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Odd Systems of Vectors and Related Lattices

Dedicated to the Memory of Lev Arkad'evich Kalužnin

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Abstract. We consider uniform odd systems, i.e. sets of vectors of constant odd norm with odd inner product, and the lattice $L(\mathcal{V})$ linearly generated by a uniform odd system $\mathcal{V}$ of odd norm $2t+1$. If $u^2 \equiv p \pmod 4$ for all $u \in \mathcal{V}$, one has $v^2 \equiv p \pmod 4$ if v^2 is odd and $v^2 \equiv 0 \pmod 4$ if v^2 is even, for any vector $v \in L(\mathcal{V})$. The vectors of even norm form a double even sublattice $L_0(\mathcal{V})$ of $L(\mathcal{V})$, i.e. $(1/\sqrt{2})L_0(\mathcal{V})$ is an even lattice. The closure of $\mathcal{V}$, i.e. all vectors of $L(\mathcal{V})$ of norm $2t+1$, are minimal vectors of $L(\mathcal{V})$ for $t = 1$, and they are almost always minimal for $t = 2$. For such t, the convex hull of vectors of the closure of $\mathcal{V}$ is an L-polytope of $L_0(\mathcal{V})$ and the contact polytope of $L(\mathcal{V})$. As an example, we consider closed uniform odd systems of norm 5 spanning equiangular lines.

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1. Introduction

We study here *odd* systems, i.e. sets of vectors with odd inner (scalar) product. An odd system is a special case of an *integral* system, which is a set of vectors with integral inner product. In particular, vectors of an integral system have integral norm (squared length), which is the inner product of a vector with itself.

Similar to an odd system, an *even* system is defined. But an even system is nothing more than an integral system multiplied by $\sqrt{2}$.

Clearly, an integral system generates an integral lattice. Such lattices arise naturally in various contexts.

The integral lattice $L(\mathcal{V})$ generated by a *uniform* integral system $\mathcal{V}$ (i.e. system of vectors of some fixed norm m) determines naturally the *closure* of the system $\mathcal{V}$, i.e. the set $\mathrm{cl}\,\mathcal{V}$ of all vectors of norm m of the lattice $L(\mathcal{V})$. In other words, $\mathrm{cl}\,\mathcal{V}$ is the set of all vectors of norm m which are integral combinations of vectors of $\mathcal{V}$.

The classical example of an integral system is a root system. For our aims, uniform root systems are important; only they occur in integral lattices. A uniform root system is a set of vectors of norm 2 with integral inner product. Each root system is a direct sum of irreducible root systems. All irreducible root systems are closed. They are completely classified; namely, a uniform irreducible root system is one of A_n, D_n, E_6, E_7 and E_8.

It is natural to consider next the case of an integral system of vectors of norm 3. Several authors approached such systems; see, for example, [13, 10]. This case is much more complicated. But closed integral systems of vectors of norm 3 with inner product ± 1 are classified completely. Note that such a system is an odd system.

It turns out that the odd condition is restricting enough. The above-mentioned odd system of norm 3 is a special case of a system of vectors spanning equiangular lines. It is known that if a set of equiangular lines has sufficiently many lines, then spanning vectors can be chosen so that they have odd norm and inner product ± 1. This odd system leads to the study of general odd systems.

An important invariant of an odd system spanning equiangular lines is the property of closure.

For an integral lattice it is important to find the minimal nonzero norm of its vectors. An integral lattice generated by an odd system of norm 3 has minimal norm 3. Similarly, 'most' integral lattices generated by odd systems of norm 5 have minimal norm 5.

2. Lattices Generated by an Odd System

DEFINITION 1. An *even* (respectively, *odd*) *integral system* $\mathcal{V}$ is a set of vectors with mutual even (respectively, odd) integral inner product. In particular, the *norm* $v^2 = vv$ of $v \in \mathcal{V}$, i.e. the inner product of v with itself, is even (respectively, odd) integral too.

An integral system is called *irreducible* if it cannot be partitioned into two subsystems such that the inner product of vectors from different subsystems is equal to 0. Otherwise, the system is called *reducible*.

Clearly, each odd system is irreducible. Of course, an odd or even system is a special case of an integral system. Moreover, division of vectors of an even system by $\sqrt{2}$ converts it to an integral system.

The *dimension* $\dim \mathcal{V}$ of an integral system $\mathcal{V}$ is the dimension of $\operatorname{span} \mathcal{V}$, the space spanned by $\mathcal{V}$.

Let $\mathcal{V}$ be an integral system, and let

$$L(\mathcal{V}) = \left\{ u : u = \sum_{v \in \mathcal{V}} z_v v,\ z_v \in \mathbb{Z} \right\}.$$

Obviously, $L(\mathcal{V})$ is an integral lattice, and if $\mathcal{V}$ is an even system then $L(\mathcal{V})$ is an even system as well. Therefore $L(\mathcal{V})$ is a special case of an even lattice. Recall that an integral lattice is called even if all its vectors have even norm.

PROPOSITION 2. *Let $\mathcal{V}$ be an odd system and let $u \in L(\mathcal{V})$ with $u = \sum_{v\in\mathcal{V}} z_v v$. Then*

$$\sum_{v\in\mathcal{V}} z_v \equiv \sum_{v\in\mathcal{V}} |z_v| \equiv u^2 \pmod 2.$$

Proof. Since $vv' \equiv 1 \pmod 2$ for $v, v' \in \mathcal{V}$, we have

$$u^2 = \sum_{v,v'\in\mathcal{V}} z_v z_{v'} vv' \equiv \sum_{v,v'\in\mathcal{V}} z_v z_{v'} = \left(\sum_{v\in\mathcal{V}} z_v\right)^2 \pmod 2.$$

This implies that $\sum_{v\in\mathcal{V}} z_v \equiv u^2 \pmod 2$. The comparison

$$\sum_{v\in\mathcal{V}} z_v \equiv \sum_{v\in\mathcal{V}} |z_v| \pmod 2$$

is obvious. □

PROPOSITION 3. *Let $\mathcal{V}$ be an odd system and let $u_i \in L(\mathcal{V})$ with $u_i = \sum_{v\in\mathcal{V}} z_v^i v,\ i = 1, 2$. Then $u_1u_2 \equiv u_1^2u_2^2 \pmod 2$.*

Proof. Using Proposition 2, we obtain

$$u_1u_2 = \sum z_v^1 z_{v'}^2 vv' \equiv \left(\sum z_v^1\right)\left(\sum z_v^2\right) \equiv u_1^2 u_2^2 \pmod 2. \qquad \square$$

For $q = 0, 1$, consider the following subsets of $L(\mathcal{V})$.

$$L_q(\mathcal{V}) = \{u \in L(\mathcal{V}) : u^2 \equiv q \pmod 2),$$
$$L^q(\mathcal{V}) = \left\{u : u = \sum_{v\in\mathcal{V}} z_v v,\ z_v \in \mathbb{Z},\ \sum_{v\in\mathcal{V}} z_v = q\right\}.$$

We call the vector $-v$ the *opposite* of vector v and denote it by v^*. $\mathcal{V}$ is called *symmetric* if for each $v \in \mathcal{V}$ its opposite belongs to $\mathcal{V}$ also. $\mathcal{V}$ is called *asymmetric* if $\mathcal{V}$ has no pair of opposite vectors.

PROPOSITION 4. *Let $\mathcal{V}$ be an odd system. Then*

(i) *$L_0(\mathcal{V})$ is an even sublattice of $L(\mathcal{V})$,*
(ii) *$L_1(\mathcal{V}) = v + L_0(\mathcal{V})$ for any $v \in \mathcal{V}$, i.e. $L_1(\mathcal{V})$ is an affine sublattice of $L(\mathcal{V})$,*
(iii) *if $\mathcal{V}$ contains a pair of opposite vectors, then $L_q(\mathcal{V}) = L^q(\mathcal{V})$. In particular, this is true for $\mathcal{V}$ symmetric.*

Proof. (i) and (ii) Using Proposition 2 we obtain

$$L_q(\mathcal{V}) = \left\{u : u = \sum_{v\in\mathcal{V}} z_v v,\ z_v \in \mathbb{Z},\ \sum_{v\in\mathcal{V}} z_v \equiv q \pmod 2\right\}.$$

It easy to see that if $u, u' \in L_1(\mathcal{V})$, then $u_0 = u - u' \in L_0(\mathcal{V})$, i.e. $u = u' + u_0$, so $L_1(\mathcal{V})$ is an affine sublattice of $L(\mathcal{V})$.

