, k. All other possible pairs of non-adjacent facets are $(T_{ij,k}, N_{ij})$, (N_{ij}, N_{ji}), (N_{ij}, N_{ki}), (N_{ij}, N_{jk}). Examples of common neighbors for those pairs are, respectively, N_{ik}, $T_{ik,j}$, N_{kj}, N_{ik}.

Remind, that in the symmetric case all triangle inequalities are facet-inducing in CUT_n and in $CUT_n^\square$ for any $n \geq 3$; the cut vectors form a single switching class, which is a clique in the 1-skeleton graph of $MET_n^\square$ (on the other hand, it is shown in Laurent [16] that every other switching class is a stable set in the 1-skeleton graph of $MET_n^\square$, that is, no two non-integral switching equivalent vertices of $MET_n^\square$ form a edge on $MET_n^\square$); two triangle inequalities are adjacent in the ridge graph of $MET_n^\square$ if and only if they are non-conflicting (see [6]).

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