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THE HYPERMETRIC CONE IS POLYHEDRAL

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The hypermetric cone H_n is the cone in the space $R^{n(n-1)/2}$ of all vectors $d=(d_{ij})_{1\le i<j\le n}$ satisfying the hypermetric inequalities: $\sum_{1\le i<j\le n} z_j z_j d_{ij} \le 0$ for all integer vectors z in Z^n with $\sum_{1\le i\le n} z_i = 1$. We explore connections of the hypermetric cone with quadratic forms and the geometry of numbers (empty spheres and L-polytopes in lattices). As an application, we show that the hypermetric cone H_n is polyhedral.

1. Introduction

Let X be a finite set with $|X| = n+1$, $X = \{0,1,2,\dots n\}$. A *metric* on X is a real valued function d defined on all pairs of points of X and satisfying the *triangle inequality*:

$$(1) \qquad d_{ij} + d_{jk} \ge d_{ik}$$

for all triples (i,j,k) of points of X. We allow $d_{ij}=0$ for some pairs (i,j); so we use the term metric for denoting what is usually called semi-metric. We set $d_{ij} = d_{ji}$ for all pairs (i,j) and $d_{ii}=0$ for all points i of X. The pair (X,d) is called a *metric space*. The family of all metrics d on X forms a cone $M(X) = M_{n+1}$, called *metric cone*. The metric cone lies in the positive orthant of the space $R^{n(n+1)/2}$; indeed, summing up two inequalities (1) corresponding to triples (i,j,k) and (j,i,k) yields the inequality $2d_{ij} \ge 0$. The metric cone M_{n+1} has full dimension $n(n+1)/2$ and its facets are defined by the $3\binom{n+1}{3}$ triangle inequalities (1).

One of the fundamental problems in the theory of metrics is the isometric embedding problem, that is, to determine conditions for a given metric space to be isometrically embeddable in a given class of spaces, say l_p-spaces. The l_1-metric on $\mathbb{R}^m$ is defined by $d(x,y) = \|x-y\|_1 = \sum_{1\le i\le m} |x_i - y_i|$. A metric space (X,d) is isometrically l_1-embeddable if there exist points $x_0, x_1, \dots, x_n$ in some space $\mathbb{R}^m$ such that $d_{ij} = \|x_i - x_j\|_1$ for all $0 \le i < j \le n$. The family of all metrics d on X which are isometrically embeddable into some l_1-space forms a cone $C(X) = C_{n+1}$, subcone of M_{n+1}, called *cut cone* (or *Hamming cone*). The cut cone C_{n+1} is

AMS subject classification code (1991): 52 A 43, 52 A 25, 05 A 99 generated by the *cut metrics* d_s for subsets S of X, where $(d_s)_{ij}=1$ if $|S\cap\{i,j\}|=1$ and $(d_s)_{ij}=0$ otherwise. Therefore, a metric d on X is isometrically l_1-embeddable if and only if d is the conic hull of cut metrics: $d=\sum_{S\subseteq X}\lambda_S d_S$ with $\lambda_S\geq 0$. The study of the l_1-embeddable finite metric spaces, i.e., of the cut cone C_{n+1}, was started in 1960 in [11]; see also e.g. [1], [2], [5], [14]. If d is rational valued, then, d is l_1-embeddable if and only if kd is embeddable into the hypercube of $\mathbb{R}^m$ for some integers k,m ([3]).

It is well-known that the cut metrics define extreme rays of the metric cone M_{n+1}. For $n\leq 3$, they are the only extreme rays of M_{n+1}; hence, the metric cone M_k and the cut cone C_k coincide for $k=2,3,4$. But there exists an extreme ray of M_5, namely the extreme ray defined by the graph metric of $K_{2,3}$, which is not a cut metric, so, the inclusion $C_5\subseteq M_5$ is strict; actually, the only type of extreme ray of M_5 which is not cut metric is the graph metric of $K_{2,3}$ ([4]). All extreme rays of M_6, M_7 are known ([17]); actually, the cut metrics are the only hypermetric extreme rays of M_n, for $n\leq 7$.

Given an integer vector $z\in\mathbb{Z}^{n+1}$, consider the following inequality:

$$\sum_{0\leq i<j\leq n} z_i z_j d_{ij} \leq 0 \tag{2}$$

For $z\in\mathbb{Z}^{n+1}$ with $\sum_{0\leq i\leq n} z_i = 1$ (resp. $\sum_{0\leq i\leq n} z_i = 0$), the inequality (2) is called *hypermetric* inequality (resp. *negative type* inequality). Let d be a real valued function defined on all unordered pairs of points of X; d is called *hypermetric* if d satisfies all hypermetric inequalities, the family of all hypermetrics d on X forms a cone $H(X)=H_{n+1}$, called *the hypermetric cone*. Hypermetric spaces were introduced in [11], [12] and, independently, in [18]. Similarly, the *negative type cone* N_{n+1} is the cone of all d satisfying all negative type inequalities. Observe that the triangle inequality (1) coincides with the hypermetric inequality (2) obtained for vector z having only three nonzero coordinates: $1,1,-1$. Therefore, the hypermetric cone H_{n+1} is a subcone of the metric cone M_{n+1}. Also, every hypermetric inequality is valid for the cut cone C_{n+1}; indeed, for each cut metric d_S, the left hand side of (2) for $z\in\mathbb{Z}^{n+1}$ with $\sum_{0\leq i\leq n} z_i=1$ takes value $z(S)(1-z(S))\leq 0$, where $z(S)=\sum_{i\in S} z_i$. Therefore, the cut cone C_{n+1} is a subcone of the hypermetric cone H_{n+1}.

So, we have three cones $C_{n+1}\subseteq H_{n+1}\subseteq M_{n+1}$, in the space $\mathbb{R}^{n(n+1)/2}$. For $k=2,3,4$, the three cones coincide: $C_k=H_k=M_k$. For $k=5,6$, we have $C_k=H_k\subsetneq M_k$ (equality $C_k=H_k$ was proved in [11] for $k\leq 5$ and in [6] for $k=6$). For $k\geq 7$, the inclusions $C_k\subseteq H_k\subseteq M_k$ are strict.

Since all cut metrics are extreme rays of the metric cone, they are also extreme rays of the hypermetric cone. Also, all hypermetric inequalities defining facets of the cut cone define facets of the hypermetric cone.

The cut cone is a polyhedral cone whose extreme rays are known: they are the cut metrics; for study of its facets, see e.g. [14], [15]. The metric cone too is a polyhedral cone, its facets are, by definition, the triangle inequalities; for study of its extreme rays, see [4], [17]. On the other hand, the hypermetric cone is defined by infinitely many hypermetric inequalities (2) and it was not known how many of them define facets. We prove here that the hypermetric cone H_n has a finite number of facets, or, equivalently, a finite number of extreme rays, i.e., we show the following result:

Theorem. *The hypermetric cone H_n is polyhedral.*

Note that the polyhedrality (as well as the list of all facets) for H_n, $n \le 6$, is implicit in [7]. On the other hand, the negative type cone N_n, $N_n \supseteq H_n$, is not polyhedral (see Remark 1).

