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A Generalization of Strongly Regular Graphs

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Abstract. Motivated from an example of ridge graphs relating to metric polytopes, a class of connected regular graphs such that the squares of their adjacency matrices are in certain symmetric Bose-Mesner algebras of dimension 3 is considered in this paper as a generalization of strongly regular graphs. In addition to analysis of this prototype example defined over $(MetP_5)^*$, some general properties of these graphs are studied from the combinatorial view point.

Keywords: strongly regular graphs, quasi-symmetric designs, ridge graphs

1. Introduction

The notion of ridge graphs is introduced in [3] for studying metric polytopes $MetP_n$ and their relatives. The complement of one of those is interesting to us in this paper. Indeed, after some modifications, a connected regular graph is found such that the square of its adjacency matrix lies in a symmetric Bose-Mesner algebra of dimension 3, though itself is not strongly regular. It therefore leads to another generalization of strongly regular graphs besides the notion of distance-regular graphs.

A connected simple graph G is called *strongly regular* with parameters v, k, λ, μ, denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$, if it consists of v vertices such that

$$|G(x) \cap G(y)| = \begin{cases} k & \text{if } x = y, \\ \lambda & \text{if } x, y \text{ are adjacent}, \\ \mu & \text{otherwise}, \end{cases}$$

where $G(x) = \{z \mid z \in V(G) \text{ is adjacent to } x\}$. This condition can be restated in terms of its adjacency matrix A as $A^2 = kI + \lambda A + \mu(J - I - A)$. As a matter of fact, its *adjacency algebra* is a symmetric Bose-Mesner algebra of dimension 3. A subspace $\mathscr{A} \subseteq M_n(\mathbf{C})$ of symmetric matrices is called a *symmetric Bose-Mesner algebra* if $I, J \in \mathscr{A}$, and $\mathscr{A}$ is closed under both the ordinary product and Hadamard product of matrices, and under the conjugate transposing as well. It is known that each symmetric Bose-Mesner algebra has a basis $\{A_i \mid 0 \le i \le d\}$ consisting of $(0,1)$-matrices such that $A_0 = I$, $\sum_{k=0}^{d} A_i = J$, and $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for suitable parameters p_{ij}^k, $0 \le i, j, k \le d$. This leads to combinational structures such as *symmetric association schemes*, and, in particular, P-polynomial association schemes, i.e., *distance-regular graphs* (refer to [1, 2] for more detail).

Some classical combinatorial structures were defined in terms of their incidence matrices through Bose-Mesner algebras. Amongst many others, recall that a $2-(v,k,\lambda)$ design $\Pi = (X, \mathscr{B})$ is called *quasi-symmetric* with sizes of intersections α, β if $M^t M = kI + \alpha A + \beta(J - I - A)$ where A is the square $(0,1)$-matrix indexed by $\mathscr{B} \times \mathscr{B}$, such that $A(B_1, B_2) = 1$ if and only if $|B_1 \cap B_2| = \alpha$. It is worth noting that $M^t M$ lies in the Bose-Mesner algebra of dimension 3 generated by A. One of the purposes of this paper is to study those $(0,1)$-matrices M, in particular those symmetric ones, such that $M^t M$ lie in symmetric Bose-Mesner algebras of dimension 3. Necessary background regarding ridge graphs is given in Section 2. As a prototype example, the complement Γ_5 of the ridge graph of metric polytopes is given in Section 3 in detail. Amongst other things, we show that its adjacency matrix lies in an imprimitive symmetric Bose-Mesner algebra of rank 5. The observations made in Section 3 leads to a generalization of strongly regular graphs in Section 4.

2. The Complement of Ridge Graphs

The notion of ridge graphs was considered in [3] for studying metric polytopes. We will recall this notion for completeness, and we then correct some misprints found in [3].

Let us prepare some notation first. For each n, define vectors u_{ijk}, v_{ijk} in the vector space $\mathbf{R}^{\binom{n}{2}}$ of dimension $\binom{n}{2}$ over real numbers indexed by $\{(p,q) \mid 1 \le p < q \le n\}$ such that

i) the (p,q)-entry of u_{ijk} $(1 \le i < j < k \le n)$ is given by

$$u_{ijk}(p,q) = \begin{cases} 1 & \text{if } (p,q) = (i,j), (i,k), \text{ or } (j,k), \\ 0 & \text{otherwise;} \end{cases}$$

ii) the (p,q)-entry of v_{ijk} $(1 \le i < j \le n, k \ne i, j)$ is given by

$$v_{ijk}(p,q) = \begin{cases} 1 & \text{if } (p,q) = (i,j), \\ -1 & \text{if } \{p,q\} = \{i,k\} \text{ or } \{j,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