(iii) Let $u \in L_q(\mathcal{V})$, $u = \sum_{v\in\mathcal{V}} z_v v$, $\sum_{v\in\mathcal{V}} z_v \equiv q \pmod 2$. Then $\sum_{v\in\mathcal{V}} z_v = 2z + q$ for some integer z. If $\mathcal{V}$ is symmetric then, adding the quantity $-z(v_0 + v_0^*)$ to $\sum_{v\in\mathcal{V}} z_v v$ for some $v_0 \in \mathcal{V}$, we obtain $\sum_{v\in\mathcal{V}} z_v = q$. □

If $\mathcal{V}$ is asymmetric then it is possible that the lattice $L^q(\mathcal{V})$ is a layer of the lattice $L_q(\mathcal{V})$.

3. Uniform Integral Systems

Let L be an integral lattice. For $k = 0, 1, 2, \ldots,$ we set

$$\mathcal{M}_k(L) = \{u \in L : u^2 = k\}.$$

Let $\mathcal{V}$ be an integral system. Call $\mathcal{V}$ *uniform* if all vectors of $\mathcal{V}$ have the same norm, and refer to this as the *norm of* $\mathcal{V}$. We consider the uniform subsets of $L(\mathcal{V})$.

Let

$$\mathrm{cl}_k(\mathcal{V}) = \mathcal{M}_k(L(\mathcal{V})).$$

We call cl_k the k-*closure* operator. If $\mathcal{V}$ is an even system, then $\mathrm{cl}_{2k+1}(\mathcal{V}) = \emptyset$. If $\mathcal{V}$ is an odd system, then, by Proposition 3, $\mathrm{cl}_{2k+1}(\mathcal{V})$ is an odd system for any integer $k \geqslant 0$. We have

$$L_0(\mathcal{V}) = \bigcup_{k=0}^{\infty} \mathrm{cl}_{2k}(\mathcal{V}), \qquad L_1(\mathcal{V}) = \bigcup_{k=0}^{\infty} \mathrm{cl}_{2k+1}(\mathcal{V}),$$

and $L_1(\mathcal{V})$ is an odd system.

Let $\mathcal{V}$ be a uniform integral system of norm k. We call $\mathcal{V}$ *closed* if $\mathcal{V} = \mathrm{cl}_k(\mathcal{V})$. Obviously, a closed system is symmetric.

A uniform integral system is called *maximal* if it cannot be enlarged without augmenting its dimension. Clearly, a maximal uniform integral system is closed. In general, the converse is not true.

For $\alpha \in \mathbb{R}$, denote by $\alpha\mathcal{V}$ the set of vectors αv for $v \in \mathcal{V}$. Obviously, $m\,\mathrm{cl}_k\,\mathcal{V} \subseteq \mathrm{cl}_t\,\mathcal{V}$ where $t = m^2 k$, and $(-1)\,\mathrm{cl}_k\,\mathcal{V} = \mathrm{cl}_k\,\mathcal{V}$. Hence the cardinality $|\mathrm{cl}_k\,\mathcal{V}|$ is an even integer for all k. We set

$$n(\mathcal{V}) = \begin{cases} \frac{1}{2}|\mathcal{V}| & \text{if } \mathcal{V} \text{ is symmetric,} \\ |\mathcal{V}| & \text{if } \mathcal{V} \text{ is asymmetric.} \end{cases}$$

PROPOSITION 5. *Let $\mathcal{V}$ be an odd system. Then* $n(\mathrm{cl}_1\,\mathcal{V}) \leqslant 1$.

Proof. Let $v_1, v_2 \in \mathrm{cl}_1\,\mathcal{V}$. Since $\mathrm{cl}_1\,\mathcal{V}$ is an odd system, $v_1 v_2 = \pm 1$. This implies that $v_2 = \pm v_1$. □

Following [7], call an odd system $\mathcal{V}$ *pillar* if there is a vector e of norm 1 such that $ve \in \{\pm 1\}$ for all $v \in \mathcal{V}$. The vector e is called the *shaft* of $\mathcal{V}$.

PROPOSITION 6. *Let $\mathcal{V}$ be a pillar odd system with shaft e. If* $\mathrm{cl}_1\,\mathcal{V} \neq \emptyset$*, then* $\mathrm{cl}_1\,\mathcal{V} = \{\pm e\}$.

Proof. Let $\mathrm{cl}_1\,\mathcal{V} \neq \emptyset$, i.e. $\mathrm{cl}_1\,\mathcal{V} = \{\pm e_1\}$. Then $e_1 = \sum_{v\in\mathcal{V}} z_v v$, where $\sum_{v\in\mathcal{V}} z_v$ is odd. Obviously, $ee_1 \leqslant 1$. But $ee_1 = \sum_v z_v(ev) = \sum_v \pm z_v$ is an odd integer. Hence, $ee_1 \in \{\pm 1\}$, i.e. $e_1 = \pm e$. □

Recall that a set of vectors is a *frame* if any two vectors of the set are either orthogonal or opposite.

PROPOSITION 7. *If $\mathcal{V}$ is an odd system, then* $\mathrm{cl}_2\,\mathcal{V}$ *is a frame.*

Proof. By Proposition 3, v_1v_2 is an even integer for any $v_1, v_2 \in \mathrm{cl}_2\,\mathcal{V}$. Hence, if $v_1 \neq \pm v_2$, then $v_1v_2 = 0$. □

Closed uniform integral systems of norm 2 are classified in [2]. Such a system is a direct sum of irreducible root systems of type A_n, D_n, E_6, E_7, E_8. The index n is the dimension of the corresponding root system. They are closed, and apart from $A_7 \subset E_7$, $A_8 \subset E_8$, $D_8 \subset E_8$ they are also maximal (see [2]). A frame is (up to a multiple) the special root system A_1^r, i.e. the direct sum of r root systems A_1 for some r. A detailed description of the root systems is given, for example, in [2].

Below, a uniform integral system $\mathcal{V}$ of norm 2 is called a *root system* if it is closed. Otherwise, $\mathcal{V}$ is simply called a *set of roots*.

Let $\mathcal{V}_4$ be a uniform even system of norm 4. Set $\beta = 1/\sqrt{2}$. Since $\beta\mathcal{V}_4$ has an integral inner product, we have the following proposition:

PROPOSITION 8. *$\beta\mathcal{V}_4$ is a root system if $\mathcal{V}_4$ is closed, and it is a set of roots if $\mathcal{V}_4$ is not closed.*

We now give two useful lemmas.

LEMMA 9. *Let $\mathcal{V}$ be an odd system and let* $v_i \in \mathrm{cl}_{2i+1}\,\mathcal{V}$*, i an integer. Let* $v_iv_{i'} = 2r + 1 > 0$ *where* $i \neq i'$*. Then* $r < (i + i')/2$.

Proof. Obviously, $v_i \neq v_{i'}$. Hence,

$$0 < (v_i - v_{i'})^2 = 2(i + i') + 2 - 2v_iv_{i'} = 2(i + i' - 2r),$$

i.e. $r < (i + i')/2$. □

For any set $\mathcal{X}$ of vectors, we set

$$a(\mathcal{X}) = \sum_{v\in\mathcal{X}} v. \tag{1}$$

LEMMA 10. *Let $\mathcal{V}$ be a uniform integral system of norm m, and $\mathcal{K} \subseteq \mathcal{V}$ be a maximal subset of vectors with mutual inner product -1. Then $\mathcal{K}$ contains at most $m + 1$ vectors, and if $|\mathcal{K}| = m + 1$, then $a(\mathcal{K}) = 0$.*

Proof. Set $k = |\mathcal{K}|$. We have

$$0 \leqslant (a(\mathcal{K}))^2 = \sum_{v\in\mathcal{K}} v^2 + \sum_{v,v'\in\mathcal{K},v\neq v'} vv' = km + k(k-1)(-1) = k(m+1-k),$$

i.e. $k \leqslant m+1$, and if $k = m+1$, then $a(\mathcal{K}) = 0$. □

We call such $\mathcal{K}$ of cardinality $m+1$ a *star*. By Lemma 10, $a(\mathcal{K}) = 0$ for $\mathcal{K}$ a star.

