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Projecting a Simplex onto Another One

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For a given n-simplex T in $\mathbb{R}^{2n-1}$, the size of a regular n-simplex in $\mathbb{R}^{2n-1}$ which orthogonally projects onto T, and the size of a regular n-simplex onto which T projects orthogonally, are described in terms of spectra of a matrix associated with T.

1. INTRODUCTION

For two n-simplices S and T in Euclidean space, we write

$$(S \to T)_N$$

if we can place S and T in $\mathbb{R}^N$ so that S is mapped to T by the orthogonal projection from $\mathbb{R}^N$ onto an n-dimensional subspace. The symbol Δ denotes the regular n-simplex of unit edge-length, and $s\Delta$ denotes the regular n-simplex of edge-length s.

For any n-simplex T, there exists an $N > 0$ and an $s > 0$ such that $(s\Delta \to T)_N$. To see this, consider, for example, a triangle T in $\mathbb{R}^2$ with vertices the row vector of the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Putting, instead of each column, all of its permutations, we obtain

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then, the triangle in $\mathbb{R}^9$ with vertices the row vectors of this matrix is equilateral and it clearly projects S onto a triangle congruent to T. Thus $(s\Delta \to T)_9$ for some s. A similar argument will clearly work for any simplex.

Now, for a given n-simplex T, how small s can be and how small N can be?

We are going to answer this question. Let $\mathbf{I}$ and $\mathbf{J}$ respectively denote the $(n+1)\times(n+1)$ identity matrix and the $(n+1)\times(n+1)$ matrix with all entries 1. Then

$$\mathbf{P} := \mathbf{I} - (1/(n+1))\mathbf{J}$$

is the matrix of the projection $\mathbb{R}^{n+1} \to V$, where

$$V = \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x} \cdot (1, 1, \dots, 1) = 0\}.$$

The *square-distance matrix* of an m-point set is an $m \times m$ matrix, the (i, j)-entry of which is the square of the distance between the points i and j.

THEOREM 1. *Let T be an n-simplex and let* $\mathbf{D}$ *be the square-distance matrix of the vertex set of T. Let $\lambda_{\max}$ be the maximum eigenvalue of* $-\mathbf{PDP}$ *with multiplicity v. Then*:
(1) $(s\Delta \to T)_N$ *for some* $s > 0 \Leftrightarrow N \geq 2n - v$;
(2) $N \geq 2n$ *and* $s \geq (\lambda_{\max})^{\frac{1}{2}} \Rightarrow (s\Delta \to T)_N$;
(3) $2n - v \leq N \leq 2n - 1$ *and* $(s\Delta \to T)_N \Rightarrow s = (\lambda_{\max})^{\frac{1}{2}}$.

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THEOREM 2. *Let T be an n-simplex and let* $\mathbf{D}$ *be the square-distance matrix of the vertex set of T. Let $\lambda_{\min}$ be the minimum positive eigenvalue of* $-\mathbf{PDP}$ *with multiplicity v'. Then*:
(1) $(T \to s\Delta)_n$ *for some* $s>0 \Leftrightarrow N \geq 2n - v'$;
(2) $N \geq 2n$ *and* $0 < s \leq (\lambda_{\min})^{\frac{1}{2}} \Rightarrow (T \to s\Delta)_N$;
(3) $2n - v' \leq N \leq 2n-1$ *and* $(T \to s\Delta)_N \Rightarrow s = (\lambda_{\min})^{\frac{1}{2}}$.

EXAMPLE. Let T be the isosceles right triangle with hypotenuse $2^{\frac{1}{2}}$. Then $-\mathbf{PDP}$ is

$$(1/3)^2\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = (2/9)\begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix}.$$

The eigenvalues of this matrix are $2, 2/3, 0$. Hence

$$(2^{\frac{1}{2}}\Delta \to T)_3 \quad \text{and} \quad (T \to (2/3)^{\frac{1}{2}}\Delta)_3.$$

2. LEMMAS

First we recall the following classical result (see, e.g., Schoenberg [2]).