The paper is devoted to the proof of the above theorem; the main steps are as follows:

- in section 3: map each hypermetric space to some positive semi-definite quadratic form and identify hypermetric spaces as generating subsets of the vertex sets of L-polytopes of the corresponding lattice (this fact was given implicitly in [7] and, independently, in [1], [2]).
- in section 4: establish a correspondance between the faces of the hypermetric cone H_{n+1} and the types of non affinely equivalent L-polytopes in $\mathbb{R}^k$, $k \le n$; then, using a theorem of Voronoi ([20]) which implies the finiteness of the number of types of L-polytopes of given dimension, obtain that H_{n+1} is polyhedral.

More precisely, Voronoi ([20]) proved that the number of non affinely equivalent lattices in the space $\mathbb{R}^k$ is finite, hence implying the finiteness of the number of non affinely equivalent L-polytopes (since any star of a lattice, i.e. all L-polytopes of the lattice tiling going through a given point, is finite). We give in section 5 a direct explicit proof of the finiteness of the number of non affinely equivalent L-polytopes in given dimension.

In the last section 6, we give some more facts on connected hypermetrics, hypermetrics arising from L-polytopes of small dimension and extremal hypermetrics.

In section 2, we recall some preliminaries on lattices, L-polytopes and empty spheres.

2. Preliminaries

In $\mathbb{R}^k$, $\|x\|$ denotes the euclidean norm of x, $\|x\|^2 = \sum_{1\le i\le k}(x_i)^2$, and $x.y$ denotes the scalar product of x,y, $x.y = \sum_{1\le i\le k} x_i y_i$.

Let $(q_1, q_2, \ldots, q_n)$ be a system of vectors of $\mathbb{R}^k$ having rank k; the $\mathbb{Z}$-module L generated by $(q_1, \ldots, q_n)$ is defined by $L = \{\sum_{1\le i\le k} z_i q_i : z \in \mathbb{Z}^n\}$. Then, L is a *lattice* if L is a discrete subgroup of $\mathbb{R}^k$, i.e. if $\beta = \min(\|q\| : q \in L - \{0\}) > 0$; in other words there exists a ball of radius $\beta > 0$ centered at each lattice point which contains no other lattice point.

A set $B = \{b_1, \ldots b_k\}$ is a *basis* of the lattice L if B *generates* L, i.e. $L = \{\sum_{1\le i\le k} z_i b_i : z \in \mathbb{Z}^k\}$, and B is a basis of $\mathbb{R}^k$. Any two bases B, B' of L are *unimodular equivalent*, i.e. $M_B = AM_{B'}$, where A is an integer matrix with determinant $|\det(A)| = 1$, and M_B (resp. $M_{B'}$) is the $k \times k$ matrix whose rows are the members of B (resp. B'). Then, the common value $|\det(B)|$ for any basis B of L is denoted as $\det(L)$.

Let S be a sphere in $\mathbb{R}^k$ with center c and radius r, $S=\{x\in\mathbb{R}^k : \|x-c\|=r\}$. One says that S is an *empty sphere* in the lattice L if $S\cap L$ generates $\mathbb{R}^k$ and $\|x-c\|\geq r$ holds for all lattice points $x\in L$; in other words, no lattice point is lying in the ball with boundary sphere S, but the lattice points lying on the sphere S generate $\mathbb{R}^k$. An empty sphere S is called *generating* if $S\cap L$ generates the lattice L. Clearly, the set $S\cap L$ is finite, since L is lattice. Then, the convex hull of the set $V=S\cap L$ is called an *L-polytope* (or *Delaunay polytope*) in the lattice L; so, the L-polytopes in a lattice L are the polytopes whose vertices are all lattice points lying on some empty sphere of L, the center of the empty sphere being then also called the center of the L-polytope.

Given a lattice point q of L, the *Voronoi polytope* $P_v(q)$ at q is the set of points which are at least as close to q as to any other lattice point, i.e. $P_v(q)=\{x\in\mathbb{R}^k : \|x-q\|\leq\|x-q'\| \text{ for all } q'\in L\}$. The vertices of the Voronoi polytopes are exactly the centers of the L-polytopes. Also, the Voronoi polytopes $P_v(q)$ for $q\in L$ form a normal tiling of the space $\mathbb{R}^k$. Another normal tiling of the space $\mathbb{R}^k$ is provided by the elementary cells $C(q)=\{q+\sum_{1\leq i\leq k} z_ib_i : 0\leq z_i\leq 1 \text{ for } 1\leq i\leq k\}$ for $q\in L$, where $B=\{b_1\ldots,b_k\}$ is a basis of L. Hence, the Voronoi polytopes and the elementary cells have the same volume, namely $\det(L)$. Another normal tiling of the space $\mathbb{R}^k$ is given by the L-polytopes in L.

For general information on lattices and the above polytopes, see [20] and e.g. [16], [9].

There is a natural way of partitioning polytopes: into classes of affinely equivalent polytopes; Voronoi called those classes *types* of polytopes. This induces a partition of L-polytopes in *types of L-polytopes.* Given two L-polytopes P,P', they have the same type if there exists an affine bijective transformation T such that $T(P)=P'$. Every type γ of L-polytopes is characterized by some integer matrix Y_γ (up to unimodular multiple).

Indeed, let P be an L-polytope in $\mathbb{R}^k$ with set of vertices V and let L be a lattice in $\mathbb{R}^k$ containing V (but P is not necessarily an L-polytope in L). Let $B=\{b_1,\ldots,b_k\}$ be a basis of L; then, for each $q\in V$, there exists $y_q\in\mathbb{Z}^k$ such that $q=\sum_{1\leq i\leq k}(y_q)_ib_i$. Denote by Q_P the $|V|\times k$ matrix whose rows correspond to the vertices of P, by M_B the $k\times k$ matrix whose rows are the members of B and by $Y_{P,B}$ the $|V|\times k$ matrix whose rows are the vectors y_q for $q\in V$. Then, the following relation holds:

$$Q_P = Y_{P,B}M_B \tag{3}$$

If B' is another basis of L, then $M_{B'}=AM_B$ for some unimodular matrix A; therefore, we deduce from (3) that $Y_{P,B'}=Y_{P,B}A^{-1}$, i.e. $Y_{P,B}$, $Y_{P,B'}$ are unimodular equivalent. On the other hand, let P' be an L-polytope which is affinely equivalent to P, i.e., $P'=T(P)$ for some affine bijective transformation T and let V' denote the set of vertices of P', then $V'=T(V)$ and the lattice $T(L)$ contains V'. Denote also by T the square matrix such that $x'=xT$ if x' is the image of x under T (x,x' being row vectors). Then, we have $M_{T(B)}=M_BT$, $Q_{P'}=Q_PT$ and, from (3), $Q_P=Y_{P,B}M_B$, $Q_{P'}=Y_{P',T(B)}M_{T(B)}$, yielding $Y_{P,B}=Y_{P',T(B)}$. Consequently, one may assume that the matrices $Y_{P,B}$ are all equal to the same integer matrix Y_γ for all

L-polytopes P of given type γ; of course, the matrix Y_γ is uniquely determined, once the basis B (called *representative* basis of type γ) has been fixed. In section 5, we shall indicate a "good" choice of the representative basis ensuring that matrix Y_γ has a "good" form (see Proposition 10).