The graph Γ_n is defined over the vectors

$$\{u_{ijk} \mid 1 \le i < j < k \le n\} \cup \{v_{ijk} \mid 1 \le i < j \le n, k \ne i, j\}$$

such that two vectors a, b are adjacent if there is an index (p, q) such that the (p, q)-entries of a and b are nonzero and their sum $a_{(p,q)} + b_{(p,q)}$ is zero. Note that Γ_3 is the complete graph K_4 of 4 vertices; and Γ_4 is a strongly regular graph $SRG(16, 9, 4, 6)$, and its complement $\overline{\Gamma_4}$ is the (4×4)-grid, *i.e.*, $L(K_{4,4})$, which is a strongly regular graph $SRG(16, 6, 2, 2)$.

Indeed, those *ridge graphs* in [3] are the complements of these graphs Γ_n $(n \ge 3)$. It is worth mentioning here that the ridge graph $\overline{\Gamma_n}$ is the skeleton of the dual metric polytope $(MetP_n)^*$, where $MetP_n$ denotes the full dimensional polytope in $\mathbf{R}^{\binom{n}{2}}$ defined by the inequalities

$$\langle x, v_{ijk} \rangle = x_{ij} - x_{ik} - x_{jk} \le 0,$$
$$\langle x, u_{ijk} \rangle = x_{ij} + x_{ik} + x_{jk} \le 2.$$

The first $3\binom{n}{3}$ homogeneous inequalities above define the cone $MetP_n$ of all semi-metrics on n points, and its dual cone is the cone of feasible multicommodity flows. The graph $\overline{\Gamma_n}$ is also the edge-graph of the dual metric polytope $(MetP_n)^*$, *i.e.*, the *ridge* (subfacet, co-edge) graph of $MetP_n$.

Proposition 2.1. ([3], Theorem 2.2) *For $n \ge 4$, the graph Γ_n is locally the bouquet of $(n-3)$ copies of (3×3)-grids with a common K_3 having parameters $v = 4\binom{n}{3}$, $k = 3(2n-5)$, $\lambda = 2(n-2)$ or 4, and*

$$\mu = \begin{cases} 2(n-1), 4, \text{ or } 0 & \text{if } n \ge 6, \\ 2(n-1), 4 & \text{if } n = 5. \end{cases}$$

Proposition 2.2. ([3], p. 362) *The ridge graph $\overline{\Gamma_n}$ is regular with valency $\frac{2(n-3)(n^2-7)}{3}$, and any non-adjacent pair of vertices have either $\frac{2(n-3)(n^2-13)}{3}$ or $\frac{2(n-3)(n^2-16)}{3} + 2$ common neighbors.*

Note that Proposition 2.1 and 2.2 correct some misprints found in Theorem 2.2 and in [3, p. 362]. Corrections for two other misprints found in [3] also given below for later reference: For $n \ge 4$, the ridge graph G'_n ([3], Theorem 2.9) of the metric cone Met_n is locally the bouquet of $(n-3)$ hexagons with a common edge having parameters $v = 3\binom{n}{3}$, $k = 2(2n-5)$, $\lambda = n-2$ or 2, and

$$\mu = \begin{cases} 2n-4, n, n-1 \text{ or } 0 & \text{if } n \ge 5, \\ 2n-4, n, n-1 & \text{if } n = 4. \end{cases}$$

The parameters for G'_n are the valency $\frac{(n-3)(n^2-6)}{2}$, and $\mu = \frac{(n-3)(n^2-12)}{2}$ or $\frac{(n-3)(n^2-14)}{2} + 1$ (line +10 in [3, p. 364]). Lines $-5 \sim -8$ in [3, p. 362] should be deleted completely.

Amongst these graphs, Γ_5 is of particular interest to us, its details will be given in the next section as a prototype for a generalization of strongly regular graphs.

3. The Prototype

Amongst the family $\{\Gamma_n \mid n \geq 4\}$ of graphs defined in the previous section, the graph Γ_5 is the focus of this section. A maximal clique partition of the vertex set of Γ_5 is given, it then leads to an imprimitive symmetric Bose-Mesner algebra of dimension 5 containing the adjacency matrix of Γ_5.