If $\mathcal{V}$ is an even system of norm $2m$, then $\beta\mathcal{V}$ is an integral system of norm m. If $\beta\mathcal{V}$ has a star $\mathcal{K}$ (of cardinality $m+1$), then $\beta^{-1}\mathcal{K}$ is a set of $m+1$ vectors of norm $2m$ with mutual inner product -2. We call such a set $\beta^{-1}\mathcal{K}$ a *star* of the even system $\mathcal{V}$.

Lemma 10 implies

COROLLARY 11. *Let $\mathcal{V}$ be an even system of norm $4t$. Then $|\mathcal{K}| = 2t+1$ for any star $\mathcal{K} \subseteq \mathcal{V}$.*

A star is a special case of a dependent set in an integral system $\mathcal{V}$. If $|\mathcal{V}| > \dim\mathcal{V}$, then there are dependencies between vectors of $\mathcal{V}$. Let $C \subseteq \mathcal{V}$ be a dependency, i.e. a set of linearly dependent vectors of $\mathcal{V}$. Then

$$\sum_{v\in C} z_v v = 0. \tag{2}$$

Equation (2) implies $\sum_{v\in C} z_v vu = 0$ for all $u \in \mathcal{V}$. Let A be the Gram matrix of the set $\mathcal{V}$. Then $z' \in \mathbb{R}^{\mathcal{V}}$, chosen to satisfy $z'_v = z_v$ for $v \in C$ and $z'_v = 0$ for $v \in \mathcal{V} - C$, is a solution to the system of equations $zA = 0$. Since A is an integral matrix, all solutions are rational (up to a multiple) so may be assumed integral.

Call the dependency in (2) *affine* if $\sum_{v\in C} z_v = 0$. By Proposition 4(iii), each dependency of $\mathcal{V}$ can be transformed into an affine dependency if $\mathcal{V}$ is symmetric.

Call the dependency in (2) *minimal* if $\sum_{v\in\mathcal{V}} y_v v \neq 0$ for all y for which $|y_v| \leqslant |z_v|$, and $|y_w| < |z_w|$ for some $w \in C$. As in matroid theory, call a minimal dependency a *circuit*. If $v \in C$ and $|z_v| = 1$ in the corresponding dependency, then the set $C - \{v\}$ is called a *broken circuit*.

We shall consider only uniform circuits and broken uniform circuits.

PROPOSITION 12. *Let $\mathcal{V}$ be a uniform integral system. Then $\mathcal{V}$ is closed if and only if $\mathcal{V}$ does not contain a broken circuit.*

Proof. Any vector $u \in (\mathrm{cl}\,\mathcal{V} - \mathcal{V})$ has the form $u = \sum_{v\in C_b} z_v v$, where $C_b \subseteq \mathcal{V}$ is a broken circuit. □

4. Sets of Vectors with Constant Norm Modulo 4

Consider an odd system $\mathcal{V}$ with vectors of norm $v^2 = 4m(v) + p$, where $m(v)$ is an integer and $p = 1$ or 3 is constant for all $v \in \mathcal{V}$. Thus, $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$.

THEOREM 13. *Let $\mathcal{V}$ be an odd system, and let $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$, $p = 1$ or 3. Then $u^2 \equiv 0 \pmod 4$ for $u \in L_0(\mathcal{V})$, and $u^2 \equiv p \pmod 4$ for $u \in L_1(\mathcal{V})$.*

Proof. Let $u = \sum_{v\in\mathcal{V}} z_v v$. We use induction on the number $s(u) = \sum_{v\in\mathcal{V}} |z_v|$. The assertion is obviously true for $s(u) = 1$. Let $u = \pm 2v_1$ or $u = v_1 \pm v_2$, $v_1 v_2 = 2k + 1$. Note that $2p \equiv 2 \pmod 4$. Hence, $u^2 = v_1^2 + v_2^2 \pm 2v_1 v_2 \equiv 2 \pm 2(2k + 1) \equiv 0 \pmod 4$. Plainly, $(\pm 2v_1)^2 \equiv 0 \pmod 4$, i.e. the assertion is true for $s(u) = 2$.

Assume the assertion is true for all u' with $s(u') < s$, and set $s(u) = s$. Let $z_{v'} \neq 0$. Without loss of generality, suppose that $z_{v'} > 0$. Then $u = u_1 + v'$, where $u_1 = \sum_{v\neq v'} z_v v + (z_{v'} - 1)v'$ and $s(u_1) = s - 1$. If s is even, then $u \in L_0(\mathcal{V})$, $u_1 \in L_1(\mathcal{V})$ and, by induction, $u_1^2 \equiv p \pmod 4$. Hence, $u^2 = u_1^2 + v'^2 + 2u_1 v' \pmod 4$. Since $u_1, v' \in L_1(\mathcal{V})$, Proposition 3 implies that $u_1 v' = 2k + 1$ for some integer k, i.e. $u^2 \equiv 2 + 4k + 2 \equiv 0 \pmod 4$. If s is odd, then $u \in L_1(\mathcal{V})$, $u_1 \in L_0(\mathcal{V})$ and, by induction, $u_1^2 \equiv 0 \pmod 4$. Proposition 3 implies $u_1 v' = 2l$, i.e. $u^2 \equiv p + 2 \times 2l \equiv p \pmod 4$. □

Theorem 13 implies the following corollary:

COROLLARY 14. *If $v^2 \equiv p \pmod 4$ for all $v \in \mathcal{V}$, then $\mathrm{cl}_k\, \mathcal{V} = \emptyset$ for $k \equiv 2, 4 - p \pmod 4$.*

Call a lattice L *double even* if L is an even system and $v^2 \equiv 0 \pmod 4$ for all $v \in L$. Obviously, L is double even if and only if βL is an even lattice ($\beta = 1/\sqrt{2}$).

COROLLARY 15. *If $\mathcal{V}$ is a uniform odd system, then the lattice $L_0(\mathcal{V})$ is a double even lattice.*

We can identify a uniform odd system $\mathcal{U}$ in a double even lattice as follows. Let L be a double even lattice and let $v_0 \in \mathcal{M}_{8k-4}(L)$, where $\mathcal{M}_k(L)$ is the set of all vectors of L of norm k as defined in Section 3. Set

$$\mathcal{M}_{4k}(v_0, L) = \{v \in \mathcal{M}_{4k}(L) : vv_0 = 4k - 2\},$$

$$u(v) = v - \tfrac{1}{2}v_0,\ \ v(u) = u + \tfrac{1}{2}v_0,$$

$$\mathcal{U}_{2k+1}(v_0, L) = \{u(v) : v \in \mathcal{M}_{4k}(v_0, L)\}.$$

Note that $u(v)$ is orthogonal to v_0 for all $v \in \mathcal{M}_{4k}(v_0, L)$.

THEOREM 16. *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent:*

(i) $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ *for a double even lattice* L,
(ii) $\mathcal{U}$ *is a closed uniform odd system of norm* $2k+1$.

Proof. (i) $\Rightarrow$ (ii). Let $v, v' \in \mathcal{M}_{4k}(v_0, L)$. Then vv' is an even integer and $|vv'| \leqslant 4k$. Since $u(v)u(v') = vv' - 2k + 1$ is odd, $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ is a uniform odd system of norm $2k+1$. It is symmetric because $v_0 - v \in \mathcal{M}_{4k}(v_0, L)$ and $u(v_0 - v) = -u(v)$.

Let $u_1 \in \mathrm{cl}_{2k+1}\,\mathcal{U}$. Since $u_1 \in L_1(\mathcal{U})$, by Proposition 4(iii),

$$u_1 = \sum_{v \in \mathcal{M}_{4k}(v_0, L), \sum z_v = 1} z_v u(v).$$

Hence, $u_1 = v_1 - \frac{1}{2}v_0$, where $v_1 = \sum_{v \in \mathcal{M}_{4k}(v_0, L), \sum z_v = 1} z_v v$. Since $\sum z_v = 1$, we have $v_1 v_0 = \sum z_v (v v_0) = (4k-2) \sum z_v = 4k - 2$. Since u_1 is orthogonal to v_0, we have $v_1^2 = u_1^2 + (\frac{1}{2}v_0)^2 = 4k$. This implies that $v_1 \in \mathcal{M}_{4k}(v_0, L)$ and $u_1 = u(v_1) \in \mathcal{U}$. This means that $\mathcal{U}$ is closed.

