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Nonrigidity Degrees of Root Lattices and their Duals

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Abstract. The nonrigidity degree of a lattice L, nrd$L$, is the dimension of the L-type domain to which L belongs. We complete here the table of nrd's of all irreducible root lattices and their duals (we give also the minimal rank of their Delaunay polytopes). In particular, the hardest remaining case of D_n^*, and the case of E_7^* are decided. As any root lattice is a direct sum of some irreducible ones, its nrd is a sum of nrd's of the summands. We describe explicitly the L-type domain $\mathcal{D}(D_n^*)$, $n \geqslant 4$. For n odd, it is a nonsimplicial, polyhedral, open cone of dimension n. For n even, it is one-dimensional, i.e. any D_{2m}^* corresponds to an edge form.

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1. Introduction

Voronoi [Vo1908] defined the partition of the cone $\mathcal{P}_n$ of positive semidefinite n-ary quadratic forms into L-type domains, which we call here L-*domains*. Forms of the same L-domain correspond to lattices that determine affinely equivalent *Voronoi partitions* of $\mathbf{R}^n$, i.e. partition into Voronoi polytopes. The partition of $\mathbf{R}^n$, which is (in both, the combinatorial and affine sense) dual to a Voronoi partition, is called the *Delaunay partition* and consists of Delaunay polytopes. In other words, two lattices have the same L-type if and only if the face posets of their Voronoi polytopes are isomorphic. (The *face poset* of a polytope P is the set of all faces of P of all dimensions ordered by inclusion.)

If the L-type of a lattice changes, then either some Delaunay polytopes are glued together to form a new Delaunay polytope, or a Delaunay polytope is partitioned into several new Delaunay polytopes. Recall that the center of a Delaunay polytope is a vertex of a Voronoi polytope. Hence, if the L-type of a lattice changes, then for each Voronoi polytope either some vertices are glued together into one vertex, or a vertex splits into several new vertices.

Voronoi proved that each L-domain is an open polyhedral cone of dimension k, $1 \leqslant k \leqslant N$, where $N = (n(n+1))/2$ is the dimension of $\mathcal{P}_n$, i.e. the dimension of the space of coefficients of $f \in \mathcal{P}_n$. An L-type having an N-dimensional domain is called *generic*. Otherwise, the L-type is called *special*.

In [BG01], a notion of *nonrigidity degree* of a form f and the corresponding lattice $L(f)$ is introduced. It is denoted by nrd$f$ and is equal to the dimension of the L-domain containing f. It is shown in [BG01] that nrd$f$ is equal to the corank of a system of equalities connecting the norms of minimal vectors of cosets $2L$ in L. Roughly speaking, nrd is the number of degrees of freedom of a Delaunay partition (more precisely, its star) to deform it affinely, while remaining a Delaunay partition.

If L is a direct sum of some lattices, then the above mentioned system is partitioned naturally into independent subsystems, each corresponding to a summand lattice of the direct sum. This implies that the nrd of a direct sum is the sum of nrd's of the summands.

Clearly, the maximal nrd is $\binom{n+1}{2}$ and it is realized only by simplicial Delaunay partitions. So, $1 \leqslant \text{nrd} f \leqslant N$, and $\text{nrd} f = N$ if f belongs to a generic L-domain. A form f and the corresponding lattice $L(f)$ are called *rigid* if $\text{nrd} f = 1$. This name was used because any affine transformation of a rigid lattice (apart from a scaling) changes its L-type. (Sometimes, a rigid form is called an *edge form*, since it lies on an extreme ray of the closure of an L-domain.) Clearly, any one-dimensional lattice is rigid. [BG01] all seven rigid lattices of dimension 5 were given and it was shown that the D_4 is the unique such lattice of dimension n, $2 \leqslant n \leqslant 4$.

The set of all L-domains is partitioned into classes of unimodularly equivalent domains. For $n \leqslant 3$, there is only one class, i.e. only one generic L-type. For $n = 4$, there are 3 generic L-types.

The L-domain $\mathcal{A}_n$ of the lattice A_n^*, which is dual to the root lattice A_n is well known. $\mathcal{A}_n$ has dimension N, i.e. it is generic. It is the unique L-domain, such that all the extreme rays of its closure are spanned by forms of rank 1. All these facts were known to Voronoi. He called the domain $\mathcal{A}_n$ *the first type domain* and one of the forms of A_n^* *the principal form of the first type*.

