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A Metric of Constant Curvature on Polycycles

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Abstract—We prove the following main theorem of the theory of (r, q)-polycycles. Suppose a nonseparable plane graph satisfies the following two conditions:

(1) each internal face is an r-gon, where $r \geq 3$;
(2) the degree of each internal vertex is q, where $q \geq 3$, and the degree of each boundary vertex is at most q and at least 2.

Then it also possesses the following third property:

(3) the vertices, the edges, and the internal faces form a cell complex.

Simple examples show that conditions (1) and (2) are independent even provided condition (3) is satisfied. These are the defining conditions for an (r, q)-polycycle.

Key words: *polycycle, cell complex, barycentric subdivision, plane graph.*

This paper completes the series of papers [1–10] concerning the so-called (r, q)-polycycles. The definition of a polycycle in this series includes three conditions. One of these three conditions turns out to be superfluous. The two others are independent. These are the defining conditions of an (r, q)-polycycle. This is the main theorem of the present paper.

Now, let us give our initial definition of an (r, q)-polycycle and recall the main properties of the (r, q)-polycycles under investigation.

1. DEFINITION AND PROPERTIES OF POLYCYCLES

A planar[1] graph G can be drawn on the plane so that no two of its edges intersect each other. Such a drawing, i.e., a plane realization of G, is called a *plane graph*. After adding all the faces to a plane graph we obtain a so-called *plane map*. All bounded faces of a plane map are the *internal faces* of the plane graph. The vertices not belonging to the unbounded face of a plane map are the *internal vertices* of the plane graph.

Definition. Suppose a plane realization of a nonseparable planar graph G satisfies the following three conditions:

(1) all internal faces are combinatorial r-gons, where $r \geq 3$;
(2) all internal vertices have the same degree q, where $q \geq 3$, and the degree of all boundary vertices is at most q and at least 2;
(3) the vertices, the edges, and the internal faces form a cell complex.

Then the graph G together with the internal faces (determined by the given plane realization of G) is called an (r, q)-*polycycle* and is denoted by $\Pi(G)$ (see [1–7]).

We must stress that the requirement of nonseparability of the graph G in the definition of an (r, q)-polycycle $\Pi(G)$ is essential. There are no two (r, q)-polycycles with a single common vertex,

[1]We borrow the terminology of graph theory (for graphs without loops and multiple edges) from [11].

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of degree less than q, forming a joint (r, q)-polycycle, since the joint graph does not satisfy the nonseparability requirement.

The number r in the definition of an (r, q)-polycycle $\Pi(G)$ is the *girth* of G. If at least one plane realization of a nonseparable planar graph G satisfies the conditions (1)–(3), where r is the girth of G and q is the maximal vertex degree, then we say that G admits an (r, q)-*polycyclic realization.*

The following version of the *uniqueness theorem* for polycyclic realizations is true: generally speaking, a nonseparable graph G admits at most one (r, q)-polycyclic realization (see [3–6]). The five exceptional cases correspond to the edge skeletons of the five Platonic solids, and the numbers of (r, q)-polycyclic realizations of G in these cases coincide with the numbers of faces of the corresponding solids. In each of these five cases, all plane realizations are (r, q)-polycyclic and pairwise isomorphic.

Thus, a graph G admitting an (r, q)-polycyclic realization, an (r, q)-polycyclic realization of G, and, finally, an (r, q)-polycycle $\Pi(G)$ are three distinct notions which determine each other in an essentially unique way. In what follows, we shall use the term *polycyle* for any of these three notions, provided the meaning is clear from the context.

Note that the notion of (r, q)-polycycle $\Pi(G)$ can be introduced not only for a finite graph G, but also for an infinite one. In the case of an infinite plane graph, we always assume that there are *finitely many* vertices and edges of the graph in any *finite* domain of the plane. This additional requirement seems to be very natural. In the case of an infinite graph G, an (r, q)-polycycle $\Pi(G)$ is an infinite simply connected domain, maybe with boundary (see below and [4–6]). In the case of a finite graph G, an (r, q)-polycycle $\Pi(G)$ is a disk, it always has a boundary, and its Euler characteristic is 1.