(ii) $\Rightarrow$ (i). Let u_0 be a vector of norm $2k-1$ which is orthogonal to the space spanned by $\mathcal{U}$. Let $\mathcal{V}_{4k} = \{v(u) : u \in \mathcal{U}\}$ with $v_0 = 2u_0$ in the definition of $v(u)$. Since $v(u)v(u') = uu' + u_0^2$ is an even integer, $\mathcal{V}_{4k}$ is an even system of vectors of norm $4k$.

Let $L = L(\mathcal{V}_{4k})$ be the lattice linearly generated by $\mathcal{V}_{4k}$. Obviously, $v_0 = 2u_0 = (u + u_0) + (-u + u_0) \in L$, $v_0^2 = 8k - 4$ and $\mathcal{V}_{4k} \subseteq \mathcal{M}_{4k}(v_0, L)$ since $v_0 v(u) = 2u_0^2 = 4k - 2$. Besides, $\mathcal{U} \subseteq \mathcal{U}_{2k+1}(v_0, L)$ since $u(v(u)) = u$. We have to prove that $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$. Let $u' \in \mathcal{U}_{2k+1}(v_0, L)$. Then $u' = v - \frac{1}{2}v_0$ for $v \in \mathcal{M}_{4k}(v_0, L)$, and $v = \sum_{u \in \mathcal{U}} z_u v(u)$ since L is generated by $v(u)$ for $u \in \mathcal{U}$, i.e. $u' = \sum_{u \in \mathcal{U}} z_u v(u) - \frac{1}{2}v_0 = \sum_{u \in \mathcal{U}} z_u u + \frac{1}{2}(\sum_{u \in \mathcal{U}} z_u - 1)v_0$. Since $u'v_0 = uv_0 = 0$, we have $\sum_{u \in \mathcal{U}} z_u = 1$. This implies that $u' \in L(\mathcal{U})$. Because $(u')^2 = 2k+1$, we have $u' \in \mathrm{cl}_{2k+1}\,\mathcal{U} = \mathcal{U}$. Hence $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$. □

The following theorem has an application to odd systems with inner product equal to ± 1.

THEOREM 17. *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent:*

(i) $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ *for a double even lattice* L *with minimal norm* $4k$,
(ii) $\mathcal{U}$ *is a closed uniform odd system of norm* $2k+1$ *and* $\mathrm{cl}_1\,\mathcal{U} = \mathrm{cl}_{2r+1}\,\mathcal{U} = \mathrm{cl}_{4r}\,\mathcal{U} = \emptyset$ *for* $0 < r < k$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 16, $\mathcal{U}$ is a closed uniform odd system of norm $2k+1$. Let $\mathcal{V} = \mathcal{M}_{4k}(v_0, L)$, $u' \in L_1(\mathcal{U})$, i.e. $u' = \sum_{v \in \mathcal{V}} z_v u(v)$ with $\sum_{v \in \mathcal{V}} z_v = 1$. Then $u' + \frac{1}{2}v_0 = \sum_{v \in \mathcal{V}} z_v v \in L^1(\mathcal{V}) \subseteq L$.

If $u' \in \mathrm{cl}_{2r+1}\,\mathcal{U}$ then, using $u'v_0 = 0$ and $v_0^2 = 8k-4$, we obtain $(u'+\frac{1}{2}v_0)^2 = 2(r+k) < 4k$ for $r < k$, a contradiction since the minimal norm of L is $4k$.

Let $u' \in L_0(\mathcal{U})$. By Proposition 4, $u' = \sum_{v\in\mathcal{V}} z_v u(v)$ with $\sum_{v\in\mathcal{V}} z_v = 0$. Hence $u' = \sum_{v\in\mathcal{V}} z_v v \in L^0(\mathcal{V}) \subseteq L$, i.e. $u'^2 \geqslant 4k$. This means that $\mathrm{cl}_{4r}\,\mathcal{U} = \emptyset$ for $r < k$.

(ii) $\Rightarrow$ (i). By Theorem 16, $\mathcal{U} = \mathcal{U}_{2k+1}(v_0, L)$ for a double even lattice L. We have to prove that the minimal norm of L is $4k$, i.e. that $\mathcal{M}_{4r}(L) = \emptyset$ for $0 < r < k$. According to the proof of (ii) $\Rightarrow$ (i) of Theorem 16, we can take $L = L(\mathcal{V})$ where $\mathcal{V} = \{u + \frac{1}{2}v_0 : u \in \mathcal{U}\}$.

Let $v \in L(\mathcal{V})$, then

$$v = \sum_{u\in\mathcal{U}} z_u\Big(u + \frac{v_0}{2}\Big) = \Big(\sum z_u\Big)\frac{v_0}{2} + \sum z_u u = z\frac{v_0}{2} + u_1,$$

where $z = \sum z_u$, $u_1 = \sum z_u u \in L(\mathcal{U})$.

Let $z = 0$, then $u_1 \in L_0(\mathcal{U})$. By Corollary 15, $L_0(\mathcal{U})$ is a double even lattice. Since $\mathrm{cl}_{4r}\,\mathcal{U} = \emptyset$ for $0 < r < k$, the minimal norm of $L_0(\mathcal{U})$ is $\geqslant 4k$.

Let $z = 1$. Then $u_1 \in L_1(\mathcal{U})$. Since $\mathrm{cl}_{2r+1}\mathcal{U} = \emptyset$ for $0 < r < k$, we have

$$v^2 = \Big(\frac{v_0}{2}\Big)^2 + u_1^2 = 2k - 1 + u_1^2 \geqslant 2k - 1 + (2k+1) = 4k.$$

It is easy to see that the norm of v with $|z| > 1$ is greater than $4k$. Since $v^2 = 4k$ for $v \in \mathcal{V}$, the minimal norm of $L(\mathcal{V})$ is equal to $4k$. □

The following lemma says that the set $\mathcal{U}_{2k+1}(v_0, L)$ of Theorem 17 spans a set of equiangular lines at angle $\arccos 1/(2k+1)$.

LEMMA 18. *Let $\mathcal{U}$ be a closed uniform odd system of norm $2k+1$. Then $uu' = \pm 1$ for all $u, u' \in \mathcal{U}$ with $u \neq \pm u'$ if and only if $\mathrm{cl}_{4r}\,\mathcal{U} = \emptyset$ for $0 < r < k$.*

Proof. Suppose there is a pair $u, u' \in \mathcal{U}$ such that $uu' = 2r+1$ with $0 < r < k$. Then $(u - u')^2 = 4(k - r)$, i.e. $u - u' \in \mathrm{cl}_{4(k-r)}\,\mathcal{U}$. □

Lemma 18 and Theorem 17 together imply the following fact, proved in [7]:

COROLLARY 19 ([7]). *Let L be a double even lattice of minimal norm $4k$ and let $v_0 \in \mathcal{M}_{8k-4}(L)$. Then the set $\mathcal{U}_{2k+1}(v_0, L)$ spans a set of equiangular lines at angle* $\arccos 1/(2k+1)$.

We denote by $L_2(\mathcal{U})$ the lattice $L(\mathcal{V}_{4k})$ (with $\mathcal{V}_{4k} = \{v(u) : u \in \mathcal{U}\}$) constructed in the proof of (ii) $\Rightarrow$ (i) of Theorem 16.

In the case of Theorem 17 the lattice $L_0(\mathcal{U})$ is the section of $L_2(\mathcal{U})$ by the affine hyperplane $H = \{x : xv_0 = 4k-2\}$. The lattice $L(\mathcal{U})$ is the projection of $L_2(\mathcal{U})$ on H.

5. Contact Polytopes and L-Polytopes of the Lattice $L_0(\mathcal{V})$

The convex hull of all minimal vectors of a lattice is called the *contact polytope* of the lattice. We denote the convex hull of $\mathcal{X}$ by conv$\mathcal{X}$.