THEOREM S. *A non-negative symmetric* $(n+1)\times(n+1)$*-matrix* $\mathbf{M} = (M_{ij})$ *with* $M_{ii} = 0$ *for all i, is the square-distance matrix of an* $(n+1)$*-point set in Euclidean space iff* $\mathbf{M}$ *is of negative type (i.e.* $-\mathbf{PMP}$ *is positive semi-definite). Furthermore, the* $(n+1)$*-point set spans a k-dimensional flat iff* $\operatorname{rank}(\mathbf{M}) = k$.

LEMMA 1. *Let T be an n-simplex with maximum edge-length $d_{\max}$ and minimum edge-length $d_{\min}$. Let* $\mathbf{D} = (D_{ij})$ *be the square-distance matrix of the vertex set of T. Then* $-\mathbf{PDP}$ *is positive semi-definite of rank n and*

$$\lambda_{\max} \geq (d_{\max})^2 \quad \text{and} \quad (d_{\min})^2 \geq \lambda_{\min} > 0,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum eigenvalue and the minimum positive eigenvalue of $-\mathbf{PDP}$, *respectively.*

PROOF OF LEMMA 1. Let $\mathbf{j} = (1, 1, \ldots, 1)$. Since $-\mathbf{jPDP} = 0$, 0 is an eigenvalue of $\mathbf{M} := -\mathbf{PDP}$ with associated eigenvector $\mathbf{j}$. Hence any eigenvector $\mathbf{x}$ associated to any other eigenvalue is orthogonal to $\mathbf{j}$; that is $\mathbf{x} = \mathbf{xP}$. Furthermore, since T cannot be contained in $(n-1)$-space, $\mathbf{M}$ is positive semi-defintie of rank n. Hence, the other eignevalues are all positive, and it follows from the property of Rayleigh quotient (see, e.g., Bellman [1, Ch. 7]) that

$$(*) \qquad \lambda_{\max} = \max_{\mathbf{vP}\neq 0} \frac{-\mathbf{vPDPv}^t}{\mathbf{vPv}^t}, \qquad \lambda_{\min} = \min_{\mathbf{vP}\neq 0} \frac{-\mathbf{vPDPv}^t}{\mathbf{vPv}^t}.$$

Now, to show that $\lambda_{\max} \geq (d_{\max})^2$, suppose that $d := d_{\max}$ is the distance between the vertices 1 and 2, i.e.

$$d^2 := (d_{\max})^2 = D_{12} = D_{21}.$$

Then, since

$$(1,-1,0,\ldots,0)\mathbf{M}(1,-1,0,\ldots,0)^t = -(-d^2, d^2, *, \ldots, *)(1,-1,0,\ldots,0)^t = 2(d_{\max})^2,$$

and

$$(1,-1,0,\ldots,0)\mathbf{P}(1,-1,0,\ldots,0)^t = 2,$$

we have $\lambda_{\max} \geq (d_{\max})^2$ by $(*)$. Similarly, $(d_{\min})^2 \geq \lambda_{\min}$ follows. □

LEMMA 2. *Let T be an n-simplex, $\mathbf{D}$ its square-distance matrix, and let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximum eigenvalue (with multiplicity ν) and the minimum positive eigenvalue (with multiplicity ν') of $-\mathbf{PDP}$. Then:*
(1) *$s^2\mathbf{P}-(-\mathbf{PDP})$ is positive semi-definite $\Leftrightarrow s^2 \geqslant \lambda_{\max}$, $(-\mathbf{PDP})-s^2\mathbf{P}$ is positive semi-definite $\Leftrightarrow s^2 \leqslant \lambda_{\min}$; and*

(2)
$$\operatorname{rank}(s^2\mathbf{P}-(-\mathbf{PDP}))=\begin{cases} n & \textit{for } s^2>\lambda_{\max},\\ n-\nu & \textit{for } s^2=\lambda_{\max},\end{cases}$$
$$\operatorname{rank}(-\mathbf{PDP}-s^2\mathbf{P})=\begin{cases} n & \textit{for } s^2<\lambda_{\min},\\ n-\nu' & \textit{for } s^2=\lambda_{\min}.\end{cases}$$