For the sake of brevity, we denote an (r, q)-polycyle $\Pi(G)$ by P. We denote by X the two-dimensional polyhedron obtained by replacing each flat r-gon in the (r, q)-polycycle $\Pi(G)$ by a two-dimensional regular r-gon of constant curvature with the angles $2\pi/q$. From the internal point of view, the polyhedron X is a metric space of *constant* curvature. Indeed, a sufficiently small disk neighborhood of an internal point of a regular r-gon is nothing but an ε-disk of constant curvature. If the neighborhood is centered at an internal point of an internal edge of X, then the ε-disk is formed by two semi-disks. If the center of the neighborhood coincides with an internal vertex of X, then such an ε-disk consists of q sectors with coinciding angles $2\pi/q$.

Hence, being glued from regular r-gons of constant curvature having angles $2\pi/q$, the polyhedron X, on the one hand, has the same cell structure as the polycycle P and, on the other hand, is a metric space (a 2-surface) of constant curvature. It is known (see the corresponding formula and theorem, e.g., in the book [12, p. 411]), the sum of internal angles of a geodesic m-gon on a 2-surface equals $(m-2)\pi$ plus the total curvature Ω of the given m-gon. It is a (rather frequently occuring) special case[2] of the well-known Gauss–Bonnet theorem.

For the geodesic polygon we take a regular r-gon of constant curvature with the angles $\varphi_i = 2\pi/q$, where $i = 1, 2, \dots, r$. By the theorem mentioned above, for such a triangle we have

$$\frac{2(r+q) - rq}{q}\pi = \Omega.$$

The polyhedron X consisting of these r-gons (meeting in q copies at each vertex), the initial polycycle P combinatorially isomorphic to it, and, finally, the corresponding parameters (r, q) are said to be *elliptic*, *parabolic*, or *hyperbolic* depending on whether $2(r+q) > rq$, or[3] $2(r+q) = rq$, or $2(r+q) < rq$, respectively.

[2]The discrete analog of the theorem for developments of positive curvature in [13] produces the same expression: the excess of the sum of angles of a geodesic m-gon (when compared to the total sum of angles of an Euclidean m-gon) coincides with its total curvature, i.e., $\sum_{i=1}^{m} \varphi_i - (m-2)\pi = \Omega$.

[3]This equality admits the geometrically more transparent form $(r-2)\pi = 2\pi r/q$ in terms of the sum of the angles of the Euclidean r-gon with the angles $2\pi/q$. In the present case, $(r, q) = (3, 6)$, or $(4, 4)$, or $(6, 3)$.

Incidently, for any values $r \geq 3$, $q \geq 3$, there is a widely used regular partition (r^q). Here (r^q) is the standard notation for a tiling of sphere $\mathbb{S}^2$, or Euclidean plane $\mathbb{R}^2$ or Lobachevsky plane $\mathbb{H}^2$, into regular r-gons with angles $2\pi/q$. In the case of the Euclidean and Lobachevsky planes, the regular partitions (r^q) are infinite (r,q)-polycycles. It happens that only these (r,q)-polycycles do not have a boundary. In the case of the sphere, any regular partition (r^q) also has no boundary. However, it is not an (r,q)-polycycle; it becomes a finite (r,q)-polycycle with boundary after a face is erased.

The existence of a regular partition (r^q) plays an essential role in the theory of (r,q)-polycycles. It implies the splitting of all polycycles into proper[4] and improper ones (helivalues). Namely, the following basic lemma was proved in [3, 5, 6]: any (r,q)-polycycle admits a cell mapping into the regular partition (r^q); such a mapping is uniquely determined by a *flag* (i.e., a vertex, an edge, and an r-gon incident to each other) and its image (the *projection*) under this mapping. The local homeomorphic mapping under consideration is (not) a global homeomorphism in the case of an (im)proper polycycle.

This crucial lemma can be reformulated (and proved) in terms of a metric of constant curvature: there is a continuous locally isometric cell mapping of the polyhedron X to the regular partition (r^q). We write this mapping as

$$f\colon X \to (r^q) \tag{1}$$

if we want to stress that under this mapping the r-gons of the polyhedron X are taken to the r-gons of the partition (r^q); we write it as

$$f\colon \mathcal{B}X \to \mathcal{B}(r^q) \tag{2}$$

if we want to underline the fact that the triangles in the first barycentric subdivision $\mathcal{B}X$ are taken to triangles in the first barycentric subdivision $\mathcal{B}(r^q)$; or we write it as

$$f\colon \mathcal{B}^2X \to \mathcal{B}^2(r^q) \tag{3}$$

if we want to underline that the triangles in the second barycentric subdivision $\mathcal{B}^2X$ are taken to triangles in the second barycentric subdivision $\mathcal{B}^2(r^q)$.