3. Mapping hypermetrics to L-polytopes

Let P_n denote the family of all vectors $a=(a_{ij})_{1\le i\le j\le n}$ of $\mathbb{R}^{n(n+1)/2}$ for which the quadratic form of $\mathbb{R}^n$, $\sum_{1\le i\le j\le n} a_{ij}x_ix_j$ for $x\in\mathbb{R}^n$, is positive semi-definite; P_n is a cone in $\mathbb{R}^{n(n+1)/2}$. We first show that the negative type cone N_{n+1} is in one-to-one correspondance with the cone P_n and that the hypermetric cone H_{n+1} can be mapped onto a subcone of P_n.

We distinguish the point 0 of $X=\{0,1,\ldots,n\}$. Consider the linear bijective transformation α of $\mathbb{R}^{n(n+1)/2}$ defined by $a=\alpha(d)$ for $d=(d_{ij})_{0\le i<j\le n}$, $a=(a_{ij})_{1\le i\le j\le n}$ in $\mathbb{R}^{n(n+1)/2}$ satisfying:

(4) $a_{ii}=d_{0i}$ for $1\le i\le n$ $\quad a_{ij}=(d_{0i}+d_{0j}-d_{ij})/2$ for $1\le i<j\le n$

or, vice versa,

(5) $d_{0i}=a_{ii}$ for $1\le i\le n$ $\quad d_{ij}=a_{ii}+a_{jj}-2a_{ij}$ for $1\le i<j\le n$

Any hypermetric inequality, i.e. inequality (2) for $z\in\mathbb{Z}^{n+1}$ with $\sum_{0\le i\le n} z_i=1$, can be transformed, using (5), into:

$$\sum_{1\le i,j\le n} z_iz_ja_{ij}-\sum_{1\le i\le n} z_ia_{ii}\ge 0 \tag{6}$$

Similarly, any negative type inequality, i.e. inequality (2) for $z\in\mathbb{Z}^{n+1}$ with $\sum_{0\le i\le n} z_i=0$, can be transformed into:

$$\sum_{1\le i,j\le n} z_iz_ja_{ij}\ge 0 \tag{7}$$

Therefore, the image $\alpha(H_{n+1})$ of the hypermetric cone (resp. the image $\alpha(N_{n+1})$ of the negative type cone) is the cone of all vectors $a\in\mathbb{R}^{n(n+1)/2}$ satisfying inequality (6) (resp. inequality (7)) for all $z\in\mathbb{Z}^n$. As a consequence, we obtain that the hypermetric cone H_{n+1} is contained in the negative type cone N_{n+1}. Indeed, if we denote by $f(z)$ the left hand side of (6), we have that $f(z)+f(-z)=2\sum_{1\le i,j\le n} z_iz_ja_{ij}$, hence implying that $\alpha(H_{n+1})\subseteq\alpha(N_{n+1})$, or equivalently, $H_{n+1}\subseteq N_{n+1}$. Observe also that $\alpha(N_{n+1})$ coincides with the cone P_n of positive semi-definite quadratic forms. Indeed, if (7) holds for all integer vectors z, then (7) holds for all rational vectors and, thus, by continuity, also for all real vectors z. Therefore, $\alpha(N_{n+1})=P_n$ and, thus, $\alpha(H_{n+1})\subseteq\alpha(N_{n+1})=P_n$.

Remark 1. It is well known that the cone P_n is not polyhedral, thus the negative type cone $N_{n+1}=\alpha^{-1}(P_n)$ too is not polyhedral.

In the remaining of the section, we consider a hypermetric d on X and its image $a=\alpha(d)$. So, a defines a positive semi-definite quadratic form. The following well-known result states that, in matrix terms, a is a Gram matrix.

Proposition 2. *Given* $a = (a_{ij})_{1 \le i \le j \le n}$, *define the matrix* $A = (a_{ij})_{1 \le i,j \le n}$ *by setting* $a_{ji} = a_{ij}$. *Assume that* A *has rank* k *and that* A *defines a positive semi-definite quadratic form. Then, there exist vectors* $q_1, \ldots, q_n$ *in* $\mathbb{R}^k$ *such that* $a_{ij} = q_i . q_j$ *for all* $1 \le i \le j \le n$. *Moreover, if* $q'_1, \ldots, q'_n$ *are other vectors of* $\mathbb{R}^k$ *such that* $a_{ij} = q'_i . q'_j$ *for all* i, j, *then* $q'_i = T(q_i)$ *for* $1 \le i \le n$, *for some transformation* T *of* $OA(\mathbb{R}^k)$. *Also, the system* $(q_1, \ldots, q_n)$ *has rank* k.

From now on, we denote by k the rank of the symmetric matrix $(a_{ij})_{1 \le i,j \le n}$ and we consider a system $(q_1, \ldots, q_n)$ of rank k of vectors of $\mathbb{R}^k$ such that:

$$a_{ij} = q_i . q_j \quad \text{for} \quad 1 \le i \le j \le n \tag{8}$$

Setting $q_0 = 0$ and using relation (5), we have:

$$d_{ij} = \|q_i - q_j\|^2 \quad \text{for} \quad 1 \le i < j \le n \tag{9}$$

Using (8), inequality (6) can be reformulated as:

$$\| \sum_{1 \le i \le n} z_i q_i \|^2 - \sum_{1 \le i \le n} z_i \|q_i\|^2 \ge 0 \tag{10}$$

For example, for the cut metric d_S where S is a subset of $X - \{0\}$, its image $a_S = \alpha(d_S)$ is defined by $(a_S)_{ij} = 1$ if $i, j \in S$ and $(a_S)_{ij} = 0$ otherwise; the matrix $((a_S)_{ij})_{1 \le i,j \le n}$ has rank $k = 1$ and the vectors q_i belong to $\mathbb{R}$ and can be chosen as $q_i = 1$ for $i \in S$ and $q_i = 0$ for $i \notin S$.

Denote by Φ_d the map from X to $\mathbb{R}^k$ defined by $\phi_d(i) = q_i$ for $i \in X$. Hence, by (9), the map ϕ_d provides an embedding of the hypermetric space (X, d) into the space $\mathbb{R}^k$ equipped with the distance $d(q, q') = \|q - q'\|^2$ for $q, q' \in \mathbb{R}^k$. We shall see below that the set $\phi_d(X)$ has a special form, namely $\phi_d(X)$ is a generating subset of the set of vertices of some L-polytope in the lattice generated by the vectors q_i for $i \in X$. The following result was given in [2]; we give the proof for the sake of completeness.