Consider the following partition of the vertex set of Γ_5 into $\{X_i \mid 1 \leq i \leq 10\}$, where

$$\begin{aligned} X_1 &= \{123, 12.3, 13.2, 23.1\}, & X_6 &= \{145, 14.5, 15.4, 45.1\}, \\ X_2 &= \{124, 12.4, 14.2, 24.1\}, & X_7 &= \{234, 23.4, 24.3, 34.2\}, \\ X_3 &= \{125, 12.5, 15.2, 25.1\}, & X_8 &= \{235, 23.5, 25.3, 35.2\}, \\ X_4 &= \{134, 13.4, 14.3, 34.1\}, & X_9 &= \{245, 24.5, 25.4, 45.2\}, \\ X_5 &= \{135, 13.5, 15.3, 35.1\}, & X_{10} &= \{345, 34.5, 35.4, 45.3\}, \end{aligned}$$

and ijk corresponds to u_{ijk}, $ij.k$ corresponds to v_{ijk} respectively for suitable i, j and $k \leq 5$. Note that each X_i gives a maximal clique of Γ_5 and there are no others. Let A_{ij} be the adjacency matrix of Γ_5 with respect to X_i (in rows) and X_j (in columns) in the orders as given respectively. Then, for each pair (i, j), A_{ij} is one of the following:

1. $\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$ for $(i, j) = (1,6), (1,9), (1,10), (2,5), (2,8), (2,10), (3,4), (3,7), (3,10), (4,8), (4,9), (5,7), (5,9), (6,7), (6,8)$ and their transposes;

2. $\begin{bmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{bmatrix}$ for $1 \leq i = j \leq 10$;

3. $\begin{bmatrix} 0&0&1&1 \\ 0&0&1&1 \\ 1&1&0&0 \\ 1&1&0&0 \end{bmatrix}$ for $(i, j) = (1,2), (1,3), (2,3), (4,5), (7,8)$ and their transposes;

4. $\begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{bmatrix}$ for $(i, j) = (2,4), (3,5), (3,6), (5,6), (8,9)$ and their transposes;

5. $\begin{array}{|cccc|}\hline 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0\\ \hline\end{array}$ for $(i,j) = (4,7),(5,8),(6,9),(6,10),(9,10)$ and their transposes;

6. $\begin{array}{|cccc|}\hline 0&0&1&1\\ 1&1&0&0\\ 0&0&1&1\\ 1&1&0&0\\ \hline\end{array}$ for $(i,j) = (1,4),(1,5),(2,6),(4,6),(7,9)$;

7. $\begin{array}{|cccc|}\hline 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0\\ 0&0&1&1\\ \hline\end{array}$ for $(i,j) = (1,7),(1,8),(2,9),(4,10),(7,10)$;

8. $\begin{array}{|cccc|}\hline 0&1&0&1\\ 1&0&1&0\\ 1&0&1&0\\ 0&1&0&1\\ \hline\end{array}$ for $(i,j) = (2,7),(3,8),(3,9),(5,10),(8,10)$;

Moreover, A_{ij} is the transpose of matrix 6 if $(i,j) = (4,1),(5,1),(6,2),(6,4),(9,7)$; the transpose of matrix 7 if $(i,j) = (7,1),(8,1),(9,2),(10,4)$ and $(10,7)$; finally the transpose of matrix 8 if $(i,j) = (7,2),(8,3),(9,3),(10,5)$ and $(10,8)$. Indeed, $A = [A_{ij}]_{1 \le i,j \le 10}$ forms an adjacency matrix of the graph Γ_5. Some information of the graph Γ_5 can be derived from the relative positions of the above eight 4×4 matrices in its adjacency matrix A; for example, the positions for those 30 copies of 4×4 zero matrix can be kept record as the Petersen graph, which gives the clique graph of Γ_5.

Proposition 3.1. *The clique graph of* Γ_5 *is the Petersen graph.*

Some matrices can be derived from the matrix A so that they turn out to be the adjacency matrices for a rank 5 imprimitive association scheme. Let

$D = [A_{ii}]_{1 \le i \le 10}$; i.e., the diagonal-like matrix with 10 copies of matrix 2), *i.e.*, $J_4 - I_4$ of order 4, along its main diagonal, and zero elsewhere;

$M = A - D$;

$N =$ the matrix obtained from M by interchanging $0, 1$ in its nonzero blocks A_{ij};

and finally,

$$S = J - (I + D + M + N).$$

Note that all these four matrices are symmetric. The following lemma can be checked easily following some computer work.