Barycentric subdivisions are discussed in [14][5]. Let us describe their specific features in our case.

A vertex, a midpoint of an edge, and the center of an r-gon in a regular partition (r^q) or in a partition of a polyhedron X into regular r-gons are the vertices of a barycentric triangle in $\mathcal{B}$. We say that these vertices are, respectively, of *type* 0, 1, or 2. The type of a vertex coincides with the dimension of the cell whose center is the vertex. Each barycentric triangle has exactly one vertex of each type. All vertices in the first barycentric subdivision of a polyhedron (r^q), which we denote by $\mathcal{B}(r^q)$, are internal.

Exactly $2r$, 4, $2q$ barycentric triangles in the partition $\mathcal{B}(r^q)$ meet at vertices of type 2, 1, 0, respectively. The meeting numbers at internal vertices of the first barycentric subdivision of a polyhedron X, which we denote by $\mathcal{B}X$, are the same. The star of barycentric triangles centered at an *internal* vertex of the partition $\mathcal{B}X$ is isometric to the corresponding *star* of barycentric triangles centered at a vertex of the same type of the subdivision $\mathcal{B}(r^q)$. The star of barycentric triangles centered at a *boundary* vertex of the partition $\mathcal{B}X$ is isometric only to a *part* of the corresponding *star* of barycentric triangles centered at a vertex of the same type of the subdivision $\mathcal{B}(r^q)$.

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[4]A proper (r,q)-polycycle is a proper subpolycycle of the partition (r^q).

[5]The statement of the main theorem of the topology of surfaces on p. 165 of the Russian translation of this book contains a misprint: one should replace $k(\geq 0)$ by either $k(>0)$ or $k(\geq 1)$.

When passing from the first barycentric subdivision $\mathcal{B}$ to the second barycentric subdivision $\mathcal{B}^2$, any barycentric triangle 012 is subdivided into triangles of the following six types:

$$0.01.012, \quad 0.02.012, \quad 1.01.012, \quad 1.12.012, \quad 2.02.012, \quad 2.12.012.$$

Here, the notation e.g., 0.01.012, denotes the triangle with vertices of types 0, 01 (a vertex of type 01 is the midpoint of an edge 01) and 012 (a vertex of type 012 is the center of a triangle 012). The triangles of these six types are situated inside the r-gon in different ways.

There are exactly seven types of vertices inside the second barycentric subdivision: 0, 1, 2, 01, 02, 12, 012. In the case of the subdivision $\mathcal{B}^2(r^q)$, all these vertices (of any type) are internal. Exactly $4q$, 8, $4r$, 4, 4, 4, or 6 barycentric triangles meet at the vertices of these types, respectively. The meeting multiplicities at internal vertices of the subdivision $\mathcal{B}^2X$ are exactly the same. The star of barycentric triangles centered at an *internal* vertex of the partition $\mathcal{B}^2X$ is isometric to the *star* of barycentric triangles centered at a vertex of the same type in the subdivision $\mathcal{B}^2(r^q)$. The star of barycentric triangles centered at a *boundary* vertex of the subdivision $\mathcal{B}^2X$ is isometric only to a *part* of the star of barycentric triangles centered at a vertex of the same type in the subdivision $\mathcal{B}^2(r^q)$.

The first barycentric subdivision $\mathcal{B}$ and the second barycentric subdivision $\mathcal{B}^2$ serve as the main tool in our proofs. We would like to attract the reader's attention to the following three important points that allowed us to construct the map f (see (1)). First of all, an (r, q)-polycycle $\Pi(G)$ has the same structure as the corresponding regular partition (r^q) in neighborhoods of the internal points. Second, regular partitions (r^q) do not have any boundary: the sphere, the Euclidean plane and Lobachevsky planes are all complete metric spaces. Third, an (r, q)-polycycle $\Pi(G)$ is simply connected: the interior of any polycycle is a connected simply connected domain. These properties are exactly those that allow one to uniquely extend a local mapping to a global one. We shall make use of these facts in the proof of the lemma below.