Proposition 3. *Consider a vector* $a = (a_{ij})_{1 \le i \le j \le n}$ *satisfying inequality (6) for all* $z \in \mathbb{Z}^n$, *i.e. vectors* $q_1, \ldots, q_n$ *of* $\mathbb{R}^k$ *satisfying inequality (10) for all* $z \in \mathbb{Z}^n$. *Then, there exists a unique vector* c *in* $\mathbb{R}^k$ *satisfying:*

$$2c . q_i = \|q_i\|^2 \quad \text{for all} \quad i = 1, 2, \ldots, n \tag{11}$$

Proof. If $k = n$, i.e. $(q_1, \ldots, q_n)$ is a system of n linearly independent vectors, then the equation (11) admits a unique solution c. Assume now that $k \le n - 1$. Let Q denote the $n \times k$ matrix whose rows are the vectors $q_1, \ldots, q_n$, U denote the vector subspace of $\mathbb{R}^n$ spanned by the columns of Q and set $b = (\|q_i\|^2)_{1 \le i \le n}$. Then, equation (11) admits a solution if and only if $b \in U$, or, equivalently, $b.t = 0$ for all $t \in U^\perp$ ($U^\perp$ being the orthogonal complement of U in $\mathbb{R}^n$). Take t in $U^\perp$, let $z(t) \in \mathbb{Z}^n$ such that $|t_i - z(t)_i| < 1$ for $i = 1, \ldots, n$ and set $\delta = t - z(t)$; so, δ belongs to the unit cube. Since, by assumption, $q_1, \ldots, q_n$ satisfy (10) for vector $z(t)$, we have that $b.z(t) = \sum_{1\le i\le n} \|q_i\|^2 z(t)_i \le \|\sum_{1\le i\le n} q_i z(t)_i\|^2$. Also, $\|\sum_{1\le i\le n} q_i z(t)_i\|^2 = \|t^T Q - \delta^T Q\|^2 = \|\delta^T Q\|^2$, since $t^T Q = 0$ because $t \in U^\perp$. Therefore, we deduce that $b.z(t) \le \|\delta^T Q\|^2$, yielding that $|b.t| \le |b.\delta| + \|\delta^T Q\|^2$; so, if $b.t \neq 0$, the left hand side of the latter inequality can be made arbitrary large while the right hand side is bounded, thus $b.t = 0$ holds. Unicity of the solution c to (11) follows from the fact that system $(q_1, \ldots, q_n)$ has full rank k. ∎

Then, from (11), $\|\sum_{1\le i\le n} z_i q_i\|^2 - \sum_{1\le i\le n} \|q_i\|^2 z_i = \|\sum_{1\le i\le n} q_i z_i - c\|^2 - \|c\|^2$, and therefore, inequality (10) can be rewritten as:

$$\|\sum_{1\le i\le n} q_i z_i - c\| \ge \|c\| \tag{12}$$

Let S_d denote the sphere in $\mathbb{R}^k$ of center c and radius $\|c\|$, so $0 \in S_d$. Then, inequality (12) means that the vector $\sum_{1\le i\le n} q_i z_i$ does not lie in the ball with boundary sphere S_d. From (11), we deduce that the vectors $q_1, \ldots, q_n$ lie on the sphere S_d. Let L_d denote the $\mathbb{Z}$-module in $\mathbb{R}^k$ generated by $q_1, \ldots, q_n$, i.e. $L_d = \{\sum_{1\le i\le n} q_i z_i : z \subset \mathbb{Z}^n\}$. The following Proposition 4 shows that L_d is a lattice. As noted above, the sphere S_d is an empty sphere in L_d and, moreover, $S_d \cap L_d$ generates L_d. The convex hull of the set $S_d \cap L_d$ is an L-polytope P_d in the lattice L_d. For example, for any cut metric $d = d_S$ for $S \subseteq X$, the lattice L_d is simply $\mathbb{Z}$ and the L-polytope P_d is the segment $[0,1]$.

Proposition 4. ([2]). *Let L be a $\mathbb{Z}$-module in $\mathbb{R}^k$ generated by vectors $q_1, \ldots, q_n$. Assume that there exists an empty sphere in L containing $q_0 = 0, q_1, \ldots, q_n$. Then, L is a lattice.*

Proof. By assumption, there exists $c \in \mathbb{R}^k$ such that:

$$\|q_i - c\| = \|c\| \quad \text{for} \quad i = 1, \ldots, n \tag{a}$$

$$\|\sum_{1\le i\le n} q_i z_i - c\| \ge \|c\| \quad \text{for} \quad z \in \mathbb{Z}^n \tag{b}$$

For $z \in \mathbb{Z}^n$, set $q(z) = \sum_{1\le i\le n} q_i z_i$; then, $q_i \mp q(z) \in L$. So, (b) yields $\|q_i \mp q(z) - c\|^2 \ge \|c\|^2$, i.e., $\|q_i - c\|^2 + \|q(z)\|^2 \mp 2(q_i - c).q(z) \ge \|c\|^2$, and using (a), we obtain that:

$$\|q(z)\|^2 \ge 2|(q_i - c).q(z)| \quad \text{for} \quad i = 1, \ldots, n \tag{c}$$

Consider the unit vectors $v_i = (q_i - c)/\|c\|$ for $i = 0, 1, \ldots, n$ and set $\beta = \mathrm{Min}(\{\mathrm{Max}(|v_i.e| : 1 \le i \le n\} : e \in \mathbb{R}^k, \|e\| = 1)$. Then, (c) implies that $\|q(z)\| \ge 2\|c\|\beta$. In order to conclude the proof, it is enough to check that $\beta \neq 0$. Suppose for contradiction that $\beta = 0$; then, one can find a sequence $(e_p)_{p\ge 1}$ of unit vectors of $\mathbb{R}^k$ such that $|v_i.e_p| \le 1/p$ for any $1 \le i \le n$, $p \ge 1$. Since the unit sphere is a compact set in $\mathbb{R}^k$, one can assume that the sequence $(e_p)_{p\ge 1}$ admits a limit e when p goes to infinity (else, replace $(e_p)_{p\ge 1}$ by a subsequence). Therefore, $\|e\| = 1$, while

$v_i.e=0$ for $i=1,\dots,n$, implying that $e=0$ since the vectors v_i span $\mathbb{R}^k$, yielding a contradiction. ∎

So far, we have established that, for any hypermetric d on X, one can find a lattice L_d, an L-polytope P_d in L_d with set of vertices V and a generating map ϕ_d (i.e., $\phi_d(X)$ generates the lattice L_d) from X to V such that $d_{ij}=\|\phi_d(i)-\phi_d(j)\|^2$ for all $i,j\in X$. Conversely, we have the following result.