The L-domain of the lattice A_n, $n \geqslant 2$, is a simplicial $(n+1)$-dimensional cone; it is described in [BG01]. So, $\text{nrd} A_n = n + 1$ for $n \geqslant 2$. Also $\text{nrd}\, Z_n = n$ for $n \geqslant 1$ and A_1, D_2, D_3 are scalings of Z_1, Z_2, A_3, respectively. So, $\text{nrd}(A_1 = A_1^*) = 1$, $\text{nrd}(D_2 = D_2^*) = 2$ and $\text{nrd}\, D_3 = 4$, $\text{nrd}\, D_3^* = 6$.

It is proved in [BG01] that the lattice D_n is rigid for $n \geqslant 4$.

The lattices E_6, E_6^*, E_7, E_7^* and $E_8 = E_8^*$ are rigid, i.e. their L-domains are one-dimensional. The rigidity of root lattices E_6, E_7 comes from [DGL92] and for E_8 it is shown in [BG01]. The rigidity of the lattice E_6^* was proved independently by Engel and Erdahl (personal communications). The rigidity of E_7^* can be proved easily by using a nice symmetric quadratic form of E_7^* given in formula (2) of [Ba94] (the lattice denoted by $\Gamma(\mathcal{A}^7)$ in [Ba94] is, in fact, the lattice E_7; this fact is not noted there).

The rigid lattices $A_1 = A_1^*$, E_6 and E_7 are first instances of *strongly rigid* lattices, i.e. such that amongst their Delaunay polytopes some are *extreme* (in the sense of [DGL92]). An extreme Delaunay polytope is a polytope, such that any its affine transformation, apart from a scaling, yields a result that is not a Delaunay polytope. The 1-simplex and unique Delaunay polytope of E_6 are the only such polytopes of dimension at most 6 (see [DD01]). A_1, $D_4 = D_4^*$, E_6, E_6^* and E_7^* are rigid lattices, having a unique type of Delaunay polytope, but only for A_1 and E_6 this polytope is extreme. In [DGL92] (see also Chapter 16 of [DL97]) 10 examples of extreme Delaunay polytopes were given: one of dimension 1, 6, 7, 22, 23 and 3, 2 of dimension 15, 16, respectively. M. Dutour (private communication in February 2003) found such (79-vertex asymmetric one) polytope of dimension 8 and, later, an infinity of extreme Delaunay polytopes. Moreover, in [DGL92] (see also Chapter 15 of [DL97]) the general notion of *rank* of a Delaunay polytope was considered, which is, roughly speaking, the number of degrees of freedom, for affine transformations, that preserve it as a Delaunay polytope. The rank of direct product of two Delaunay polytopes is the sum of their ranks (Proposition 15.1.10 in [DL97]; so, for example, the rank of n-cube is $n = \text{nrd}\, Z_n$. Clearly, the maximal rank, as the maximal nrd, is $\binom{n+1}{2}$, which is realized only by n-simplices. The rank of the n-cross-polytope is $\binom{n}{2} + 1$; the rank of the half-n-cube is 6, 7 for $n = 3, 4$ and n for $n \geqslant 5$. In general, $\text{nrd}\, L \leqslant \text{rank}(P)$ for any Delaunay polytope P of L. Rigid lattices D_n, $n \geqslant 5$, and E_8 have each two types of Delaunay polytope: the n-cross-polytope, the half-n-cube and 8-cross-polytope, 8-simplex, respectively. The Delaunay polytopes of A_n are Johnson n-polytopes $J(n+1, k)$; such a polytope is n-simplex if $k = 1, n$ and it has rank $n+1$ if $1 < k < n$. The lattices D_n^* have (see [CS91]) a unique type of Delaunay polytope: the *join of m-cubes* for $n = 2m$ (its vertices are the vertices of two m-cubes in complementary m-spaces and the *separate join of m-cubes* for $n = 2m + 1$ (again the join of two m-cubes, but now their centers are separated by the vector $(0^m, \frac{1}{4}, 0^m)$, which is orthogonal to both m-spaces). The rank of this polytope is $2m + (m+1)^2 = (m+2)^2 - 3$ for $n = 2m + 1$, since $2m + 2$ elements of its affine basis are divided equally between two m-cubes, giving $(m+1)^2$ independent distances, while each m-cube contribute m independent distances. For even n some dependencies appear; we give exact formula in Table I.