Let $\mathcal{V}$ be an odd system and let H_k be the space spanned by $\mathrm{cl}_k\,\mathcal{V}$. Set

$$m = \min\{k > 0 : \mathrm{cl}_{2k}\,\mathcal{V} \neq \emptyset\}.$$

Then the minimal norm of the lattice $L_0(\mathcal{V})$ is $2m$, and $\mathrm{conv}(\mathrm{cl}_{2m}\,\mathcal{V})$ is the contact polytope of $L_0(\mathcal{V})$.

An L-polytope of a lattice is the convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full affine rank. Let

$$t = \min\{k : \mathrm{cl}_{2k+1}\,\mathcal{V} \neq \emptyset\}.$$

Then the sphere circumscribing $\mathrm{cl}_{2t+1}\,\mathcal{V}$ with squared radius $2t+1$ is empty in $L_1(\mathcal{V})$. Recall that squared Euclidean distance of any point of $L_1(\mathcal{V})$ from the origin is odd, and $L_1(\mathcal{V})$ is a translation of the lattice $L_0(\mathcal{V})$. Any vector of $\mathrm{cl}_{2t+1}\,\mathcal{V}$ can be taken as the translation vector. Hence, the translation of $\mathrm{conv}(\mathrm{cl}_{2t+1}\,\mathcal{V})$ is an L-polytope of the sublattice $L_0(\mathcal{V}) \cap H_{2t+1}$.

PROPOSITION 20. *Let $\mathcal{V}$ be an odd system.*

(i) *If $v^2 \equiv 3 \pmod 4$ for all $v \in \mathcal{V}$, then* $\mathrm{conv}(\mathrm{cl}_3\,\mathcal{V})$ *is an L-polytope of the lattice $L_0(\mathcal{V}) \cap H_3$ and the contact polytope of the lattice $L(\mathcal{V})$.*

(ii) *If $v^2 \equiv 1 \pmod 4$ for all $v \in \mathcal{V}$ and* $\mathrm{cl}_1\,\mathcal{V} = \emptyset$*, then* $\mathrm{conv}(\mathrm{cl}_5\,\mathcal{V})$ *is an L-polytope of the lattice $L_0(\mathcal{V}) \cap H_5$.*

(iii) *If $\mathcal{V}$ is a uniform odd system of norm* 5 *with* $\mathrm{cl}_1\,\mathcal{V} = \emptyset$*, and $vv' = \pm 1$ for distinct $v, v' \in \mathrm{cl}_5\,\mathcal{V}$, then* $\mathrm{conv}(\mathrm{cl}_5\,\mathcal{V})$ *is an L-polytope of $L_0(\mathcal{V})$ and the contact polytope of $L(\mathcal{V})$. Moreover,* $\mathrm{conv}(\mathrm{cl}_8\,\mathcal{V})$ *is the contact polytope of $L_0(\mathcal{V})$.*

Proof. (i) and (ii) are implied by Corollaries 14 and 15.

(iii) Since $\mathcal{V} \subseteq \mathrm{cl}_5\,\mathcal{V}$, it follows that $L(\mathcal{V}) \subset H_5$. By (ii), $\mathrm{conv}(\mathrm{cl}_5\,\mathcal{V})$ is an L-polytope of $L_0(\mathcal{V})$. By Corollary 14 and Lemma 18, $\mathrm{cl}_k\,\mathcal{V} = \emptyset$ for $k = 2, 3, 4, 6, 7$. Hence $\mathrm{cl}_5\,\mathcal{V}$ and $\mathrm{cl}_8\,\mathcal{V}$ are sets of minimal vectors of the lattices $L(\mathcal{V})$ and $L_0(\mathcal{V})$, respectively. □

6. Closed Uniform Odd Systems of Norm 3

Note that uniform systems of norms 3 and 5 are the first members of the sequences of odd systems with norm $3 \pmod 4$ and norm $1 \pmod 4$, respectively.

Note also that, for $k = 1$, Theorems 16 and 17 coincide. In fact, on the one hand, the norm of any vector of a double even lattice is a multiple of 4. Hence, if $\mathcal{U}_3(v_0, L) \neq \emptyset$ then the minimal norm of L is equal to 4. On the other hand, there is no r satisfying (ii) of Theorem 17, and $\mathrm{cl}_1\mathcal{U} = \emptyset$ by Corollary 14.

For a uniform odd system of norm 3, Proposition 20(i) gives the following analogue of Proposition 1 of [10].

COROLLARY 21. *Let $\mathcal{U}$ be a uniform odd system of norm* 3. *Then the minimal norm of the lattice $L(\mathcal{U})$, generated by $\mathcal{U}$, is* 3.

Let $\mathcal{U}$ be a uniform odd system of norm 3. Then the lattices $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$ are double even lattices of minimal norm 4, and $L_2(\mathcal{U})$ is generated by a uniform even system $\mathcal{M}_4$ of norm 4. Hence, $L_2(\mathcal{U})$ is, up to the multiple $\beta = 1/\sqrt{2}$, a root lattice.

In contrast to $\beta L_2(\mathcal{U})$, the lattice $\beta L_0(\mathcal{U})$ is not always generated by the root system $\beta\,\mathrm{cl}_4\mathcal{U}$. Theorem 22 below shows that $\beta L_2(\mathcal{U})$ is an irreducible root lattice. Since all irreducible root lattices are known, this theorem allows one to classify closed uniform odd systems of norm 3.

THEOREM 22. *Let $\mathcal{U}$ be a closed uniform odd system of norm* 3. *Then the lattice $\beta L_2(\mathcal{U})$ is an irreducible root lattice.*

Proof. Recall that $\mathcal{U} = \mathcal{U}_3(v_0, L)$ for $L = L_2(\mathcal{U})$. Here $v_0 \in L$ is a vector of norm 4, i.e. $v_0 \in \mathcal{M}_4(L)$. Hence, the set of vectors $\mathcal{M}_4(v_0, L) \cup \{v_0\}$, generating L, is irreducible since $vv_0 \neq 0$ for all $v \in \mathcal{M}_4(v_0, L)$. This means that the lattice βL is generated by an irreducible subset of roots. Hence, $\beta L = \beta L_2(\mathcal{U})$ is an irreducible root lattice. □

Theorems 17 and 22 together imply the following known fact (see, for example, Theorem 1 of [10]).

COROLLARY 23. *There is a one-to-one correspondence between closed uniform systems of norm* 3 *and irreducible root systems.*

It is obvious that a uniform odd system of norm 3 spans a set of equiangular lines at angle $\arccos \frac{1}{3}$.

The closed uniform odd systems $\mathcal{U}$ of norm 3 and corresponding lattices $L(\mathcal{U})$, $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$, if they have known names, are given in Table I.

The lattices A_5^{+2}, D_6^{+2} and E_7^{+2} of Table I are described in [4] and [5].

Recall that an odd system $\mathcal{U}$ is called pillar if there is a vector e of norm 1 such that $ve \in \{\pm 1\}$ for all $v \in \mathcal{U}$.

Denote by $\mathcal{U}_n^Q$ the system $\mathcal{U}$ corresponding to the root lattice Q_n where $Q = A, D$ or E. Note that $\mathcal{U}_n^A$ and $\mathcal{U}_n^D$ are pillar with $e = \beta(e_1 + e_2)$, where $\{e_i : 1 \leqslant i \leqslant n+1\}$ is an orthonormal basis such that the roots of A_n (respectively, D_n) are $\pm(e_i - e_j)$ (respectively, $\pm(e_i \pm e_j)$), $1 \leqslant i, j \leqslant n+1$. The unit vector e does not belong to $\mathrm{span}\,\mathcal{U}_n^A$ but belongs to $\mathrm{span}\,\mathcal{U}_n^D$. The vectors $u \in \mathcal{U}_n^A$ have

Table I.