2. MAIN THEOREM ABOUT POLYCYCLES

Let us now formulate the main theorem of the paper. It states that the three conditions (1), (2), (3) on a nonseparable plane graph G in the definition of an (r, q)-polycycle $\Pi(G)$ are dependent.

Theorem. *Conditions* (1) *and* (2) *imply condition* (3).

Before proving the theorem, let us state a trivial corollary, which we consider as being very important in the theory of polycycles. The main goal of the proof of the theorem is the following corollary, and it must be understood as a simplified definition of a polycycle.

Corollary. *An* (r, q)*-polycycle* $\Pi(G)$ *is defined by two conditions* (1), (2), *on a nonseparable plane graph* G.

Thus, condition (3) in the initial definition of a polycycle is superfluous. Simple examples show that conditions (1) and (2) are independent, even when condition (3) is satisfied. Therefore, the simplified definition of a polycycle is, in a sense, final.

Proof of the theorem. Now, let us start the proof of the theorem[6]. We call a nonseparable plane graph G satisfying the two conditions (1) and (2), with added internal faces, a *generalized*[7] polycycle and denote it by P_*. Replacing each flat r-gon in P_* by a regular two-dimensional r-gon of constant curvature with angles $2\pi/q$, we obtain, instead of a generalized polycycle P_*, a polyhedron equipped with a metric of constant curvature, which we denote by X_*.

[6]If the graph G under consideration is finite and 3-connected, then our theorem follows from the Steinitz theorem about the existence of a *convex* polyhedron with the edge skeleton G.

[7]This is the first attempt to generalize the initial notion of polycycle given in the definition.

For this polyhedron, conditions (1) and (2) also are satisfied. By condition (1), each face of the polyhedron X_*, being a two-dimensional combinatorial r-gon, is a disk. Therefore, different angles attached to a vertex of X_* belong to distinct r-gons. Now, condition (2) can be reformulated as follows: the number of r-gons meeting at an internal vertex of X_* is q, and all these q r-gons form a simple circuit; the number of r-gons meeting at a boundary vertex of X_* is j, where $1 \le j \le q-1$, and all these j r-gons form a simple chain. An internal edge of the polyhderon X_* belongs to two and only two r-gons; a boundary edge belongs to a single r-gon. Our goal is to prove that the vertices, the edges, and the faces of the polyhedron X_* form a cell complex[8] (the cells are closed). Roughly speaking, this is a complete analog of a triangulation.

Take the first barycentric subdivision of the polyhedron X_*. According to our agreement, we denote it by $\mathcal{B}X_*$. Let us show that it is a triangulation. Indeed, q edges of type 01 with pairwise distinct other ends of type 1, q edges of type 02 with pairwise distinct other ends of type 2, as well as $2q$ barycentric triangles of the polyhedron $\mathcal{B}X_*$, where $q \ge 3$, meet at an internal vertex of type 0. All these $2q$ triangles form a star centered at 0. The intersection of any two triangles is either an edge or a vertex. The star centered at a boundary vertex of type 0 is isometric to a part of a star centered at an internal vertex of type 0. Neither 2-gons nor loops are incident to vertices of type 0. The same can be said about stars centered at vertices of type 1 or 2, with the exception of the fact that vertices of type 2 can be internal only. Therefore, the barycentric triangles of the polyhedron $\mathcal{B}X_*$ form a triangulation.

Main lemma. *There is a continuous locally isometric cell map*

$$f_*\colon \mathcal{B}X_* \to \mathcal{B}(r^q). \qquad (2_*)$$

Proof. The polyhedra $\mathcal{B}X_*$ and $\mathcal{B}(r^q)$ are given by their triangulations. They consist of pairwise isometric triangles. The map f_* of $\mathcal{B}X_*$ to $\mathcal{B}(r^q)$ is constructed as follows (see, e.g., [1]). Take an isometric mapping of a given triangle in $\mathcal{B}X_*$ onto a given triangle in $\mathcal{B}(r^q)$ preserving the types of the vertices. This mapping can be extended uniquely to a mapping of the neighboring (having a common edge) triangle in $\mathcal{B}X_*$ (if there is one) onto the neighboring (having a common edge) triangle in $\mathcal{B}(r^q)$ (there always is one). Now, let us take an arbitrary triangle in the polyhedron $\mathcal{B}X_*$. Connect it to the initial triangle by a chain of triangles such that any two neighboring triangles in the chain have a common edge. Such a chain exists even for an infinite graph, due to the additional requirements mentioned above. Moving along this chain step by step, we will finally construct a mapping of the last triangle in the chain onto a triangle in $\mathcal{B}(r^q)$. Let us show that this mapping is independent of the chosen chain connecting the given triangle with the initial one.