In this note we describe explicitly the L-domain for the lattice D_n^* which is dual to the root lattice D_n. This L-domain is special (i.e. not generic) and has dimension n for odd n and dimension 1 for even n. This special L-domain is a facet of the closure of several generic L-domains.

This work completes the computation of nonrigidity degree of root lattices and their duals. In a sense, it is an addition to the work [CS91], where Delaunay and Voronoi polytopes for irreducible root lattices and their duals are listed. We present

Table I.

L	A_1	$A_n n \geqslant 2$	$A_n^* n \geqslant 2$	D_4	$D_n n > 4$	$D_{2m}^* m \geqslant 3$	$D_{2m+1}^* m \geqslant 2$	E_6	E_6^*	E_7	E_7^*	E_8
nrd$L$	1	$n+1$	$\frac{n(n+1)}{2}$	1	1	1	$2m+1$	1	1	1	1	1
mrk$L$	1	$n+1$	$\frac{n(n+1)}{2}$	7	n	$(m+1)^2-2$	$(m+2)^2-3$	1	19	1	21	29

the values of nrd and *mrk* L, i.e. minimal rank of Delaunay polytopes of L, for these lattices L in Table I.

2. The Cone $\mathcal{G}_n$

Let $\{e_i : i \in I_n\}$ be a set of mutually orthogonal vectors of norms (i.e. of squared lengths) $e_i^2 = 2\gamma_i$, where $I_n = \{1, 2, \dots, n\}$. For $S \subseteq I_n$, let $e(S) = \sum_{i \in S} e_i$ and $\gamma(S) = \sum_{i \in S} \gamma_i$. We introduce the vector b of norm α as follows:

$$b = \frac{1}{2}\sum_{i \in I_n} e_i = \frac{1}{2}e(I_n), \quad \text{where } b^2 = \alpha = \frac{1}{2}\sum_{i \in I_n} \gamma_i = \frac{1}{2}\gamma(I_n). \tag{1}$$

Let $\bar{\gamma}$ be the vector with the coordinates $\{\gamma_i : 1 \leqslant i \leqslant n\}$. Consider the lattice $L(\bar{\gamma})$ generated by the vector b and any $n-1$ vectors e_i. If $\gamma_i = 1$ for all i, and $n \geqslant 4$, then $L(\bar{\gamma}) = D_n^*$.

We take as a basis of $L(\bar{\gamma})$ the vector b and the vectors e_i for $1 \leqslant i \leqslant n-1$. Then the coefficients of the quadratic form $f_{\bar{\gamma}}$ corresponding to this basis are as follows:

$$a_{ii} = e_i^2 = 2\gamma_i, \quad 1 \leqslant i \leqslant n-1, \quad a_{ij} = e_i e_j = 0, \quad 1 \leqslant i, j \leqslant n-1, \quad i \neq j, \tag{2}$$

$$a_{nn} = b^2 = \alpha, \; a_{in} = e_i b = \gamma_i, \quad 1 \leqslant i \leqslant n-1. \tag{3}$$

This form has the following explicit expression:

$$f_{\bar{\gamma}}(x) = \left(x_n b + \sum_{i=1}^{n-1} x_i e_i \right)^2 = \alpha x_n^2 + 2\sum_{1}^{n-1} \gamma_i x_i^2 + 2\sum_{1}^{n-1} \gamma_i x_i x_n. \tag{4}$$

In the basis $\{e_i : i \in I_n\}$, each vector of $L(\bar{\gamma})$ has integer or half-integer coordinates.

Let n be odd, say $n = 2m+1$. Suppose that the parameters γ_i satisfy the following $\binom{n}{m}$ inequalities:

$$\sum_{i \in S} \gamma_i < \alpha, \quad S \subset I_n, \quad |S| = m. \tag{5}$$

Recall that the scalar product of basic vectors of a lattice L form a Gram matrix. This Gram matrix defines a quadratic positive form $f(L)$ related to L. A small perturbation of L move the form $f(L)$ in the cone $\mathcal{P}_n$ of all quadratic positive forms. Recall also that the face poset of the Voronoi polytope of L determines the L-type of both L and $f(L)$. The connected domain of $\mathcal{P}_n$ containing $f(L)$ and all forms with the same L-type as $f(L)$, is, by definition, the L-domain of the lattice L and of the form $f(L)$.