PROOF. Regard the matrices $\mathbf{P}$ and $-\mathbf{PDP}$ as linear transformations of the vector space

$$V=\{\mathbf{x}\in\mathbb{R}^{n+1}\colon \mathbf{x}\cdot(1,1,\ldots,1)=0\}.$$

Then, (1) follows from the fact that $\mathbf{P}$ is the identity transformation of V, and (2) follows from $\operatorname{rank}(-\mathbf{PDP})=n$. □

3. PROOFS OF THEOREMS 1 AND 2

PROOF OF THEOREM 1. (1) Suppose that there is an $s>0$ such that $(s\Delta\to T)_N$. Then we can place $s\Delta$ and T in $\mathbb{R}^N$ so that $T=p(s\Delta)$, where

$$p\colon \mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^{N-n}\to\mathbb{R}^n$$

is the projection onto the first component. Projecting the vertices of $s\Delta$ onto the second component $\mathbb{R}^{N-n}$, we have $n+1$ (not necessarily all distinct) points with square-distance matrix $\mathbf{D}'=s^2(\mathbf{J}-\mathbf{I})-\mathbf{D}$. Hence the matrix $\mathbf{D}'$ must be of negative type, i.e. $-\mathbf{PD}'\mathbf{P}=s^2\mathbf{P}-(-\mathbf{PDP})$ is positive semi-definite. Hence $s^2\geqslant\lambda_{\max}$ by Lemma 2(1), and hence $-\mathbf{PD}'\mathbf{P}$ has rank $\geqslant n-\nu$. Therefore, $N-n\geqslant n-\nu$ by Theorem S. Note that if $s^2>\lambda_{\max}$, then $-\mathbf{PD}'\mathbf{P}$ has rank n by Lemma 2(2), and hence $N-n\geqslant n$ by Theorem S.

(2) Now suppose that $N\geqslant 2n$ and $s\geqslant(\lambda_{\max})^{\frac{1}{2}}$. Then since $\lambda_{\max}\geqslant(d_{\max})^2$ by Lemma 1, $\mathbf{D}':=s^2(\mathbf{J}-\mathbf{I})-\mathbf{D}$ is a non-negative matrix, and since $-\mathbf{PD}'\mathbf{P}=s^2\mathbf{P}-(-\mathbf{PDP})$ is positive semi-definite with rank $\leqslant n$ by Lemma 2, there exist $n+1$ points $y_1,\ldots,y_{n+1}$ in $\mathbb{R}^n$ with the square-distance matrix

$$s^2(\mathbf{J}-\mathbf{I})-\mathbf{D}.$$

If we denote the vertices of T by $x_1,\ldots,x_{n+1}$, then the $n+1$ points $(x_1,y_1),\ldots,(x_{n+1},y_{n+1})$ in $\mathbb{R}^n\times\mathbb{R}^n=\mathbb{R}^{2n}$ span a regular simplex of edge-length s which clearly projects onto T. Hence $(s\Delta\to T)_{2n}$, and hence $(s\Delta\to T)_N$.

(3) Finally, suppose that $s>(\lambda_{\max})^{\frac{1}{2}}$ and $(s\Delta\to T)_N$. Then, as noted in the proof of (1), we must have $N-n\geqslant n$, i.e. $N\geqslant 2n$. Hence, if

$$2n-\nu'\leqslant N\leqslant 2n-1 \qquad\text{and}\qquad (s\Delta\to T)_{2n-1}$$

then $s=(\lambda_{\max})^{\frac{1}{2}}$. □

The proof of Theorem 2 is almost identical, and is omitted.

4. Concluding Remark

Let S and T be two n-simplices. Let sT denote the enlargement of T with ratio $s > 0$. Then the extremal values of s, and the minimum value of N such that

$$(sT \to S)_N \qquad \text{or} \qquad (S \to sT)_N$$

can be presented in terms of the maximum root or the minimum positive root (and their multiplicities) of the equation

$$\det(s\mathbf{P}\mathbf{D}_T\mathbf{P} - \mathbf{P}\mathbf{D}_S\mathbf{P}) = 0,$$

where $\mathbf{D}_T$ and $\mathbf{d}_S$ are the square-distance matrices of T and S, respectively.

References

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