Proposition 5. *Let P be an L-polytope with vertex set V, $0\in V$, and let ϕ be a map from X to V. Set $d_{ij}=\|\phi(i)-\phi(j)\|^2$ for all $i,j\in X$; then, d is a hypermetric on X.*

Proof. Since P is an L-polytope in some lattice L, let S be the empty sphere in L such that $V=S\cap L$. Let c denote the center of the sphere S; since $O\in V$, S has radius $\|c\|$. Take integers z_i for $i\in X$ such that $\sum_{i\in X} z_i=1$. Note that $\sum_{i,j\in X,i<j} z_iz_jd_{ij}=(\sum_{i,j\in X} z_iz_jd_{ij})/2$. Also, $\sum_{i,j\in X} z_iz_jd_{ij}=\sum_{i,j\in X} z_iz_j\|\phi(i)-\phi(j)\|^2=\sum_{i,j\in X} z_iz_j\|(\phi(i)-c)-(\phi(j)-c)\|^2=\sum_{i,j\in X} z_iz_j(\|\phi(i)-c\|^2+\|\phi(j)-c\|^2-2(\phi(i)-c).(\phi(j)-c))=2\|c\|^2\,(\sum_{i,j\in X} z_iz_j)-2\|\sum_{i\in X} z_i(\phi(i)-c)\|^2=2(\|c\|^2-\|(\sum_{i\in X} z_i\phi(i))-c\|^2)\leq 0$. Hence, d is indeed a hypermetric on X. ∎

In particular, any L-polytope P with vertex set V yields the hypermetric d_P on V defined by: $d_P(q,q')=\|q-q'\|^2$ for all $q,q'\in V$.

So, given a pair (P,ϕ) where P is an L-polytope and ϕ is a map from X to the vertex set of P, we obtain a hypermetric d on X; in turn, we obtain from d the pair (P_d,ϕ_d) as indicated above. In order to ensure that both pairs $(P,\phi),(P_d,\phi_d)$ coincide (up to orthogonal transformation), it suffices to assume that ϕ is a generating map. So, we have the following statement. There is a one-to-one correspondance between:

(i) the hypermetrics d on X, $|X|=n+1$

and

(ii) the pair (P,ϕ) where P is an L-polytope in $\mathbb{R}^k$, $k\leq n$, and ϕ is a generating map from X to the vertex set of P (P being defined up to orthogonal transformation) satisfying: $d_{ij}=\|\phi(i)-\phi(j)\|^2$ for all $i,j\in X$.

4. The hypermetric cone is polyhedral

For any hypermetric d of H_{n+1} we define its *annulator Ann(d)* (corresponding to the notion of root figure of [16]) by $\text{Ann}(d)=\{z\in\mathbb{Z}^{n+1}: z\neq u_0,u_1,\dots,u_n,\ \sum_{0\leq i<j\leq n} z_iz_jd_{ij}=0\}$, where $u_i=(0,\dots,0,1,0,\dots,0)$ for $0\leq i\leq n$ denote the coordinate vectors of $\mathbb{R}^{n+1}$ corresponding to the trivial hypermetric inequalities (with all zero coefficients). So, Ann(d) corresponds to the set of (nontrivial) hypermetric inequalities which are satisfied at equality by d. Clearly, $\text{Ann}(d)\neq\emptyset$ if and only if d lies on the boundary of the hypermetric cone H_{n+1}. Let $F(d)$ denote the smallest face of H_{n+1} containing d. Then, $F(d)=\cap_{z\in\text{Ann}(d)}F_z\cap H_{n+1}$, where F_z denotes the hyperplane of $\mathbb{R}^{n(n+1)/2}$ defined by the equation $\sum_{0\leq i<j\leq n} z_iz_jx_{ij}=0$. Each face of H_{n+1} is of the form $F(d)$ for some $d\in H_{n+1}$. Therefore, in order to

show that the hypermetric cone H_{n+1} is polyhedral, we have to prove that there is a finite number of distinct faces $F(d)$, i.e., equivalently, that there is a finite number of distinct annulators $\mathrm{Ann}(d)$ for $d\in H_{n+1}$.

We now show that the number of annulators is indeed finite. Consider $d \in H_{n+1}$ and its image $a=\alpha(d)$. Recall that hypermetric inequalities are equivalent to inequalities (6) and further to inequalities (12). We correspondingly define the annulator of a by $\mathrm{Ann}(a) = \{z \in \mathbb{Z}^n : z \neq 0, u_1,\dots,u_n$ and $\sum_{1\le i\le j\le n} z_i z_j a_{ij} - \sum_{1\le i\le n} z_i a_{ii} = 0\} = \{z\in\mathbb{Z}^n : z\neq 0, u_1,\dots,u_n$ and $\|\sum_{1\le i\le n} z_i q_i - c\| = \|c\|\}$. Setting $\mathbb{Z}(q) = \{z\in\mathbb{Z}^n : q=\sum_{1\le i\le n} z_i q_i\}$ for $q\in L_d$ and denoting by V the set of vertices of the L-polytope P_d, one has clearly that:

$$\mathrm{Ann}(a) \cup \{0, u_1, \dots, u_n\} = \cup_{q\in V}\mathbb{Z}(q) \tag{13}$$

Let Q_V denote the $|V|\times k$ matrix whose rows correspond to the vertices $q\in V$ of P_d and let Q denote the $n\times k$ matrix whose rows correspond to $q_1,\dots,q_n$; so every row of Q is a row of Q_V and Q may have repeated rows. Assume that the L-polytope P_d is of type γ; as mentioned in section 2, for some basis B of $\mathbb{R}^k$, we have that: $Q_V = Y_\gamma M_B$ (recall relation (3)).

Let Y be the $n\times k$ integer matrix such that $Q=YM_B$. Denote by y_q the rows of Y_γ for $q\in V$ and by $y_1,\dots,y_n$ the rows of Y. Note that relation $q=\sum_{1\le i\le n} z_i q_i$ is equivalent to relation $y_q=\sum_{1\le i\le n} z_i y_i$; so $\mathbb{Z}(q)=\{z\in\mathbb{Z}^n : y_q=\sum_{1\le i\le n} z_i y_i\}$ for all $q\in V$. Let z_q be an element of $\mathbb{Z}(q)$ and set $K(d)=\{z\in\mathbb{Z}^n : \sum_{1\le i\le n} z_i y_i=0\}$; then, $z(q)=\{z=z_q+z' : z'\in K(d)\}$. Note that z_q depends only on $y_1,\dots,y_n,y_q$ and $K(d)$ depends only on $y_1,\dots,y_n$; hence, $\mathbb{Z}(q)$ depends only on $y_1,\dots,y_n,y_q$. Therefore, from (13), $\mathrm{Ann}(a)$, i.e. $\mathrm{Ann}(d)$, is completely determined by Y_γ and $(y_1,\dots,y_n)$.

In other words, every annulator is completely determined by a pair (γ,θ) where γ is a type of L-polytopes in $\mathbb{R}^k$ with $k\le n$ and θ is a map from $\{1,2,\dots,n\}$ to the set of rows of matrix Y_γ. Therefore, since the number of such maps θ is obviously finite and since the number of types of L-polytopes of given dimension is finite, we deduce that there is a finite number of annulators $\mathrm{Ann}(d)$ for $d\in H_{n+1}$ and thus that the hypermetric cone H_{n+1} is polyhedral.