Denote by $\mathcal{G}_n$ the n-dimensional domain determined in the space of variables γ_i, $i \in I_n$, by the inequalities (5). Since these inequalities are linear and homogeneous (recall that $\alpha = \frac{1}{2}\gamma(I_n)$), $\mathcal{G}_n$ is an open polyhedral cone. Since $\gamma(S) + \gamma(I_n - S) = 2\alpha$, the inequalities (5) imply the following inequalities:

$$\gamma(T) > \alpha, \quad T \subset I_n, \quad |T| = m+1. \tag{6}$$

For a set T of cardinality $|T| = m + 1$, let $T = S \cup \{i\}$, where $|S| = m$. Then (5) and (6) imply

$$\alpha < \gamma(T) = \gamma(S) + \gamma_i, \quad \text{i.e. } \gamma_i > \alpha - \gamma(S) > 0.$$

Hence the cone $\mathcal{G}_n$ lies in the positive orthant of $\mathbf{R}^n$.

Consider the closure $\mathrm{cl}\mathcal{G}_n$ of the cone $\mathcal{G}_n$. Obviously, $\mathrm{cl}\mathcal{G}_n$ is defined by the non-strict version of inequalities (5). So, using that $2\alpha = \gamma(I_n)$, we have

$$\mathrm{cl}\mathcal{G}_n = \{\bar{\gamma} : \gamma(S) - \gamma(I_n - S) \leqslant 0, \ S \subset I_n, |S| = m\}. \tag{7}$$

Note that the zero vector belongs to $\mathrm{cl}\mathcal{G}_n$. The automorphism group of $\mathrm{cl}\mathcal{G}_n$ is isomorphic to the group of all permutations of the set I_n.

Obviously, the hyperplanes supporting facets of $\mathrm{cl}\mathcal{G}_n$ are contained among the hyperplanes defined by the equalities

$$\gamma(S) = \gamma(I_n - S) = \alpha = \frac{1}{2}\gamma(I_n), \quad S \subset I_n, \quad |S| = m. \tag{8}$$

Note that the equality $\gamma(S_1) = \alpha$ can be transposed into the equality $\gamma(S_2) = \alpha$ by the automorphism group, for any $S_1, S_2 \subset I_n$ with $|S_1| = |S_2| = m$. Hence, each of the equations of (8) determines a facet of $\mathrm{cl}\mathcal{G}_n$.

PROPOSITION 1. *Let n be odd and $n = 2m + 1 \geqslant 5$, i.e. $m \geqslant 2$. Then the closure of $\mathcal{G}_n$ has the following $2n$ extreme rays:*

$$\bar{\gamma}^k_q = \{\gamma_i = \gamma \geqslant 0, i \in I_n - \{k\}, \gamma_k = 2q\gamma\}, \quad q = 0, 1, k \in I_n.$$

Proof. Let $\bar{\gamma} \in \mathrm{cl}\mathcal{G}_n$ be fixed. Then $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ defines a partition of the set I_n as follows. Let the coordinates γ_i take k distinct values $0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_k$, where k is an integer between 1 and n. For $1 \leqslant j \leqslant k$, set $S_j = \{i \in I_n : \gamma_i = \beta_j\}$ and $s_j = |S_j|$. Then $\sum_{j=1}^k s_j = n = 2m + 1$ and $I_n = \bigcup_{j=1}^k S_j$ is the above-mentioned partition.

Consider the values of $\gamma(S)$ for $S \subset I_n$, $|S| = m$. $\gamma(S)$ takes a maximal value for the following sets S. Let j_0 be such that $\sum_{j=1}^{j_0-1} s_j < m + 1$, but $\sum_{j=1}^{j_0} s_j \geqslant m + 1$. Then $\sum_{j=j_0+1}^k s_j \leqslant m$. Let $S_{\max}(T) = T \bigcup_{j=j_0+1}^k S_j$, where $T \subseteq S_{j_0}$, $|T| = t_0$ and $t_0 := m - \sum_{j=j_0+1}^k s_j$. Obviously, $\gamma(S)$ takes the maximal value $t_0\beta_{j_0} + \sum_{j=j_0+1}^k s_j\beta_j$ if $S = S_{\max}(T)$ for any $T \subseteq S_{j_0}$ of cardinality $|T| = t_0$.