Lemma 3.2. *The ordinary products among* $\{I, D, M, N, R\}$ *are given in the following table:*

$\bullet$	I	D	M	N	S
I	I	D	M	N	S
D	D	$3I+2D$	$M+2N$	$2M+N$	$3S$
M	M	$M+2N$	$12I+2M+4(J-I-M)$	$8D+4M+2N+4S$	$4M+4N+4S$
N	N	$2M+N$	$8D+4M+2N+4S$	$12I+2M+4(J-I-M)$	$4M+4N+4S$
S	S	$3S$	$4M+4N+4S$	$4M+4N+4S$	$12I+12D+4M+4N$

The Lemma 3.2 shows that the vector space spanned by $\{I, D, M, N, S\}$ over complex numbers is closed under the ordinary product of matrices, and hence it carries the structure of a symmetric Bose-Mesner algebra. The following theorem follows immediately from the fact that

$$\begin{aligned} A &= D + M; \\ A^2 &= 4J + 11I + 2D + 4N; \\ A^3 &= 76J - 10I + 31D + 21M \quad \text{and} \\ A^4 &= 1224J + 261I + 52D + 104N. \end{aligned}$$

Theorem 3.3. *The adjacency algebra of the graph* Γ_5 *is the symmetric Bose-Mesner algebra of dimension* 5 *generated by I, D, M, N and S.*

As a consequence, $\{I, D, M, N, R\}$ form the adjacency matrices of a symmetric association scheme of 4 classes. Note also that

$$N^2 = M^2 = 12I + 2M + 4(J - I - M) \tag{$*$}$$

in the table, which can be interpreted in terms of the notion of strongly regular graphs. Let $G(M)$ be the graph with M as its adjacency matrix, then $G(M)$ is a 12-regular graph such that each pair of vertices have 2 or 4 common neighbors depending on whether they are adjacent or not, and hence $G(M)$ is a strongly regular graph SRG$(40, 12, 2, 4)$ isomorphic to the polar graph $O_5(3)$ as defined below. Let W be a vector space of dimension 5 over $GF(3)$ equipped with an orthogonal form

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The polar graph $O_5(3)$ is defined on the set of all 1-dimensional isotropic subspaces of W, and two vertices x, y are adjacent if and only if x and y span an isotropic subspace of dimension 2.

Corollary 3.4. *The graph* $G(M)$ *isomorphic to the polar graph* $O_5(3)$ *is a strongly regular graph* $SRG(40, 12, 2, 4)$ *and hence its adjacency algebra is a symmetric Bose-Mesner algebra of dimension* 3 *generated by* $\{I, M, J - I - M\}$.

The equation $(^*)$ also shows that the graph $G(N)$ with adjacency matrix N is 12-regular such that each pair of vertices has either 2 or 4 common neighbors depending on whether they are adjacent in the graph $G(M)$, rather than in $G(N)$ itself. Hence $G(N)$ shares similar properties posed for strongly regular graphs, though not strongly regular. This observation lead to a generalization of strongly regular graphs.

4. Quasi Strongly Regular Graphs

Distance-regular graphs can be seen as a generalization of strongly regular graphs by releasing diameter bound. Following the prototype example Γ_5 and the two associated graphs $G(M)$ and $G(N)$ given in Section 3, another generalization of strongly regular graphs is considered in this section.

Let us give the definition first, and we then consider this generalization in a broader sense. A connected graph Γ with adjacency matrix M is called *quasi strongly regular* if

$$M^2 = kI + \lambda A + M(J - I - A)$$

for some symmetric $(0, 1)$-matrix A where $\langle I, J, A\rangle$ form a Bose-Mesner algebra of dimension 3. For example, the graph $G(N)$ is such a proper example.

Let us now consider this generalization in terms of incidence matrices of some incidence structures. Let M be a $(0, 1)$-matrix of order $v \times b$, indexed by $X \times \mathscr{B}$, such that $MJ = rJ$, $JM = kJ$, and

$$M^tM = kI + xA + y(J - I - A).$$

This matrix M can be interpreted as an incidence matrix of a k-uniform, r-regular incidence structure $\Pi = (X, \mathscr{B})$ such that any pair of distinct blocks have either x or y points in common. It is interesting to note that some classical combinatorial structures can be derived by posing some additional conditions over MM^t. For example, M is an incidence matrix of a *quasi symmetric* $2 - (v, k, \lambda)$ design with sizes x and y of intersection between any two blocks if $MM^t = rI + \lambda(J - I)$, and then as a consequence, both MM^t and M^tM lie in a symmetric Bose-Mesner algebra of dimension 3 generated by I, J and A. The above observation for $(0, 1)$-matrices and designs provides a motivation for studying those matrices $M \in \mathscr{M}_{n\times m}(\mathbf{C})$ with constant column and row sums such that either M^tM or MM^t lie in some Bose-Mesner algebras together with their potential combinatorial structures.