Finiteness of the number of types of L-polytopes plays a crucial role in the proof of polyhedrality of the hypermetric cone. As we mentioned in section 2, it follows from a result of Voronoi ([20]); but, we give a direct explicit proof of this fact in the next section 5.

Remark 6. (i) A hypermetric d lies in the interior of H_{n+1} if and only if $\mathrm{Ann}(d)=\emptyset$, i.e., $0,q_1,\dots,q_n$ are pairwise distinct and they are the only lattice points lying on the empty sphere S_d, or equivalently, the L-polytope P_d is a $(n+1)$-simplex of $\mathbb{R}^n$ and the map ϕ_d is one-to-one.

(ii) A hypermetric d of H_{n+1} belongs to the cut cone C_{n+1} if and only if the associated lattice L_d can be embedded in a grid ([1]).

(iii) Let d be a hypermetric of H_{n+1} such that $d_{ij}\equiv 0 \pmod{2m}$ for all i,j, for some integer m. Then, $\|x\|^2\equiv 0 \pmod{2m}$, $x.y\equiv 0 \pmod{m}$ and $c.x\equiv 0 \pmod{m}$ for all $x,y\in L_d$; for $m=1$, this means that L_d is an even lattice and that the center c of the empty sphere S_d belongs to the dual lattice of L_d, as remarked in [1].

(iv) Let P be an L-polytope in $\mathbb{R}^n$ with vertex set V, $V_0=\{q_0=0,q_1,\dots,q_n\}\subseteq V$ be a subset of $n+1$ affinely independent vertices of P and d be the hypermetric on X, $|X|=n+1$, defined by $d_{ij}=\|q_i-q_j\|^2$ for $i,j\in X$. Clearly, $|\mathbb{Z}(q)|=1$ holds for every vertex q of P, since $K(d)=\{0\}$. Hence, from relation (13), we deduce that $|\mathrm{Ann}(a)|+n+1=|V|$. Hence, if d defines an extreme ray of H_{n+1}, then $|\mathrm{Ann}(a)|\geq n(n+1)/2-1$, yielding that $|V|\geq(n+1)(n+2)/2-1$. See [13] for a detailed study of L-polytopes associated with hypermetrics defining extreme rays.

5. Finiteness of the number of types of L-polytopes

In this section, we give a direct explicit proof of the finiteness of the number of types of L-polytopes in $\mathbb{R}^k$. The main idea is to show that every L-polytope is affinely equivalent to an L-polytope defined by an integer matrix having a special form and that there is a finite number of such matrices.

We first give two results on L-polytopes, namely upper bounds on their number of vertices and on their volume. Let L be a lattice in $\mathbb{R}^k$ and P be an L-polytope in L with vertex set V. From Proposition 5, if we set $d(q,q')=\|q-q'\|^2$ for all $q,q'\in V$, then d is a hypermetric on V. In particular, d satisfies the triangle inequality, i.e., for $q_1,q_2,q_3\in V$, $\|q_i-q_j\|^2+\|q_i-q_k\|^2\geq\|q_j-q_k\|^2$ holds for any permutation (i,j,k) of (1,2,3). This implies that the triangle in $\mathbb{R}^k$ whose vertices are q_1,q_2,q_3 has no obtuse angles. The next result implies that $|V|\leq 2^k$, so any L-polytope in $\mathbb{R}^k$ has at most 2^k vertices.

Proposition 7. ([10]). *Let V be a finite set in $\mathbb{R}^k$ such that any three points of V form a triangle with no obtuse angles. Then $|V|\leq 2^k$ holds.*

Proof. Given two points q,q' of V, denote by H_q (resp. $H_{q'}$, $H_{q,q'}$) the hyperplane going through q (resp. $q',(q+q')/2$) and orthogonal to the segment $[q,q']$ and denote by $R(q,q')$ the region lying between the hyperplanes H_q and $H_{q'}$. By assumption, any other point q'' of V must lie in the region $R(q,q')$, since the triangle with vertices q,q',q'' has no obtuse angles. Let P denote the convex hull of V. For $q\in V$, let h_q denote the 1/2-fold homothety with center q, i.e. $h_q(x)$ is defined by $h_q(x)-q=(x-q)/2$; so, h_q maps $H_{q'}$ into $H_{q,q'}$. Since P lies in the region $R(q,q')$, its image $h_q(P)$ lies in the region between hyperplanes H_q and $H_{q,q'}$; thus, the hyperplane $H_{q,q'}$ separates $h_q(P)$ and $h_{q'}(P)$. Since $\cup_{q\in V}h_q(P)\subseteq P$, we have that $\mathrm{vol}(P)\geq\mathrm{vol}(\cup_{q\in V}h_q(P))=\sum_{q\in V}\mathrm{vol}(h_q(P))=|V|\ \mathrm{vol}(P)/2^k$, yielding the bound $|V|\leq 2^k$. ∎

We now give an upper bound on the volume of an L-polytope of $\mathbb{R}^k$. Recall that the volume of any Voronoi polytope of a lattice L is equal to $\det(L)$.

Proposition 8. *Let P be an L-polytope in a lattice L in $\mathbb{R}^k$. Then, vol$(P)\leq 2^k$ det(L) holds.*

Proof. Let $P_v(0)$ denote the Voronoi polytope at point 0. We can assume w.l.o.g. that 0 is a vertex of P. Let h_0 denote the 1/2-fold homothety with center 0, so $h_0(x)=x/2$. We have that $h_0(P)\subseteq P_v(0)$.

Indeed, take a vertex q of P and suppose for contradiction that $q/2\notin P_v(0)$. Take a hyperplane H supporting a facet of $P_v(0)$ which separates $P_v(0)$ and $q/2$;

H is of the form $H_{0,q'}$, i.e., H is the hyperplane going through $q'/2$ and orthogonal to the segment $[0,q']$, for some $q' \in L$. Clearly, $q \notin R(0,q')$, i.e., q does not lie in the region between the two parallel hyperplanes to H through 0 and q', implying that $q.q' > \|q'\|$. From the proof of Proposition 7, since $q \notin R(0,q'), q'$ is not a vertex of P and thus $\|q'-c\| > \|c\|$, i.e. $\|q'\|^2 > 2c.q'$, where c denotes the center of the L-polytope P. On the other hand, $\|q-c\| = \|c\|$, i.e. $\|q\|^2 = 2c.q$. Then, one checks that $\|q-q'-c\| < \|c\|$, contradicting the fact that $q-q'$ is a lattice point. Indeed $\|q-q'-c\|^2 - \|c\|^2 = \|q-q'\|^2 - 2c.(q-q') = \|q\|^2 + \|q'\|^2 - 2q.q' - 2c.(q-q') < 2(\|q'\|^2 - q.q') < 0$.