$\beta L_2(\mathcal{U}) =$	A_n	D_n	E_6	E_7	E_8
dim$\mathcal{U}$	$n-2$	$n-1$	5	6	7
$n(\mathcal{U})$	$n-2$	$2(n-2)$	10	16	28
$\beta L_0(\mathcal{U})$			A_5	D_6	E_7
$\beta L(\mathcal{U})$			A_5^{+2}	D_6^{+2}	E_7^{+2}

the form $u = \pm\beta(e_1 + e_2 - 2e_i)$, $3 \leqslant i \leqslant n+1$, and those of $\mathcal{U}_n^D$ have the form $u = \pm\beta(e_1 + e_2 \pm 2e_i)$, $3 \leqslant i \leqslant n+1$.

The lattice $\beta L_0(\mathcal{U}_n^Q)$ is the section of Q_n by the affine hyperplane $H = \{x : x(e_1 + e_2) = 2\}$. The lattice $\beta L(\mathcal{U}_n^Q)$ is the projection of Q_n onto the affine hyperplane H. The lattices $\beta L_0(\mathcal{U}_n^A)$ and $\beta L_0(\mathcal{U}_n^D)$ are *not* root lattices, because every basis for either lattice contains a vector of norm at least 4.

7. Pillar Uniform Odd Systems of Norm 5

Now we consider closed uniform odd systems of norm 5. Let $\mathcal{U}$ be such a system. According to Proposition 8, $\beta\,\mathrm{cl}_4\mathcal{U}$ is a root system. We try to understand when the root system $\mathcal{V} = \beta\,\mathrm{cl}_4\mathcal{U}$ determines uniquely the system $\mathcal{U}$. For example, if e is a vector of norm 1 which is orthogonal to all roots of $\mathcal{V}$, then the set of vectors $\mathcal{U} = \{\sqrt{2}v \pm e : v \in \mathcal{V}\}$ is a closed uniform odd system of norm 5, with $\mathrm{cl}_1\mathcal{U} = \{\pm e\}$ and $\mathrm{cl}_4\mathcal{U} = \sqrt{2}\mathcal{V} \cup \{\pm 2e\}$. Moreover, $\mathcal{U}$ is pillar and, as we show below, every closed odd system $\mathcal{U}$ of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$ has such a form.

Recall that a uniform odd system $\mathcal{U}$ is called pillar if there is a vector e of norm 1 such that $ue \in \{\pm 1\}$ for all $u \in \mathcal{U}$. By Proposition 6, if $\mathcal{U}$ is pillar with shaft e and $\mathrm{cl}_1\mathcal{U} \neq \emptyset$, then $\mathrm{cl}_1\mathcal{U} = \{\pm e\}$. The assertion of Proposition 6 can be reversed for a uniform odd system of norm 5. Moreover, an odd system $\mathcal{U}$ of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$ is a special case of a pillar odd system.

PROPOSITION 24. *Let $\mathcal{U}$ be a uniform odd system of norm* 5, *and let e be a vector of norm* 1. *If $\mathcal{U}' = \mathcal{U} \cup \{e\}$ is an odd system, then $\mathcal{U}$ is pillar.*

Proof. By the definition of $\mathcal{U}'$, $e \in \mathrm{cl}_1\mathcal{U}'$. Let $u \in \mathrm{cl}_5\mathcal{U}'$ be such that $ue > 0$. By Lemma 9, $ue = 2r + 1$, where r is an integer and $0 \leqslant r < \frac{1}{2}$, i.e. $r = 0$. Hence $ue = \pm 1$ for all $u \in \mathrm{cl}_5\mathcal{U}'$, in particular, for all $u \in \mathcal{U}$. □

Note that Proposition 24 is not true for odd systems of norm > 5.

In this section we show that a pillar odd system of norm 5 is related to a set of roots. Let $\mathcal{U}$ be a pillar uniform odd system of norm 5 with shaft e. For $u \in \mathcal{U}$, set

$$v(u) = u - (ue)e, \tag{3}$$

$$\mathcal{V}_4(\mathcal{U}) = \{v(u) : u \in \mathcal{U}\}. \tag{4}$$

It is easy to verify that $\mathcal{V}_4(\mathcal{U})$ is a uniform even system of norm 4, and $ve = 0$ for all $v \in \mathcal{V}_4(\mathcal{U})$. If $\mathcal{U}$ is closed, then $\mathcal{V}_4(\mathcal{U})$ is symmetric since $v(-u) = -v(u)$. The equality (3) defines a linear map v: $\mathcal{U} \to \mathcal{V}_4(\mathcal{U})$ which is the projection of $\mathcal{U}$ on the hyperplane which is orthogonal to the vector e.

Obviously each vector $u \in \mathcal{U}$ has the form $u = (ue)e + v(u)$, where $v(u)$ has norm 4. If $\mathrm{cl}_1\mathcal{U} \neq \emptyset$, then $v(u) \in \mathrm{cl}_4\mathcal{U}$.

By Corollary 11, any star $\mathcal{K} \subseteq \mathcal{V}_4(\mathcal{U})$ contains 3 vectors with mutual inner product -2.

LEMMA 25. *Let $\mathcal{V}_4(\mathcal{U})$ be closed. Then, $\mathcal{V}_4(\mathcal{U})$ contains no star if and only if it is a frame.*

Proof. If $\mathcal{V}_4(\mathcal{U})$ is a frame, then it contains no pair of vectors with inner product ± 2, and therefore contains no star. Suppose $\mathcal{V}_4(\mathcal{U})$ is not a frame and contains two vectors v, v' with inner product $vv' = -2$. Then $(v + v')^2 = 4$, and $v + v' \in \mathcal{V}_4(\mathcal{U})$ since $\mathcal{V}_4(\mathcal{U})$ is closed, whence $\{v, v', -(v + v')\}$ is a star. □

Below we consider closed pillar uniform odd systems of norm 5 with shaft e. We have the following three cases:

(1) $\mathrm{cl}_1\mathcal{U} \neq \emptyset$,
(2) $\mathrm{cl}_1\mathcal{U} = \emptyset$, and $\mathcal{V}_4(\mathcal{U})$ is closed,
(3) $\mathrm{cl}_1\mathcal{U} = \emptyset$, and $\mathcal{V}_4(\mathcal{U})$ is not closed.

We consider the cases (1) and (2) in detail.

Recall that $\mathcal{V}_4(\mathcal{U})$ and $\mathrm{cl}_4\mathcal{U}$ consist of vectors of norm 4. Proposition 26 below describes a relation between these sets.

PROPOSITION 26. *Let $\mathcal{U}$ be a closed uniform pillar odd system of norm* 5 *with shft e. Then*

(i) $2e \in \mathrm{cl}_4\mathcal{U}$ *if and only if there exists $v \in \mathcal{V}_4(\mathcal{U})$ such that $v \pm e \in \mathcal{U}$, and if* $2e \in \mathrm{cl}_4\mathcal{U}$ *then* $\mathcal{U} = \{v \pm e : v \in \mathcal{V}_4(\mathcal{U})\}$,
(ii) $\mathrm{cl}_1\mathcal{U} = \emptyset$ *if and only if* $\mathcal{V}_4(\mathcal{U}) \cap \mathrm{cl}_\triangle\mathcal{U} = \emptyset$.

Proof. (i) Let $u \in \mathcal{U}$. Then, by (3), $u = v(u) + (ue)e$. Now if $2e \in \mathrm{cl}_4\mathcal{U}$, then the vector $u' = v(u) - (ue)e$ belongs also to $\mathcal{U}$ since $u' = u - (ue)2e$ has norm 5 and $\mathcal{U}$ is closed. Conversely, if $u^\pm = v \pm e \in \mathcal{U}$, then $u^+ - u^- = 2e \in \mathrm{cl}_4\mathcal{U}$.

(ii) $v(u) \in \mathcal{V}_4(\mathcal{U})$ belongs to $\mathrm{cl}_4\mathcal{U}$ if and only if $\pm e = \pm(u - v(u)) \in \mathrm{cl}_1\mathcal{U}$. □

Proposition 27 below completely characterizes $\mathcal{U}$ with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$. Recall that $\beta = \frac{1}{\sqrt{2}}$.