For given $\bar{\gamma} \in \mathrm{cl}\mathcal{G}_n$, let $\mathcal{S}_{\max}(\bar{\gamma})$ be the system of equations of type (8), where $S = S_{\max}(T)$ for all $T \subseteq S_{j_0}$ with $|T| = t_0$. If $\bar{\gamma}$ is an extreme ray of $\mathrm{cl}\mathcal{G}_n$, then $\mathcal{S}_{\max}(\bar{\gamma})$ determines uniquely up to a multiple the vector $\bar{\gamma}$. It is not difficult to see that $\mathcal{S}_{\max}(\bar{\gamma})$ can uniquely determine $\bar{\gamma}$ only if $k = 2$. Consider this case in detail.

If $k = 2$, we have $I_n = S_1 \cup S_2$ and $n = 2m + 1 = s_1 + s_2$. Let $s_2 \leqslant m$, i.e. $j_0 = 1$. Then $\mathcal{S}_{\max}(\bar{\gamma})$ consists of the following equations

$$\gamma(T \cup S_2) = \gamma(T) + \gamma(S_2) = \alpha = \gamma(S_1 - T), \quad T \subseteq S_1, \quad |T| = m - s_2.$$

Note that γ_i for $i \in S_2$ belongs to the above system only as a member of the sum $\gamma(S_2) = \sum_{i \in S_2} \gamma_i$. Hence, such a system can determine the coordinates γ_i, $i \in S_2$, only if $s_2 = 1$.

Now the above system implies that γ_i takes the same value, say γ, for all $i \in S_1$. In fact, let $i_1, i_2 \in S_1$, $i_1 \in T_1$, $i_1 \notin T_2$, $i_2 \notin T_1$, $i_2 \in T_2$, for some $T_1, T_2 \subset S_1$ of cardinality $m-1$. Such T_1 and T_2 exist, since $|T_j| = m-1 \geqslant 1$. Subtracting the equation of the above system for $T = T_2$ from the equation for $T = T_1$, we obtain the equality $\gamma_{i_1} - \gamma_{i_2} = \gamma_{i_2} - \gamma_{i_1}$, i.e. $\gamma_{i_1} = \gamma_{i_2}$.

In this case, the above system, where $S_2 = \{k\}$, gives $\gamma_k = \gamma(S_1) - 2\gamma(T) = s_1\gamma - 2(m-1)\gamma = 2\gamma$. We obtain the extreme ray $\bar{\gamma}_1^k$.

Now, let $s_2 > m$, i.e. $s_1 < m+1$ and $j_0 = 2$. A similar analysis shows that $s_1 = 1$, say $S_1 = \{k\}$, and γ_i take the same value, say γ, for all $i \in S_2$. This gives $\gamma_k = 0$, and we obtain the extreme ray $\bar{\gamma}_0^k$. The result follows.

The facet defined by the equation $\gamma(S) = \alpha$, $|S| = m$, contains the following $n = 2m+1$ extreme rays: $\bar{\gamma}_0^k$, $k \notin S$, $\bar{\gamma}_1^k$, $k \in S$. Each facet has the following geometrical description. The $m+1$ rays $\bar{\gamma}_0^k$, $k \notin S$, form an $(m+1)$-dimensional simplicial cone. Similarly, the m rays $\bar{\gamma}_1^k$, $k \in S$, form an m-dimensional cone. Both these cones intersect by the ray $\{\bar{\gamma} : \gamma_i = (m+1)\gamma, i \in S, \gamma_i = m\gamma, i \notin S\}$. Hence, the cone hull of these two cones is a cone of dimension $(m+1) + m - 1 = 2m$. This cone is just a facet of $\mathcal{G}_n$ for $n = 2m+1$.

Let n be even, $n = 2m$. In this case, all the inequalities (5) imply the following set of inequalities:

$$\gamma(I_n - S) = \gamma(T) > \alpha, \quad T = I_n - S \subset I_n, \quad |T| = m.$$

We see that this system of inequalities contradicts to the system (5). This means that the open cone $\mathcal{G}_n$ for even $n = 2m$ is empty. But the solution of the set of equalities (8) is not empty. Namely, it has the solution $\gamma_i = \gamma \geqslant 0$ for all $i \in I_n$. In other words, $\mathrm{cl}\mathcal{G}_n$ is the following ray

$$\mathrm{cl}\mathcal{G}_{2m} = \{\bar{\gamma} : \gamma_i = \gamma \geqslant 0, i \in I_{2m}\}.$$

3. The Domain $\mathcal{D}_n$

Denote by $\mathcal{D}_n$ the domain of forms $f_{\bar{\gamma}}$, where $\bar{\gamma}$ belongs to $\mathcal{G}_n$.

