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ON t-DISTANCE SETS OF $(0, \pm 1)$-VECTORS

ABSTRACT. We consider sets of $(0, \pm 1)$-vectors in R^n, having exactly s non-zero positions. In some cases we give best or nearly best possible bounds for the maximal number of such vectors if all the pairwise scalar products belong to a fixed set D of integers. The investigated cases include $D = \{-d, d\}$, which corresponds to equiangular lines.

1. INTRODUCTION

For given integers n and $s, n \geqslant s \geqslant 1$, we consider sets of $(0, \pm 1)$-vectors, $V = \{v_1, \dots, v_m\}$ in the n-dimensional Euclidean space R^n, such that each vector has s non-zero coordinates, and consequently they have all the same length $\sqrt{s}$. For two vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ we define the scalar product$(\mathbf{x}, \mathbf{y})$ in the usual way: $(\mathbf{x}, \mathbf{y}) = \Sigma_{1 \leqslant i \leqslant n} x_i y_i$.

For a subset D of $\{-s, -s+1, \dots, 0, 1, \dots, s-1\}$ let us make the following

DEFINITION 1.1. *The set* $V = \{v_1, \dots, v_m\}$ *is an element of* $V(n, s, D)$ *if*

(i) v_i *is a* $(0, \pm 1)$*-vector of dimension* n *for* $i = 1, \dots, m$;

(ii) $(v_i, v_i) = s$ *for* $i = 1, \dots, m$;

(iii) $(v_i, v_j) \in D$ *for* $1 \leqslant i < j \leqslant m$.

Let us denote by $m(n, s, D)$ the maximum cardinality of $V \in V(n, s, D)$. For $V \in V(n, s, D)$, we set $\Omega(V) = \{v/\sqrt{s} : v \in V\}$, the homothetical image of V on the unit sphere, Ω^n.

If $(v_i, v_j) = d$, then the Euclidean distance of $v_i/\sqrt{s}$ and $v_j/\sqrt{s}$ is $\sqrt{2(1 - d/s)}$; hence $\Omega(V)$ is a $|D|$-distance set on Ω^n, i.e. these points span at most $|D|$ different non-zero distances. Delsarte *et al.* [3] have shown:

THEOREM 1.2. *If X is a t-distance subset of Ω^n, then*

$$|X| \leqslant \binom{n+t-1}{t} + \binom{n+t-2}{t-1}.$$

For $(0, \pm 1)$-vectors we have the following immediate

COROLLARY 1.3. $m(n, s, D) \leqslant \binom{n+|D|-1}{|D|} + \binom{n+|D|-2}{|D|-1}$.

Since this bound is independent of s and depends only on the cardinality of D, it is not surprising that it can be improved for particular cases.

Let $V \in V(n, s, \{d\})$ with $d > 0$. We call V a *sunflower* if there are d positions $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n$, such that all the vectors in V have the same non-zero (i_j)th coordinates for $1 \leqslant j \leqslant d$.

THEOREM 1.4. (Deza and Frankl [5]). *Suppose* $V \in V(n, s, \{d\})$. *Then*

Geometriae Dedicata **14** (1983) 293–301. 0046–5755/83/0143–0293$01. 35.
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(i) $d < 0$: $|V| \leqslant [1 - s/d]$, moreover, the same bound holds if $V \in V(n, s, \{-s, \ldots, d\})$.

(ii) $d = 0$: $|V| \leqslant n$ (the case of equality corresponds to so-called weighing matrices, which are just Hadamard matrices if $s = n$).

(iii) $d > 0$, $d \geqslant s/2$: $|V| \leqslant \max\{(s-d)(d+2),\ n-d\}$, and in the case $|V| > (s-d)(d+2)$ the set V is a sunflower.

(iv) $d > 0$, $d < s/2$: $|V| \leqslant \max\{(s-d)^2 + (s-d) + 1,\ n-d\}$, and in the case $|V| > (s-d)^2 + (s-d) + 1$ the set V is a sunflower. Let us mention that all the above bounds are best possible for large n, e.g. for $n > s^2$. In case (iii) there always exists a V with $|V| = (s-d)(d+2)$ which is not a sunflower. In case (iv) nonsunflowers with cardinality $(s-d)^2 + (s-d) + 1$ exist if there is a projective plane of order $s-d$ or a Hadamard matrix of order $s-d-1$.

Part (i) of the above theorem was proved by Delsarte, *et al.* [3] in a more general setting; case (ii) is trivial.