Let us begin by verifying this assertion for the so-called elementary chain of triangles in the partition $\mathcal{B}X_*$. By definition, an *elementary closed chain* consists of

(a) two triangles having a common internal edge of the partition $\mathcal{B}X_*$, with a passage from one triangle to the second one and back;

(b) all triangles meeting at an internal vertex of the partition $\mathcal{B}X_*$, with the circular path around this vertex.

The common vertex of the triangles in a chain of type (b) can be of type 0, 1, or 2. All the triangles in the chain form a star centered at this vertex. This star is homeomorphic to the disk.

The common edge of the triangles in a chain of type (a) can have type 01, 02 or 12. The two triangles adjacent to an edge of any of these types together form a disk as well.

Thus, a polyhedron $\mathcal{B}X_*$ has elementary closed chains of triangles of six types only. The triangles of an elementary closed chain of any of the six types form a disk. It is clear that an

[8]See the definition of a cell complex, e.g., in [14, p. 267 of the Russian translation]. Here it suffices to recall the following. A zero-dimensional cell is a point. The boundary of a one-dimensional cell is a pair of points, while the cell itself is a segment. The boundary of a two-dimensional cell is a circle, while the cell itself is a disk. The intersection of any two cells is a cell.

elementary closed chain in the polyhedron $\mathcal{B}X_*$ is isometric to an elementary closed chain of the same type in the polyhedron $\mathcal{B}(r^q)$. Moreover, an isometry mapping of an elementary chain in $\mathcal{B}(r^q)$ can be uniquely reconstructed from its restriction to a single triangle. This means that a round trip along an elementary closed chain returns us to the initial triangle of the chain in $\mathcal{B}X_*$ together with the initial mapping of this triangle onto a triangle in the polyhedron $\mathcal{B}(r^q)$.

Now, after understanding the situation with an elementary closed chain, let us return to the case of an arbitrary chain. If we insert or erase an elementary closed chain either of type (a) or (b) anywhere in an arbitrary chain, then the mapping corresponding to the new chain coincides with the original one.

By means of transformations of types (a) and (b) any chain connecting two given triangles can be transformed into any other such chain. This follows from the simple connectedness of the polyhedron $\mathcal{B}X_*$. Therefore, the mapping constructed by means of a chain is independent of the chain.

Thus, an isometric mapping of a chosen triangle in a polyhedron $\mathcal{B}X_*$ onto a chosen triangle in $\mathcal{B}(r^q)$ preserving the types of vertices extends uniquely to an isometric mapping of any other triangle of the same polyhedron, i.e., to an isometry of the entire polyhedron. The lemma is proved. $\square$

Remark. The mapping (2_*) takes the star of triangles at a vertex of type 2 of $\mathcal{B}X_*$ to the star of triangles at a vertex of type 2 of $\mathcal{B}(r^q)$. In other words, f_* takes an r-gon X_* to an r-gon of (r^q). This simple conclusion plays an essential role in the proof of the theorem.

Before completing the proof of the theorem, let us recall that the partition (r^q) possesses the following remarkable property: the girth of the skeleton of (r^q), considered as a graph, is r, i.e., the length of a minimal edge circuit in (r^q) is r (see [1]).

This statement can be easily verified in the elliptic case. Two adjacent edges of a tetrahedron always belong to the same triangle. Two adjacent edges of the same face of an octahedron (or, of an icosahedron) enter the boundary triangle of the face, while if they do not belong to the same face, then they enter an edge circuit of length at least 4. Three successive edges of a cube (a dodecahedron) belonging to the same face enter the boundary quadrangle (pentagon) of the face, while if they do not belong to the same face, then they enter an edge circuit of length at least 6.

The verification in the parabolic or hyperbolic cases proceeds as follows. Take a simple edge circuit in the partition (r^q). By the Jordan theorem, this circuit bounds a finite domain in the plane. This domain contains at least one two-dimensional r-gon of (r^q). Draw the rays from the center of this r-gon through its vertices. These rays divide the central angle into r sectors, each sector based on its own side of the r-gon. Any line connecting two external points of the boundary rays of a sector is longer than the side of the r-gon. Therefore, the number of edges in any edge circuit containing the r-gon and not coinciding with it is greater than r. Hence the girth of the skeleton of the partition (r^q) is r.