We prove the following theorem.

THEOREM 1. *Let n be odd, $n = 2m+1$. The domain $\mathcal{D}_n$ is an L-domain. It lies in an n-dimensional space which is an intersection of $\binom{n}{2}$ hyperplanes given by the following equalities:*

$$a_{ij} = 0, \ \ 1 \leqslant i < j \leqslant n-1, \quad 2a_{in} = a_{ii}, \ \ 1 \leqslant i \leqslant n-1. \tag{9}$$

The domain $\mathcal{D}_n$ is cut from this space by the following inequalities:

$$\sum_{i \in S} a_{ii} < 2a_{nn}, \quad S \subset I_{n-1}, \quad |S| = m, \tag{10}$$

$$2a_{nn} < \sum_{i \in T} a_{ii}, \quad T \subset I_{n-1}, \quad |T| = m+1. \tag{11}$$

There is a one-to-one correspondence between $\mathcal{D}_n$ and the cone $\mathcal{G}_n$ given by the equalities (2) *and* (3).

In particular, the closure of $\mathcal{D}_n$ has $2n$ extreme rays f_0^k, f_1^k, $k \in I_n$, with the coefficients a_{ij} of these forms defined as follows (*where the term $a_{kk}(f_{0,1}^k)$ should be omitted if $k = n$*)*:*

$$a_{ii}(f_0^k) = 2\gamma,\ \ i \in I_{n-1},\ \ i \neq k,\quad a_{kk}(f_0^k) = 0,\quad a_{nn}(f_0^k) = m\gamma;$$
$$a_{ii}(f_1^k) = 2\gamma,\ \ i \in I_{n-1},\ \ i \neq k,\quad a_{kk}(f_1^k) = 4\gamma,\quad a_{nn}(f_1^k) = (m+1)\gamma;$$

$a_{ij}(f_{0,1}^k)$ for $i \neq j$ are defined by Equations (9).

The inequalities (10) *and* (11) *define facets of the closure $\mathrm{cl}\mathcal{D}_n$. All facets are domains of equivalent L-types, each having n extreme rays f_1^k, $k \in S$, f_0^k, $k \notin S$, $S \subset I_{n-1}$, $|S| = m$, or $S = I_n - T$ and T is as in* (11).

If n is even, $n = 2m$, then $\mathrm{cl}\mathcal{D}_n$ is one-dimensional. The ray $\mathrm{cl}\mathcal{D}_{2m}$ is the intersection of the $\binom{n}{2}$ hyperplanes (9) *and the $n-1$ hyperplanes given by the following equalities:*

$$2a_{nn} = ma_{ii},\quad 1 \leqslant i \leqslant n-1. \tag{12}$$

Proof. We will proceed as follows. For a function $f_{\bar\gamma}$ given by (4), we find the Voronoi polytope. Take in attention that the inequalities (10) and (11), in terms of the parameters α and γ_i take the form (5) for $n \notin S$ and $n \in S$, respectively. We show that the face-poset of the Voronoi polytope does not change if the parameters of $f_{\bar\gamma}$ change, such that they satisfy (5).

On the other hand, we show if at least one of inequalities (5) holds as equality for parameters of a function $f_{\bar\gamma}$, then the L-type of $f_{\bar\gamma}$ differs from the L-type of $f_{\bar\gamma} \in \mathcal{D}_n$. This will mean that $\mathcal{D}_n$ is an L-domain.

For to find the Voronoi polytope of $f_{\bar\gamma}$ given by (4), consider the cosets of $2L$ in the lattice $L = L(\bar\gamma)$. Let $v = x_n b + \sum_{i \in I_{n-1}} x_i e_i$ be a vector of $L(\bar\gamma)$. Then this vector belongs to the coset $Q(S, z)$, where $S \subseteq I_{n-1}$ is the set of indices of odd coordinates x_i and the number $z \in \{0, 1\}$ indicates the parity of the b-coordinate x_n of the vector v. Note that the vector $e(I_n) = 2b$ belongs to the trivial coset $Q(\emptyset, 0) = 2L$. Hence, the vectors $e(S)$ and $e(I_n - S)$ belong to the same coset for any $S \subseteq I_n$. This coset is $Q(S, 0)$ if $n \notin S$, and $Q(I_n - S, 0)$ if $n \in S$. In particular, e_n belongs to $Q(I_{n-1}, 0)$, and it is minimal in this coset. Moreover, we have $b - e(S) = -(b - e(I_n - S))$. So, the 2^n vectors $b - e(S)$, $S \subseteq I_n$, are partitioned into 2^{n-1} pairs of opposite vectors.