2. Statement of the Results

Most of our results are best or nearly best possible upper bounds for the case $n > n_0(s)$.

Our first theorem shows that in the case $D = \{d,\ d+1, \ldots, d+t-1\}$, $d_0 \geqslant 0$, $t \geqslant 2$, for $m(n, s, D)$ the same bound holds as for (0, 1)-vectors.

THEOREM 2.1. *Suppose d, t are non-negative integers, $t \geqslant 2$. Then for $n > n_0(s)$ we have*

$$m(n, s, \{d, d+1, \ldots, d+t-1\}) \leqslant \binom{n-d}{t} \Big/ \binom{s-d}{t}.$$

Note that in view of Theorem 1.4 the condition $t \geqslant 2$ cannot be omitted. The bound is best possible in the sense that equality can be attained whenever a Steiner-system $S(n-d, s-d, t)$ exists, i.e. a family of $(s-d)$-subsets of an $(n-d)$-set X, such that every t-subset of X is contained in exactly one member of the family. The same bound for (0, 1)-vectors was proved in [4].

By a simple linear algebraic argument we deduce the following:

THEOREM 2.2. *Suppose a, b are integers, $0 \leqslant a < b$, and $d \equiv a \pmod b$ for every $d \in D$, but $s \not\equiv a \pmod b$. Then*

$$m(n, s, D) \leqslant n + 1.$$

For the case $D = \{-d, d\}$ we have

THEOREM 2.3. *Suppose $d > 0$, and $n > n_0(s)$. Then $m(n, s, \{-d, d\}) \leqslant n + 1$, unless s/d is an odd integer. In this latter case $m(n, s, \{-d, d\}) \leqslant 2(n-d)$ holds.*

The interest in the case $D = \{-d, d\}$ is explained by the following:

As before, let $\Omega(V)$ be the homothetical image of $V \in V(n, s, \{-d, d\})$ on the unit sphere Ω^n. Then for any two points A, B in $\Omega(V)$ the angle between the lines OA, OB (O is the origin) is the same, arc cos d/s. Thus $\Omega(V)$ corresponds to a set of *equiangular* lines (in the elliptic space E^{n-1}, whose points are the lines through the origin, and whose the distance is defined as the angle of these lines, it is just an equidistant point set).

The problem of determining the maximum number of equiangular lines in R^n has been extensively studied. For $0 \leqslant a \leqslant 1$ let us denote by $v_a(n)$ the maximum number of equiangular lines in R^n, with angle arc cos a.

THEOREM 2.4. (Lemmens and Seidel [9]).

(i) $a = 1/3$: $v_{1/3}(n) = 2(n-1)$ *for* $n = 3, 4$ *and* $n \geqslant 15$
$v_{1/3}(n) = 28$ *for* $7 \leqslant n \leqslant 14$, $v_{1/3}(5) = 10$, $v_{1/3}(6) = 16$.
(ii) $\max_a v_a(n) \geqslant 2^k(2^{2k} + 2^k + 1)$ *for* $n = 2^{2k} + 2^k + 1$;

$\max_a v_a(n) = v_{1/3}(n)\,(v_{1/5}(n))$ *for* $4 \leqslant n \leqslant 14$ $(15 \leqslant n \leqslant 43)$, *respectively*.

Let us note that the bounds in (i), for the cases $n = 3$, $6 \leqslant n \leqslant 14$, and also the lower bound in (ii) can be attained by $(0, \pm 1)$-vectors. For example, for $7 \leqslant n \leqslant 14$ we take the seven lines, each containing three points, of the projective plane of order 2, and replace each line by the following four vectors $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$, putting zeros in the positions which are not contained in the corresponding line. Again, for odd $n \geqslant 15$ we can attain equality in (i) with $(0, \pm 1)$-vectors: all the vectors have 1 in the first position, v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3} are non-zero in the positions $2i$ and $2i+1$ having $(1, 1), (1, -1), (-1, 1), (-1, -1)$ in these positions, respectively. On the other hand, for $n = 4, 5$ and even $n > 15$ the bounds of (i) cannot be obtained by $(0, \pm 1)$-vectors.

Let us mention (cf. [9]) that any system of unit vectors with pairwise scalar products a, b; $a + b < 0$, in R^n can be lifted to a system of equiangular lines in R^{n+1}. For $(0, \pm 1)$-vectors a similar statement holds: if $V \in V(n, s, \{a, b\})$, with $a + b = -2c$, c positive integer, then by adding in V to each vector c new 1's in positions $n + 1$ through $n + c$ we obtain $V \in V(n + c, s + c, \{a + c, -(a + c)\})$.