Let M be a $(0,1)$-matrix indexed by $X \times \mathscr{B}$, such that $MJ = rJ$, $JM = kJ$ and

$$\begin{aligned} M^tM &= (k-\beta)I + (\alpha-\beta)A + \beta J \\ &= kI + \alpha A + \beta(J - I - A), \end{aligned} \tag{$*$}$$

and

$$\begin{aligned} MM^t &= (r-\mu)I + (\lambda-\mu)B + \mu J \\ &= rI + \lambda B + \mu(J - I - B) \end{aligned} \tag{$**$}$$

for distinct α, β and for some square matrices A (indexed by $\mathscr{B} \times \mathscr{B}$) and B (indexed by $X \times X$). Two graphs can be associated with M naturally. The one with B as an incidence matrix is called the *point graph* of Π, the other with A as an incidence matrix is called the *block graph* of Π. Note that the graphs with $J - I - B$ or $J - I - A$ as their adjacency matrices are simply the complements of those under consideration.

Lemma 4.1. *Let $\Pi = (X, \mathscr{B})$ be an incidence structure with an incidence matrix M satisfying conditions* $(^*)$ *and* $(^{**})$*, then its block graph is strongly regular if and only if M^tMA (and hence M^tBM) belongs to the algebra $\langle I, J, A\rangle$ generated by I, J and A.*

Proof. Since

$$A = \frac{1}{(\alpha-\beta)}\big(M^tM - (k-\beta)I - \beta J\big),$$

then

$$A^2 = \frac{1}{(\alpha-\beta)^2}\big(M^tMM^tM - (k-\beta)^2 I + (\beta^2 b - 2k\beta r)J - 2(k-\beta)(\alpha-\beta)A\big) \tag{*}$$

where $b = |\mathscr{B}|$. Substituting

$$\begin{aligned} &M^tM(M^tM) \\ &\quad = (\alpha-\beta)M^tMA + (k-\beta)^2 I + \big(\beta(k-\beta) + \beta rk\big)J + (\alpha-\beta)(k-\beta)A, \\ &M^t(MM^t)M \\ &\quad = (\lambda-\mu)M^tBM + (r-\mu)(k-\beta)I + \big(\beta(r-\mu) + \mu k^2\big)J + (\alpha-\beta)(r-\mu)A, \end{aligned}$$

into $(^*)$, we have

$$A^2 = \frac{1}{(\alpha-\beta)^2}\big((\alpha-\beta)M^tMA + \beta(k-\beta-kr) + \beta^2 b - (\alpha-\beta)(k-\beta)A\big)$$
$$= \frac{1}{(\alpha-\beta)^2}\big((\lambda-\mu)M^tBM + \big((r-\mu)(k-\beta) - (k-\beta)^2\big)I$$
$$+ \big(\beta(r-\mu) + \mu k^2 + (\beta^2 b - 2k\beta r)\big)J$$
$$+ \big((\alpha-\beta)(r-\mu) - 2(k-\beta)(\alpha-\beta)\big)A\big).$$

It follows that $A^2 \in \langle I, J, A\rangle$ if and only if M^tMA (and hence M^tBM) $\in \langle I, J, A\rangle$. ∎

Note also that the entry of the matrix M^tMA at (B_1, B_2) is the number of flags (x, C) such that $x \in B_1 \cap C$ and $C \cap B_2 \neq \emptyset$; and $M^tBM(B_1, B_2)$ is the number of adjacent pairs (x, y) with $x \in B_1$ and $y \in B_2$. Theorem 4.2 is an immediate consequence of Lemma 4.1.

Theorem 4.2. *Let $\Pi = (X, \mathscr{B})$ be an incidence structure with an incidence matrix M satisfying conditions $(^*)$, $(^{**})$, and $MB \in \langle M, J\rangle$ (or equivalently $AM \in \langle M, J\rangle$), then $A^2 \in \langle I, J, A\rangle$, $B^2 \in \langle I, J, B\rangle$ and hence both the point graph and the block of graph Π are strongly regular.*

For a symmetric M, $M^tM = MM^t = M^2 = kI + \lambda A + \mu(J - I - A)$, and hence M can be interpreted as an adjacency matrix of a k-regular graph such that any two distinct vertices has λ or μ common neighbors. Such a graph is called a *Deza graph* in [4] if, in addition, it is connected. As a corollary of Theorem 4.2, we have

Corollary 4.3. *If Γ is a connected simple graph with an adjacency matrix M such that $M^2 = kI + \lambda A + \mu(J - I - A)$ with MA or AM lying in the space $\langle M, J\rangle$ spanned by M and J, then Γ is quasi strongly regular.*

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