Note that $e^2(S) = \sum_{i \in S} e_i^2 = 2\gamma(S)$ and, according to (1), $\gamma(S) + \gamma(I_n - S) = 2\alpha$. Recall that $\mathcal{D}_n$ is the domain of $f_{\bar\gamma}$, where $\bar\gamma$ belongs to $\mathcal{G}_n$. Hence, γ_i, $i \in I_n$, satisfy (5). Taking in attention (5), we see that, for $|S| \leqslant m$, the norm of $e(S)$ is less than the norm of $e(I_n - S) = e(T)$ for $|T| \geqslant m+1$, $S, T \subset I_n$.

If $f_{\bar\gamma}$ go to the boundary of $\mathcal{D}_n$, then the sets of minimal vectors of some cosets change. At first we describe the simple cosets, which are constant on the closure of $\mathcal{D}_n$. The norm of minimal vectors of a coset is called also *norm* of the coset.

These are the following cosets:

The n cosets $Q(\{i\},0)$, $i\in I_{n-1}$, and $Q(I_{n-1},0)$ of norms $2\gamma_i$, $i\in I_n$, with minimal vectors e_i, $i\in I_{n-1}$ and $e_n=2b-e(I_{n-1})$, respectively.

The 2^{n-1} cosets $Q(S,1)$ of norm α with minimal vectors $b-e(S)$, $S\subseteq I_{n-1}$.

The 2^{n-1}-n non-simple cosets $Q(S,0)$, $S\subset I_n$, $1<|S|\leqslant m$, have norms $\gamma(S)$ with minimal vectors $\sum_{i\in S}\varepsilon_i e_i$, where $\varepsilon_i\in\{\pm 1\}$. If $\alpha=\gamma(S)$, then $|S|=m$ and the coset $Q(S,0)$ contains also the vector $\sum_{i\in I_n-S}\varepsilon_i e_i$.

Recall that the minimal vectors of simple cosets determine facets of the Voronoi polytope. Consider a point $x\in\mathbf{R}^n$ in the basis $\{e_i : i\in I_n\}$, $x=\sum_{i\in I_n}x_ie_i$. Then x belongs to the Voronoi polytope P of $L(\bar\gamma)$ if the inequalities

$$-\frac{v^2}{2}\leqslant xv\leqslant\frac{v^2}{2}$$

hold for all minimal vectors v of simple cosets of $L(\bar\gamma)$. Using (2), (3) and the identity $b=\frac{1}{2}\sum_{i\in I_n}e_i$, we obtain the following system of inequalities describing the Voronoi polytope of $L(\bar\gamma)$:

$$-\frac{1}{2}\leqslant x_i\leqslant\frac{1}{2},\quad i\in I_n, \tag{13}$$

$$-\frac{1}{2}\alpha\leqslant\sum_{i\in I_n}\gamma_i\varepsilon_i^Tx_i\leqslant\frac{1}{2}\alpha,\quad \varepsilon_i^T\in\{\pm 1\},\quad i\in I_n. \tag{14}$$

Here the inequality (14) is given by the minimal vector $\frac{1}{2}\sum_{i\in I_n}\varepsilon_i^Te_i$ of $Q(T,1)$ such that $\varepsilon_i^T=-1$ if $i\in T$, and $\varepsilon_i^T=1$ if $i\notin T$.

Note that $\sum_{i\in I_n}\gamma_i\varepsilon_ix_i$ is the linear function on ε_ix_i taking maximal value if $\varepsilon_ix_i\geqslant 0$ for all $i\in I_n$. Hence, the right-hand inequality in (14) holds as equality for a vertex x only if $\varepsilon_ix_i>0$ for $x_i\neq 0$.

An analysis of the system (13), (14) shows, that for each vertex x, there is the opposite vertex $-x$, and x has the following coordinates:

$$x_i=\frac{1}{2}\varepsilon_i, i\in S\subseteq I_n, |S|=m, x_k=\frac{\varepsilon_k}{2\gamma_k}(\alpha-\gamma(S)), x_l=0,\qquad\text{for } l\in I_n-(S\cup\{k\}). \tag{15}$$

There are $(m+1)\binom{n}{m}$ positive vertices of this type. Taking in attention signs, we obtain $2^{m+1}\binom{n}{m}$ vertices of the Voronoi polytope. Denote the vertex (15) by $x(k;S,\varepsilon)$.