For the proof of Theorem 2.3 we need Ramsey's theorem in its most elementary form (cf. Ramsey [12]):

THEOREM 2.5. *For any two positive integers $k, l > 2$, there exists an integer $R(k, l)$ such that in every graph with $R(k, l)$ or more vertices there is either a complete subgraph on k or an empty subgraph on l vertices.*

We have one more theorem, on $V(n, s, D)$'s in general.

THEOREM 2.6. *Let f denote the number of non-negative elements of D. Then $m(n, s, D) \leqslant c(s)n^f$, where $c(s)$ is a constant depending only on s.*

The proof of this theorem is running along the same lines as that of Theorem 4 in Deza *et al.* [4]. As the bounds on $c(s)$ are rather poor, we did not include the proof.

3. THE PROOF OF THEOREM 2.1

First we need some definitions. Let, as always, $V \in V(n, s, D)$, $1 \leqslant i \leqslant n$. Then let $\deg_i^+(V)$ or simply $\deg_i^+$ ($\deg_i^-$) denote the number of vectors in V which have $+1$ (-1) in the ith position. Moreover, let V_i^+ (V_i^-) be the set of these $\deg_i^+$ ($\deg_i^-$) vectors replacing the ith coordinate by 0.

For any integer j, define $D - j = \{c - j : c \in D\}$.

The following two observations are immediate:

(i) V_i^+ and V_i^- can be viewed as $V \in V(n-1, s-1, D-1)$;

(ii) $\Sigma_{1 \leqslant i \leqslant n} |V_i^+| + |V_i^-| = \Sigma_{1 \leqslant i \leqslant n} \deg_i^+ + \deg_i^- = s|V|$.

Using (i) and (ii) we deduce

LEMMA 3.1. $m(n, s, D) \leqslant (2n/s)m(n-1, s-1, D-1)$.

When iterating this lemma j times, we obtain

COROLLARY 3.2. $m(n\, s, D) \leqslant 2^j\left(\binom{n}{j}\Big/\binom{s}{j}\right)m(n-j, s-j, D-j)$.

For a vector v let $S(v)$ denote its *support*:

$$S(v) = \{i : 1 \leqslant i \leqslant n,\ v \text{ is non-zero in the } i\text{th position}\}.$$

For a given subset T of $\{1, 2, \ldots, n\}$ we define

$$V(T) = \{v \in V : T \subseteq S(v)\}.$$

PROPOSITION 3.3. *Let $\bar{d}$ be the largest element in D and T an arbitrary $(\bar{d}+1)$-element subset of $\{1, 2, \ldots, n\}$. Then*

$$|V(T)| \leqslant 2^{\bar{d}+1}(s - \bar{d}).$$

Proof. In fact there are $2^{\bar{d}+1}$ possibilities for the restriction of v to T for $v \in V(T)$. On the other hand, vectors having the same restriction have scalar product at most -1 on $\{1, \ldots, n\} - T$. Thus in view of Theorem 1.4(i) there are at most $s - \bar{d}$ of them, and the statement of the proposition follows.

Now we take a V satisfying the assumptions of Theorem 2.1.

PROPOSITION 3.4. *If* $|V| > \binom{s^2}{d+1} 2^{d+t}(s-d-t+1)\binom{n-d-1}{t-1}$ then we can find vector $v_1, v_2, \ldots, v_{s+1} \in V$ such that $C = S(v_1) \cap S(v_2)$ is a d-element set and we have $C = S(v_i) \cap S(v_j)$ for every $1 \leqslant i < j \leqslant s+1$..

Proof. Let us choose an arbitrary $v_1 \in V$, and define $T_1 = S(v_1)$. Suppose $v_1, \ldots, v_r$ and $T_1, \ldots, T_r$ have already been defined. Then choose $v_{r+1} \in V$ such that $|S(v_{r+1}) \cap T_r| \leqslant d$, set $T_{r+1} = T_r \cup S(v_{r+1})$. First we have to show that this is possible. Let us suppose the contrary.