PROPOSITION 27. *Let $\mathcal{U}$ be a closed uniform odd system of norm* 5*. The following statements are equivalent.*

(i) *$\mathcal{U}$ contains three vectors u_1, u_2, u_3 such that $u_1u_2 = -1$, $u_1u_3 = u_2u_3 = 3$,*
(ii) $\mathrm{cl}_1\mathcal{U} \neq \emptyset$,

(iii) $\mathcal{U}$ *is pillar with shaft* e *and* $\mathrm{cl}_4\mathcal{U} = \mathcal{V}_4(\mathcal{U}) \cup \{\pm 2e\}$,
(iv) $\mathcal{U}$ *is pillar with shaft* e, $2e \in \mathrm{cl}_4\mathcal{U}$, *and* $\mathcal{V}_4(\mathcal{U})$ *contains a star.*

Proof. (i) $\Rightarrow$ (ii). It is easy to see that the vector $u_1 + u_2 - u_3$ has norm 1, i.e. $u_1 + u_2 - u_3 \in \mathrm{cl}_1\mathcal{U}$.

(ii) $\Rightarrow$ (iii). Since $e \in \mathrm{cl}_1\mathcal{U}$, $\mathcal{U}$ is pillar and clearly $\mathcal{V}_4(\mathcal{U}) \cup \{\pm 2e\} \subseteq \mathrm{cl}_4\mathcal{U}$. Let $v \in \mathrm{cl}_4\mathcal{U}$. Then $ve \in \{0, \pm 2\}$ (see Proposition 3). Let $ve = 2$. Then $(v-e)^2 = 1$, i.e. $v - e \in \mathrm{cl}_1\mathcal{U}$. This implies $v = 2e$. Hence, if $v \neq \pm 2e$, $ve = 0$. It follows that $u(v) = v + e \in \mathrm{cl}_5\mathcal{U}$. Since $\mathcal{U}$ is closed, $v + e = u$ for some $u \in \mathcal{U}$, and $ue = 1$, i.e. $v = v(u)$, where $v(u)$ is as in (3). Hence, $\{v(u) : u \in \mathcal{U}\} = \mathrm{cl}_4\mathcal{U} - \{\pm 2e\}$.

(iii) $\Rightarrow$ (iv). Since $\mathcal{V}_4(\mathcal{U}) \subseteq \mathrm{cl}_4\mathcal{U}$ and e is orthogonal to all $v(u)$, $\mathcal{V}_4(\mathcal{U})$ is closed. By Proposition 8, $\mathcal{V}_4(\mathcal{U})$ is a root system. Suppose that $\mathcal{V}_4(\mathcal{U})$ is a frame, i.e. $\mathcal{V}_4(\mathcal{U}) = \bigcup_i \{\pm 2e_i\}$. Then, by Proposition 26(i), $\mathcal{U} = \bigcup_i \{\pm(e \pm 2e_i)\}$. The inclusion $e \in L(\mathcal{U})$ implies

$$\begin{aligned} e &= \sum_i z_i^+(e + 2e_i) + \sum_i z_i^-(e - 2e_i) \\ &= \sum_i (z_i^+ + z_i^-)e + \sum_i (z_i^+ - z_i^-)2e_i. \end{aligned}$$

Since all e_i and e are mutually orthogonal, $z_i^+ - z_i^- = 0$. Hence, $z_i^+ = z_i^-$ and $2\sum_i z_i^+ = 1$. But this is a contradiction since z_i^+ is an integer. Now by Lemma 25, $\mathcal{V}_4(\mathcal{U})$ contains a star.

(iv) $\Rightarrow$ (i). Since $2e \in \mathrm{cl}_4\mathcal{U}$, by Proposition 26, $v \pm e \in \mathcal{U}$ for all $v \in \mathcal{V}_4(\mathcal{U})$. Let $\mathcal{K} = \{v_1, v_2, v_3\} \subseteq \mathcal{V}_4(\mathcal{U})$ be a star. Then the vectors $u_1 = e + v_1$, $u_2 = e + v_2$, $u_3 = e - v_3$ belong to $\mathcal{U}$. It is easy to see that these vectors satisfy (i). □

As we saw at the beginning of this section, any root system determines an odd system of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$. Since $\beta\mathcal{V}_4(\mathcal{U})$ is a root system, Proposition 26 implies the following characterization of odd systems of norm 5 with $\mathrm{cl}_1\mathcal{U} \neq \emptyset$.

THEOREM 28. *There is one-to-one correspondence between closed uniform odd systems* $\mathcal{U}$ *of norm* 5 *with* $\mathrm{cl}_1\mathcal{U} \neq \emptyset$ *and root systems* $\beta\mathcal{V}$. *This correspondence is such that if* $\mathrm{cl}_1\mathcal{U} = \{\pm e\}$, *then* $\mathrm{cl}_4\mathcal{U} = \{\pm 2e\} \cup \mathcal{V}$ *and* $\mathcal{U} = \{v \pm e : v \in \mathcal{V}\}$, *where* e *is orthogonal to the space spanning* $\mathcal{V}$.

Now we consider the case $\mathrm{cl}_1\mathcal{U} = \emptyset$ and distinguish the case when $\mathcal{V}_4(\mathcal{U})$ is closed.

PROPOSITION 29. *Let* $\mathcal{U}$ *be a closed uniform pillar odd system of norm* 5 *with shaft* e *and* $\mathrm{cl}_1\mathcal{U} = \emptyset$. *If* $\mathcal{V}_4(\mathcal{U})$ *is closed, then* $\mathcal{V}_4(\mathcal{U})$ *is a frame* $\bigcup_{i=1}^k \{\pm 2e_i\}$ *and exactly one of the following is true.*

(i) $\mathcal{U} = \bigcup_{i=1}^k \{\pm(e \pm 2e_i)\}$ *and* $\mathrm{cl}_4\mathcal{U} = \{\pm 2e\}$,
(ii) $\mathcal{U} = \bigcup_{i=1}^k \{\pm(e + 2e_i)\}$ *and* $\mathrm{cl}_4\mathcal{U} = \emptyset$.

Proof. If $\mathcal{V}_4(\mathcal{U})$ is closed, then $\beta\mathcal{V}_4(\mathcal{U})$ is a root system. According to Proposition 27(iv) and Lemma 25, the equality $\mathrm{cl}_1\mathcal{U} = \emptyset$ implies that $\mathcal{V}_4(\mathcal{U})$ is a frame, say $\mathcal{V}_4(\mathcal{U}) = \bigcup_1^k\{\pm 2e_i\}$.

For each $v \in \mathcal{V}_4(\mathcal{U})$, we have that at least one of the vectors $v+e$, $v-e$ belongs to $\mathcal{U}$.

If there is $v \in \mathcal{V}_4(\mathcal{U})$ such that $v+e, v-e \in \mathcal{U}$, then by Proposition 26, $\mathcal{U} = \bigcup_1^k\{\pm(e \pm 2e_i)\}$.

Conversely, if $\mathrm{cl}_4\mathcal{U} \neq \emptyset$ and $v \in \mathrm{cl}_4\mathcal{U}$, then $v = \sum_i z_i^+(e+2e_i) + \sum_i z_i^-(e-2e_i) = \sum_i(z_i^+ + z_i^-)e + \sum_i(z_i^+ - z_i^-)2e_i$ and $4 = v^2 = (\sum_i(z_i^+ + z_i^-))^2 + 4\sum_i(z_i^+ - z_i^-)^2$. This implies that $(z_i^+ - z_i^-)^2 = 0$, i.e. $z_i^+ = z_i^-$. Hence, $v = 2\sum_i z_i^+ e = \pm 1$, i.e. $v = \pm 2e$.

Now let $\mathrm{cl}_4\mathcal{U} = \emptyset$. Then the above argument implies that only one vector from $v \pm e$ belongs to $\mathcal{U}$. Since $-u \in \mathcal{U}$ if $u \in \mathcal{U}$ and $-v \in \mathcal{V}_4(\mathcal{U})$ if $v \in \mathcal{V}_4(\mathcal{U})$, we can suppose, reversing the sign of v if necessary, that $v + e \in \mathcal{U}$ for all $v \in \mathcal{V}_4(\mathcal{U})$. This means that $\mathcal{U} = \bigcup_{i=1}^k\{\pm(2e_i + e)\}$. □

The case of pillar $\mathcal{U}$ with $\mathrm{cl}_1\mathcal{U} = \emptyset$ and $\mathcal{V}_4(\mathcal{U})$ not closed is much more complicated. In this case, $\beta\mathcal{V}_4(\mathcal{U})$ is a set of roots but not a root system which is closed.