In any standard regular partition (r^q) any closed edge circuit of length r bounds a face. Exactly two edge circuits of the shortest length r pass through each edge of the partition (r^q).

In a polyhedron X_*, each closed edge circuit of length r also bounds a face. Generally speaking, this face is internal. It can be external only in the five exceptional cases. Two internal faces are adjacent to each internal edge of the polyhedron X_*. A single internal face is adjacent to each boundary edge. Now, let us study the form of a connected component of the intersection of two r-gons in a polyhedron X_*. Suppose the intersection of two r-gons in a polyhedron X_* is nonempty. Let us show, by *reductio ad absurdum*, that each connected component of the intersection is either a vertex or an edge. Suppose a connected component of the intersection of two r-gons in X_* contains at least two edges. Consider the common vertex of two edges belonging to the same connected component of the intersection. Two successive edges of each of the r-gons are adjacent to this vertex, and both two-dimensional r-gons are adjacent to each of these edges. Therefore, this vertex is an internal vertex of X_*. Its degree is 2, which contradicts assumption (2) of the theorem asserting that $q \geq 3$. Therefore, a connected component of a nonempty intersection of two r-gons can be either an edge or a vertex.

Now, let us study the number of connected components of the intersection of any two r-gons of the polyhedron X_*. We are going to show, by arriving at a contradiction, that the intersection has only one connected component. Suppose the intersection of two r-gons has at least two connected components. Take two common vertices of two r-gons, belonging to distinct connected components of their intersection. These two vertices divide the circumference of the first r-gon into two chains x and y, and the circumference of the second r-gon into two chains u and v. Without loss of generality, one can suppose that the length (the number of edges) of x is not greater than the length of u (otherwise we reverse the notation). Then the circuit xv is not longer than the circuit uv. Since the length of the circuit uv is r, the length of xv is at most r. Let us show that it equals r. In order to do this, consider the mapping f_* constructed in the lemma. This mapping takes any edge circuit γ in X_* to an edge circuit $f_*(\gamma)$ in (r^q), of the same length as γ. And since the length of $f_*(\gamma)$ is at least r, the length of γ also is at least r. In particular, the circuit xv, of length at most r, is taken by f_* to a circuit $f_*(xv)$, of length at most r. And since the girth of the edge skeleton of (r^q) is r, the length of the circuit $f_*(xv)$ is at least r. Hence the length of $f_*(xv)$ is r. Therefore, the length of xv also is r. The circuit xv must bound a face, either internal, or external one. The last case is realized only for a Platonic solid with the interior of one face erased. But, in this case, the intersection of any two internal faces is connected, which does not fall into the case under consideration. The only possibility left is that the circuit xv bounds an internal face. In this case, x is an edge which is a connected component of the intersection of two faces with the boundaries xy and xv. Similarly, the chain v is an edge which is a connected component of the intersection of two faces with the boundaries xv and uv. The two edges x and v form a digon, i.e., the girth of the edge skeleton of the polyhedron X_* is 2, which contradicts assumption (1) of the theorem asserting that $r \geq 3$. Therefore, the intersection of two r-gons can have only one connected component.

Thus, a nonempty intersection of two r-gons in X_* is always connected and is either an edge or a vertex.

Since the graph G has no loops or multiple edges, the intersection of any pair of its edges is either empty or consists of a single vertex.

To complete the proof of the theorem, we must study the intersection of an r-gon and an edge. It suffices to study only the case in which the edge has two common vertices with the r-gon, but does not belong to it. Obviously, this edge belongs to some other r-gon of X_* (and there is one such r-gon if the edge is external for X_*, or two if the edge is internal for X_*). Hence we obtain two r-gons with two common vertices. Because of the argument above, they have a common edge connecting these two vertices, whence there are two distinct edges connecting these two vertices. Once again, the girth is 2, and we arrive at a contradiction with assumption (1) of the theorem asserting that $r \geq 3$. Therefore, if the edge has two common vertices with an r-gon, then this edge necessarily belongs to this r-gon.