Since $h_0(P) \subseteq P_v(0)$, we deduce that $\text{vol}(h_0(P)) = \text{vol}(P)/2^k \leq \text{vol}(P_v(0)) = \det(L)$, i.e. $\text{vol}(P) \leq 2^k \det(L)$. ∎

As we saw in section 2, each type γ of L-polytopes is specified by some integer matrix Y_γ once the representative basis B has been chosen. We see below that, if one suitably chooses the representative basis B, then the matrix Y_γ has a special form and there is only a finite number of such matrices. We first recall a well-known fact on lattices.

Proposition 9. (see e.g. [8] p.11–13). *Let L, L' be two lattices of $\mathbb{R}^k$ such that $L' \subseteq L$. For any basis $A = \{a_1, \ldots, a_k\}$ of L', there exists a basis $B = \{b_1, \ldots, b_k\}$ of L such that:*

(9i) $a_i = v_{i1}b_1 + v_{i2}b_2 + \ldots + v_{ii}b_i$ *for* $i = 1, 2, \ldots, k$

where $(v_{ij})_{1 \leq j \leq i \leq k}$ *are integers satisfying:*

(9ii) $0 \leq v_{ij} < v_{ii}$ *for all* $1 \leq j < i \leq k$.

Proposition 10. *Every type γ of L-polytopes in $\mathbb{R}^k$ is characterized by an integer matrix Y_γ of the following form:*

(10i) *there exists a $k \times k$ submatrix D of Y_γ which is lower triangular and satisfies condition (9ii)*

(10ii) $p = |\det(D)|$ *is the maximum possible value of the absolute value of the determinant of any $k \times k$ submatrix of Y_γ.*

For any given value of p, there is a finite number of such matrices Y_γ.

Proof. Let P be an L-polytope of type γ in the lattice L of $\mathbb{R}^k$ with vertex set V. Denote by Q_P the $|V| \times k$ matrix whose rows correspond to the vertices of V (in the canonical basis). Let V_0 be a subset of V of size k such that the $k \times k$ submatrix Q_0 of Q_P, corresponding to the members of V_0, has the largest possible value of the absolute value of its determinant. Applying Proposition 9 to the lattice L and the sublattice L' with basis V_0, we deduce the existence of a basis B of L such that:

(a) $$Q_0 = D\, M_B$$

where M_B is the matrix with rows the members of B (in the canonical basis) and D is a lower triangular integer matrix satisfying (9ii).

We can suppose, for instance, that Q_0 is the submatrix of Q_P formed by its first k rows. The matrix $Y = Q_P(M_B)^{-1}$ is an integer matrix; the submatrix of Y formed by its first k rows is the matrix D; denote by D' the submatrix of Y formed by its last $|V| - k$ rows.

Note that $p = |\det(D)| = |\det(Q_0)|/|\det(M_B)| = |\det(Q_0)|/\det(L)$; hence, by choice of Q_0, the absolute value of the determinant of any $k \times k$ submatrix of Y is less than or equal to p. Therefore, choosing as representative basis of type γ the basis B, we have that $Y = Y_\gamma$, thus stating (10i), (10ii).

Consider the matrix YD^{-1}; its $k \times k$ upper part is the identity matrix. Let r_{il} be a nonzero element of the matrix YD^{-1} with $k+1 \leq i \leq |V|$, $1 \leq l \leq k$. Let C denote the matrix obtained from D by replacing the l-th row of D by the l-th row of Y. Then, $|\det(CD^{-1})| = |r_{il}|$ and thus, $|r_{il}| = |\det(C)|/|\det(D)| = |\det(C)|/p$ where $|\det(C)|$ is an integer between 1 and p. Therefore, all elements of matrix YD^{-1} are rational numbers of the form a/p with $1 \leq |a| \leq p$. Since, moreover, YD^{-1} is a $|V| \times k$ matrix with $|V| \leq 2^k$ (from Proposition 7), we obtain that, for a given value of p, there is a finite number of such matrices YD^{-1}. On the other hand, the matrix D is an integer matrix satisfying condition (ii) of Proposition 13 with $\det(D) = |v_{11}.v_{22}..v_{kk}| = p$. Therefore, for a given value of p, there is only a finite number of such matrices D and, consequently, for a given value of p, there is a finite number of possibilities for the matrix Y. This concludes the proof of Proposition 10. ∎

In view of Proposition 10, in order to prove the finiteness of the number of types of L-polytopes in $\mathbb{R}^k$, it suffices to give an upper bound on p which depends only on k. Indeed, $p \leq k!2^k$ holds. For this, let S denote the simplex whose vertices are the rows of matrix Q_0 (see the proof of Proposition 10). Then, S is contained in the L-polytope P, implying that $\text{vol}(S) \leq \text{vol}(P)$. But, $\text{vol}(P) \leq 2^k \det(L)$, from Proposition 8, and $\text{vol}(S) = |\det(Q_0)|/k! = p \det(L)/k!$. Therefore, we deduce that $p \leq 2^k k!$.

6. Concluding remarks

Given an integral valued function d defined on all pairs of points of X, d is called *connected* if the graph with vertex set X and edges the pairs (i,j) such that $d_{ij} = 1$ is connected. The following results on connected hypermetrics and connected negative type distance functions were proved in [19]; they are a specification of the results given in section 3.

(a) Assume that d is connected; then, d is of negative type if and only if the corresponding $\mathbb{Z}$-module $\mathbb{Z}(\sqrt{2}q_1, \ldots, \sqrt{2}q_n)$ is a root lattice.

(b) Assume that d is connected; then, d is hypermetric if and only if the lattice $\mathbb{Z}(\sqrt{2}q_1, \ldots, \sqrt{2}q_n)$ is a root lattice and the associated L-polytope has its center at a point of the dual lattice.

Furthermore, all L-polytopes in irreducible root lattices and whose center is a point of the dual lattice are described in [19]; in graph terms, they are the Johnson graph for the root lattice A_n, the halfcube and cocktail party graph (corresponding to the cross polytope β_n) for the root lattice D_n, the Gosset graph on 56 vertices in E_7, the Schläfli graph on 27 vertices in E_6. One can check that, among the above graphs, the corresponding hypermetrics do not belong to the cut cone precisely for the last two graphs.

All L-polytopes (up to affine equivalence) in $\mathbb{R}^k$, $k=2,3,4$, are known. For $k=1$, the only L-polytope is the 2-simplex, i.e., the segment $[0,1]$ (corresponding to the cut metrics). For $k=2$, the only L-polytopes are the 3-simplex (triangle) and the 2-dimensional cube (square). For $k=3$, the only L-polytopes are the 4-simplex (α_3), the pyramid, the prism, the octahedron (β_3), and the 3-dimensional cube (γ_3). For $k=4$, [16] gives the complete list of L-polytopes, there are 19 of them. We checked that the hypermetrics defined by all L-polytopes in $\mathbb{R}^k$, $k \leq 4$, belong, in fact, to the cut cone. This is not the case for $k=6$. Actually, Grishukhin found all (non cuts) extreme rays of H_7 coming from the Schläfli polytope; there are exactly 26 of them (see [13]). We refer to [13] for a detailed treatment of extreme L-polytopes, i.e., L-polytopes corresponding to extreme rays of the hypermetric cone; in particular, several examples of extreme L-polytopes coming from the root lattices E_6, E_7, from the Barnes–Wall lattice Λ_{16} and the Leech lattice Λ_{24} are described.