The form of the vertex $x(k;S,\varepsilon)$ shows that some vertices can be glued if and only if the equality $\alpha=\gamma(S)$ holds for some set S. If $\alpha=\gamma(S)$, then $x_k(k;S,\varepsilon)=0$, and the $m+1$ vertices $x(l;S,\varepsilon)$, $l\in I_n-S$, are glued into one vertex. This means that if $\alpha=\gamma(S)$ for some S, then L-type of $f_{\bar\gamma}$ changes. So, we proved that the inequalities (5), i.e. the inequalities (10) and (11) hold for $f\in\mathcal{D}_n$.

Now, we show that $\mathcal{D}_n$ lies in the intersection of the hyperplanes (9). It is proved in [BG01] that the equations of the hyperplanes in the intersection of which an L-type domain lies are given by some linear forms on norms of minimal vectors of cosets of $2L$ in L. Some of such linear forms are obtained by equating norms of minimal vectors of a non-simple coset. There are L-type domains for which linear forms of last type are sufficient for to describe the space, where this L-type domain lies. This is so in our case.

In fact, it is sufficient to consider the non-simple cosets $Q(S,0)$ for $|S|=2$, i.e. to equate the norms of vectors e_i+e_j and e_i-e_j, $i,j\in I_n$. The equality $(e_i+e_j)^2=(e_i-e_j)^2$ implies $e_ie_j=0$, i.e. $a_{ij}=0$, $0\leqslant i<j\leqslant n-1$. We obtain the first equalities in (9). For $j=n$, we have $e_ie_n=-e_i(2b-e(I_{n-1}))=0$. Since $e_ie_j=0$, this equality is equivalent to $2be_i=e_i^2$. We obtain the second equalities in (9). If we set $be_i=\gamma_i$, $b^2=\alpha$, we obtain the original function $f_{\bar\gamma}$.

Now, let $n=2m$ be even. In this case, for $f\in \mathrm{cl}\mathcal{D}_n$, all cosets of $2L$ in L (excluding the cosets $Q(S,0)$ for $|S|=m$) are the same as in the odd case. But, for $|S|=m$, $Q(S,0)$ contains beside the vectors $\sum_{i\in S}\varepsilon_ie_i$ also the vectors $\sum_{i\in I_n-S}\varepsilon_ie_i$. Since norms of these vectors are $\gamma(S)$ and $\gamma(I_n-S)$, respectively, the equating of these norms gives the system (8). This system has the unique solution $\gamma_i=\gamma$ for all $i\in I_n$.

Hence, for $n=2m$, $\alpha=\frac{1}{2}\gamma(I_n)=\frac{1}{2}(\gamma(S)+\gamma(I_n-S))=\gamma(S)=m\gamma$. Taking in attention (2) and (3), we can rewrite this equality as $2a_{nn}=ma_{ii}$ for any $i\in I_n$. So, we obtain (12). This means that $\mathrm{cl}\mathcal{D}_n$ is a ray, which lies in the intersection of the hyperplanes given by the Equations (10), (11) and (12). Any $f\in\mathrm{cl}\mathcal{D}_n$ is a rigid (i.e. edge) form.

So, Theorem 1 is proved. □

Recall that $L(\bar\gamma)=D_n^*$ if $\gamma_i=1$ for all $i\in I_n$. Since, for this $\bar\gamma$, the parameters of f_γ satisfy (5), where $\alpha=m+\frac{1}{2}$, this implies the following

COROLLARY. *$\mathcal{D}_n$ is the L-domain of D_n^*. In particular, the lattice D_{2m}^*, $m\geqslant 2$, is rigid.*

Remark. Note that, for n odd, $n=2m+1$, the extreme rays f_0^k have rank $n-1=2m$. These forms are forms of lattices isomorphic to $\gamma D_{n-1}^*=\gamma D_{2m}^*$.

We saw in the proof of Theorem 1 that the Voronoi polytope of the lattice D_n^* is an n-cube whose vertices are cut by hyperplanes (14). A description of the Voronoi polytope of D_n^* can be found in [CS91].

Theorem 1 is a generalization of the result of [EG01], where the L-domain of the lattice D_5^* is described in detail.

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