8. Closed Odd Systems of Norm 5 Spanning Equiangular Lines

Let $\mathcal{U}$ be a closed uniform odd system of norm 5 such that $uu' = \pm 1$ for distinct $u, u' \in \mathcal{U}$. Since $\mathrm{cl}_3\mathcal{U} = \emptyset$ by Corollary 14, and $\mathrm{cl}_4\mathcal{U} = \emptyset$ by Lemma 18, Theorem 17 can be reformulated as follows:

THEOREM 30. *Let $\mathcal{U}$ be a set of vectors. The following assertions are equivalent:*

(i) $\mathcal{U} = \mathcal{U}_5(v_0, L)$ *for a double even lattice L of minimal norm* 8 *with $v_0 \in L$ of norm* 12,

(ii) $\mathcal{U}$ *is a closed uniform odd system of norm* 5 *such that $uu' = \pm 1$ for $u, u' \in \mathcal{U}$, $u \neq \pm u'$, and $\mathrm{cl}_1\mathcal{U} = \emptyset$, i.e. $\mathcal{U}$ spans a set of equiangular lines at angle* $\arccos\frac{1}{5}$.

As an example, we consider regular uniform odd systems of norm 5 spanning equiangular lines. These systems are in one-to-one correspondence with regular two-graphs with minimal eigenvalue -5. In Table II below we give dimensions for which such regular two-graphs are known (see [11]).

In each dimension, we know only one closed odd system. This closed system corresponds to the two-graph with doubly-transitive automorphism group. The corresponding lattices $L(\mathcal{U})$, $L_0(\mathcal{U})$ and $L_2(\mathcal{U})$, which can be identified up to a multiple γ with known lattices, are given in Table II. The Q-lattices (excluding Q_{10}) are described in [4]. The lattice Q_{10} is mentioned in [6]. The lattices Λ_{16} and

Table II.

$\dim \mathcal{U}$	$2t+1$	5	10	13	15	21	22	23
$n(\mathcal{U})$	$2t$	6	16	26	36	126	176	276
$\gamma L(\mathcal{U})$	$A^*_{2t+1} = A^{+(2t+2)}_{2t+1}$	$A^*_5 = A^{+6}_5$		$Q_{13}(2)^{+2}$				$Q_{23}(6)^{+2}$
$\gamma L_0(\mathcal{U})$	$A^{+(t+1)}_{2t+1}$	A^{+3}_5	Q_{10}	$Q_{13}(2)$				$Q_{23}(6)$
$\gamma L_2(\mathcal{U})$					Λ_{16}			Λ_{24}

Λ_{24} are the Barnes–Wall and Leech lattices, respectively (see [3]). For $\dim \mathcal{U} = 5$ we have $\gamma = 1/\sqrt{6}$, while $\gamma = \beta = 1/\sqrt{2}$ for other values of $\dim \mathcal{U}$.

The lattice $\gamma L_2(\mathcal{U})$ for $\dim \mathcal{U} = 10$ is a sublattice of the Barnes–Wall lattice Λ_{16}. Similarly, the lattices $\gamma L_2(\mathcal{U})$, for $\dim \mathcal{U} = 21$ and 22, are sublattices of the Leech lattice Λ_{24}.

If $\dim \mathcal{U} = 5$, then $\mathcal{U}$ is a star. Let $\mathcal{U}(t)$ denote the star of norm $2t+1$. In Table II we include lattices related to $\mathcal{U}(t)$ as well. For $\mathcal{U} = \mathcal{U}(t)$, $\gamma = 1/\sqrt{2t+2}$.

Baranovskii in [1] studies the L-polytope affinely spanned by $\mathcal{U}(t)$ and the lattice $\gamma L_0(\mathcal{U}(t))$. In particular, he proves that the lattice $\gamma L_0(\mathcal{U}(4))$ has minimal covering radius among all known integral lattices of dimension 9.

Note that other regular uniform odd systems with inner product ± 1 (equivalently, regular two-graphs) are known only for dimensions 13 and 15. There are exactly three nonisomorphic systems of dimension 13. With the aid of a computer, E. Spence [12] discovered 227 nonisomorphic regular two-graphs on 36 points. These two-graphs are represented by odd systems of norm 5 and dimension 15. Hence, one knows 227 systems of dimension 15. These odd systems are partitioned into classes of systems having the same closure. Moreover, according to Lemma 18, $\mathrm{cl}_4 \mathcal{U} \neq \emptyset$ for all these systems $\mathcal{U}$. Hence, odd systems with distinct closures are distinguished by the root system $\beta\, \mathrm{cl}_4 \mathcal{U}$. Following [3], we denote below by A^k_n the direct sum of k copies of the root system A_n.

The three odd systems of dimension 13 have two closures with $\beta\, \mathrm{cl}_4 \mathcal{U} = A_{12}$ and $\beta\, \mathrm{cl}_4 \mathcal{U} = A^3_4$.

We find 10 different closures of odd systems $\mathcal{U}$ of dimension 15 (with $\mathrm{cl}_4 \mathcal{U} \neq \emptyset$) (see [8] and [9]):

for $\mathcal{U}$ related to Latin squares, $\beta\, \mathrm{cl}_4 \mathcal{U} = A^3_5$, and A^9_1,

for $\mathcal{U}$ related to Steiner triple systems, $\beta\, \mathrm{cl}_4 \mathcal{U} = A_{14},\ A^7_1,\ A_6A_7,\ A_2A^3_3$,
for $\mathcal{U}$ discovered by E. Spence (see [11]), $\beta\, \mathrm{cl}_4 \mathcal{U} = A^7_1,\ A^9_1 D_4,\ A_1A_3D_4D_6,\ D_7E_7,\ A_5D_{10}$.

These odd systems are described in detail in [8] and [9].

References

1. Baranovskii, E. P.: Perfect lattices $\Gamma(\mathcal{A}^n)$, and covering density of $\Gamma(\mathcal{A}^9)$, *Europ. J. Combin.* **15** (1994), 317–323.

2. Cameron, P. J., Goethals, J. M., Seidel, J. J. and Shult, E. E.: Line graphs, root systems, and elliptic geometry, *J. Algebra* **43** (1976), 305–327.
3. Conway, J. H. and Sloane, N. J. A.: *Sphere Packing, Lattices and Groups*, Grundl. Math. Wiss. 290, Springer-Verlag, Berlin, 1987.
4. Conway, J. H. and Sloane, N. J. A.: Low-dimensional lattices. II. Subgroups of $\mathrm{GL}(n, Z)$, *Proc. Roy. Soc. London A* **419** (1988), 29–68.
5. Conway, J. H. and Sloane, N. J. A.: The cell structures of certain lattices, in: P. Hilton *et al.* (eds), *Miscellanea Mathematica*, Springer-Verlag, 1991, pp. 71–107.
6. Conway, J. H. and Sloane, N. J. A.: On lattices equivalent to their duals, *J. Number Theory* **48** (1994), 373–382.
7. Deza, M. and Grishukhin, V. P.: L-polytopes and equiangular lines, *Discrete Appl. Math.* **55** (1995), 181–214.
8. Deza, M. and Grishukhin, V. P.: Two-graphs on 36 points from the even unimodular lattices $E_8 \oplus E_8$ and D_{16}^+, Rapport de Recherche LIENS-95-9, Ecole Normale Supérieure, Paris, 1995.
9. Grishukhin, V. P.: Regular two-graphs from the even unimodular lattice $E_8 \oplus E_8$, Preprint 95–108, Universität Bielefeld, 1995.
10. Neumaier, A.: On norm three vectors in integral Euclidean lattices, *Math. Z.* **183** (1983), 565–574.
11. Seidel, J. J.: More about two-graphs, in: J. Nesetril and M. Fiedler (eds), *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity*, Ann. Discrete Math., Elsevier, Amsterdam, 1992, pp. 297–308.
12. Spence, E.: Regular two-graphs on 36 vertices, *Linear Algebra Appl.* **226–228** (1995), 459–497.
13. Shult, E. and Yanushka, A.: Near n-gons and line systems, *Geom. Dedicata* **9** (1980), 1–72.