We conclude with a first necessary condition for an L-polytope to be extreme. A lattice L in $\mathbb{R}^k$ is called *reducible* if $\mathbb{R}^k$ is the orthogonal sum of two subspaces R_1 and R_2 such that the projection $L_i = p_i(L)$ of L on R_i is a non trivial (i.e., distinct from $\{0\}$) sublattice of L for $i=1,2$.

Proposition 11. *Let L be a lattice and assume that the hypermetric given by some L-polytope in L defines an extreme ray, then L is irreducible.*

Proof. Let L be a reducible lattice in $\mathbb{R}^k$, i.e., L is the orthogonal sum of R_1, R_2 and the projection $L_i = p_i(L)$ of L on R_i is a sublattice of L, for $i=1,2$. Let P be an L-polytope in L and let S be the empty sphere of L containing the vertices of P. W.l.o.g. we can suppose that 0 is a vertex of P, so if c denotes the center of sphere S, its radius is $\|c\|$. Let $P_i = p_i(P)$ denote the projection of P on R_i, S_i denote the sphere in R_i of center c_i and radius $\|c_i\|$, where $c_i = p_i(c)$, for $i=1,2$, so $c=c_1+c_2$ and $\|c\|^2 = \|c_1\|^2 + \|c_2\|^2$.

We verify that P_i is an L-polytope in the lattice L_i for $i=1,2$. First, S_1 is an empty sphere in L_1. Indeed, take $x_1 \in L_1$; since $x_1 \in L$, $\|x_1 - c\|^2 \geq \|c\|^2$ holds, implying that: $\|x_1 - c_1\|^2 + \|c_2\|^2 \geq \|c\|^2 = \|c_1\|^2 + \|c_2\|^2$, and thus, $\|x_1 - c_1\| \geq \|c_1\|$. Then, we check that P_1 coincides with the convex hull of $S_1 \cap L_1$. Indeed, if $x \in S_1 \cap L_1$, then $x \in L$ and, by a similar argument as before, $x \in S$, yielding that $\mathrm{Conv}(S_1 \cap L_1) \subseteq P_1$. Now, take a vertex x of P, $x = x_1 + x_2, x_i \in P_i$. From the fact that $\|x-c\|^2 = \|x_1 - c_1\|^2 + \|x_2 - c_2\|^2 = \|c_1\|^2 + \|c_2\|^2$ and $\|x_i - c_i\|^2 \geq \|c_i\|^2$ for $i=1,2$, we deduce that $\|x_1 - c_1\| = \|c_1\|$ and thus $x_1 \in S_1 \cap L_1$.

Let d denote the hypermetric given by the L-polytope P; d is defined on the vertex set: $V = S \cap L$ of P by $d(x,y) = \|x-y\|^2$ for all $x, y \in V$. Setting $d_i(x,y) = \|x-y\|^2$ for all $x, y \in p_i(V)$, we obtain a hypermetric d_i on $p_i(V)$ for $i=1,2$. Clearly, $d = d_1 + d_2$ holds; so d is the sum of two hypermetrics and hence d does not define an extreme ray. ∎

Therefore, for finding extreme rays of the hypermetric cone, it is sufficient to consider L-polytopes in irreducible lattices. In particular, if an L-polytope of dimension k gives an extreme ray, then its set of vertices cannot be a subset of the set of vertices of some m-dimensional cube with $m \geq k$; indeed, recall that every metric which is embeddable into some cube belongs to the cut cone and hence is conic hull of cut metrics.

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References

[1] P. Assouad: Sous espaces de L^1 et inégalités hypermétriques, *Comptes Rendus de l'Académie des Sciences de Paris* **294** (1982), 439–442.

[2] P. Assouad: Sur les inégalités valides dans L_1, *European Journal of Combinatorics* **5** (1984), 99–112.

[3] P. Assouad, and M. Deza: Espaces métriques plongeables dans un hypercube: aspects combinatoires, *Annals of Discrete Mathematics* **8** (1980), 197–210.

[4] D. Avis: On the extreme rays of the metric cone, *Canadian Journal of Mathematics* **32**(1) (1980), 126–144.

[5] D. Avis, and M. Deza: L_1-embeddability, complexity and multicommodity flows, *Networks* **21** (1991), 595–617.

[6] D. Avis, and Mutt: All the facets of the six point Hamming cone, *European Journal of Combinatorics* **10** (1989), 309–312.

[7] E. P. Baranovskii: Simplexes of L-decompositions of Euclidian spaces, *Matematicheskie Zametki* (in Russian) **10**(6) (1971), 659–670; English translation in *Mathematical notes* **10** (1971), 827–834.

[8] J. W. S. Cassels: *An introduction to the geometry of numbers*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 99, Springer Verlag, 1959.

[9] J. H. Conway, and N. J. A. Sloane: *Sphere packings, lattices and groups*, Grundlehren der mathematischen Wissenschaften 290, Springer Verlag, 1987.

[10] L. Danzer, and B. Grünbaum: Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee, *Math. Zeitschrift* **79** (1962), 95–99.

[11] M. Deza: On the Hamming geometry of unitary cubes, *Doklady Akad. Nauk. SSSR* (in Russian) **134** (1960), 1037–1040; English translation in *Soviet Physics Dokl.* **5** (1961), 940–943.

[12] M. Deza: Matrices de formes quadratiques non négatives pour des arguments binaires, *Comptes Rendus de l'Académie des Sciences de Paris* **277** (1973), 873–875.

[13] M. Deza, V. P. Grishukhin, and M. Laurent: Extreme hypermetrics and L-polytopes, in Sets, Graphs and Numbers, Budapest, 1991, volume 60 of Colloquia Mathematica Societatis János Bolyai (1992), 157–209.

[14] M. Deza, and M. Laurent: Facets for the cut cone I, *Mathematical Programming* **56** (2) (1992), 121–160.

[15] M. Deza, and M. Laurent: Facets for the cut cone II: clique-web inequalities, *Mathematical Programming* **56** (2) (1992), 161–188.

[16] R. M. Erdahl, and S. S. Ryshkov: The empty sphere, *Canadian Journal of Mathematics* **34** (1987), 794–824.

[17] V. P. Grishukhin: Computing extreme rays of the metric cone for seven points, *European Journal of Combinatorics* **13** (1992), 153–165.

[18] J. B. Kelly: Hypermetric spaces, in: Lecture Notes in Mathematics 490 (1974), 17–31.

[19] P. Terwiliger, and M. Deza: Classification of finite connected hypermetric spaces, *Graphs and Combinatorics* **3** (1987), 293–298.

[20] G. F. VORONOI: Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Deuxième mémoire, *J. Reine Angew. Math.* **134** (1908) 198–287, **136** (1909), 67–178.

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