Hence our partition of the polyhedron X_* into regular r-gons isomorphic to a generalized polycycle P_* is a cell complex. The proof of the theorem is complete. □

Since condition (3) follows from conditions (1) and (2), the subscript $*$ in the mapping (2_*) can be omitted, and it coincides with the mapping (2). The generalized polycycle P_* coincides with our initial polycycle P.

3. ON AN ATTEMPT TO GENERALIZE THE NOTION OF POLYCYCLE

The attempt to generalize the notion of polycycle in the previous section did not lead to its extension. Now, we are going to make another attempt. We start with an attempt to extend the notion of face.

If the plane graph is connected, then any of its internal faces is a simply connected domain. We assumed previously that the closure of this domain is simply connected as well. Now, we assume that the closure of the simply connected domain under consideration is not simply connected. Choose an orientation of the plane. Let us go along the boundary of an interior face of the plane graph following the orientation. We pass along any edge such that the face is on both sides of it twice: first in one direction, then in the opposite one. After a complete path along the boundary of the internal face we obtain an edge circuit. Once again we denote the number of edges of this path by r.

It turns out that assumption (1) in our theorem can be replaced by the following weaker condition:

$(1')$ each internal face of the plane graph G is bounded by a closed edge path of length r, where $r \geq 3$.

The boundary of the internal face in condition $(1')$ need not be a simple edge circuit.

Suppose a nonseparable plane graph G satisfies conditions $(1')$ and (2). Then the following statement is true.

Proposition. *Assumptions* $(1')$, (2) *imply assumptions* (1), (2).

Proof. We call a nonseparable plane graph G satisfying assumptions $(1')$, (2) together with its internal faces a *generalized*[9] polycycle. Denote it by P_{**}. After replacing each flat r-gon in P_{**} by a two-dimensional regular r-gon of positive curvature with the angles $2\pi/q$, we obtain, instead of the generalized polycycle P_{**}, a polyhedron X_{**}. A two-dimensional r-gon in the polyhedron X_{**} is not necessarily a disk, since some of its boundary points can be identified. Moreover, the first barycentric subdivision $\mathcal{B}X_{**}$ is not necessarily a triangulation. However, the second subdivision $\mathcal{B}^2X_{**}$ is a triangulation, as a simple verification shows.

Pick a barycentric triangle of the second barycentric subdivision $\mathcal{B}^2X_{**}$. Let us map this triangle onto a triangle of the same type in the subdivision $\mathcal{B}^2(r^q)$ preserving the types of the triangle vertices. Extend this mapping to a continuous locally isometric cell mapping

$$f_{**}\colon \mathcal{B}^2X_{**} \to \mathcal{B}^2(r^q). \qquad (3_{**})$$

The proof of the existence of the mapping (3_{**}) coincides with that in the case of the mapping (2_*). The mapping f_{**} takes the edge skeleton of X_{**} to the edge skeleton of the partition (r^q). Would the two-dimensional r-gon of the polyhedron X_{**} have an external self-tangency, a part of its boundary starting and ending at the self-tangency point would form a closed edge circuit γ whose length is less than r. Then the image $f_{**}(\gamma)$ would also be shorter than r. This is impossible, however, since the girth of the edge skeleton of the partition (r^q) is r. Therefore, no two-dimensional r-gons in the polyhedron X_{**} possess self-tangency. The proposition is proved. $\square$

Thus, any two-dimensional r-gon of the polyhedron X_{**} under study is a disk. The edge path in assumption $(1')$ is in fact an edge circuit. The subscript $**$ is superfluous. The mapping (3_{**}) coincides with (3). Once again, the polycycle P_{**} coincides with the initial polycycle P. Due to the above proposition, this way of weakening the assumptions of the theorem is just a formal one.

Remark. A two-dimensional convex polygon of constant curvature is a disk. After identifying some boundary points of the disk preserving the constancy of the curvature, it remains a disk no longer, and it cannot be extended to a simply connected metric space of constant curvature[10]. This fact explains why the formal weakening of the assumptions of the theorem did not lead to a real extension of the notion under study, the notion of a polycycle.

[9] This is the second attempt to generalize the initial notion of polycycle in the definition.

[10] Recall that in order to simplify the study of polycycles we replace an arbitrary (r, q)-polycycle P by a combinatorially isomorphic two-dimensional polyhedron X which is a *simply connected* metric space of *constant* curvature.

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