This means that for every $v \in V$ we have $|S(v) \cap T_r| \geqslant d + 1$. Let T_0 be an arbitrary $(d+1)$-subset of T_{r+1}. Set $\mathscr{A}_0 = \{S(v) - T_0 : T_0 \subseteq S(v),\ v \in V\}$. We consider $\mathscr{A}_0$ as a multiset, i.e. each set may occur several times. However, Proposition 3.3 yields that every $(t-1)$-subset of $\{1, 2, \ldots, n\} - T_0$ can be contained in at most $2^{d+t}(s-d-t+1)$ of the $S(v)$. Thus we derive

$$|\mathscr{A}_0| \leqslant 2^{d+t}(s-d-t+1)\binom{n-d-1}{t-1}.$$

As $|T_r| = d + r(s-d) \leqslant s^2$, there are at most $\binom{s^2}{d+1}$ choices for T_0. Thus

$$|V| \leqslant \binom{s^2}{d+1} 2^{d+t}(s-d-t+1)\binom{n-d-1}{t-1};$$

a contradiction.

Since $(v_1, v_{r+1}) \geqslant d$, $|S(v_1) \cap S(v_{r+1})| \geqslant d$ holds. Thus $|S(v_{r+1}) \cap T_r| = d$ holds. Consequently, $S(v_{r+1}) \cap T_r = S(v_{r+1}) \cap S(v_1)$. By definition $T_r = S(v_1) \cup \ldots \cup S(v_r)$, implying $S(v_{r+1}) \cap S(v_i) \subseteq S(v_{r+1}) \cap S(v_1)$ for $2 \leqslant i \leqslant r$. But $|S(v_{r+1}) \cap S(v_i)| \geqslant d$ for $2 \leqslant i \leqslant r$. Thus for $2 \leqslant i \leqslant r$, $S(v_{r+1}) \cap S(v_1) = S(v_{r+1}) \cap S(v_i)$ holds. We claim that all these pairwise intersections are in fact $C = S(v_1) \cap S(v_2)$. Indeed, $|C| = d$ and $C \supseteq (S(v_{r+1}) \cap S(v_1)) \cap (S(v_{r+1}) \cap S(v_2)) = S(v_{r+1}) \cap S(v_1)$. Since the set on the right-hand side is also a d-element set, we must have equality. Thus we have shown that choosing $v_1, \ldots, v_{s+1}$ in the above way, they will satisfy the proposition.

For $n > n_0(s)$ we have

$$\binom{n-d}{t} \Big/ \binom{s-d}{t} > \binom{s^2}{d+1} 2^{d+t}(s-d-t+1)\binom{n-d-1}{t-1}.$$

Thus we may assume the existence of $v_1, \ldots, v_{s+1} \in V$, satisfying Proposition 3.4. As $(v_1, v_i) \geqslant d$ and $|S(v_1) \cap S(v_i) = C| = d$, v_1 and v_i must be identical on C $(2 \leqslant i \leqslant s+1)$.

Now we are going to show that every $v \in V$ coincides with v_1 on C. Let us denote by w_i the $(0, \pm 1)$-vector which is zero on C but coincides with v_i everywhere else $(1 \leqslant i \leqslant s+1)$. The sets $S(w_1), \ldots, S(w_{s+1})$ are pairwise disjoint and their number is $s+1$. Thus no s-element set intersects all of them. In particular, for every $v \in V$ we can find i, $1 \leqslant i \leqslant s+1$, such that $S(w_i) \cap S(v) = \varnothing$. Consequently, $S(v_i) \cap S(v) \subseteq C$. As $|C| = d$ and $(v, v_i) \geqslant d$, v and v_i must be identical on C. But v_i coincides with v_1 on C, proving our claim.

Thus we have shown the existence of a d-subset C such that all the vectors in V have the same non-zero positions on C. Define for $v \in V$ the vector w which is zero on C but coincides with v everywhere else. Let W denote the collection of these w's. Then $|W| = |V|$ and $W \in V(n-d,\ s-d,\ \{0, 1, \dots, t-1\})$. Thus it will be sufficient to prove the theorem for the particular case $d = 0$.

From now on we assume $d = 0$. Denote $\deg_i^+ = d_i^+$, $\deg_i^- = d_i^-$.

If for every i, $1 \leqslant i \leqslant n$, $d_i^+ d_i^- = 0$, then we can replace -1's by $+1$'s in all the vectors without altering the scalar products. We obtain a system of (0, 1)-vectors, which can also be viewed as a family of s-element subsets of $\{1, \dots, n\}$. The fact that any two different vectors have scalar product less than t yields that any two sets in the family have at most $t-1$ element in common, i.e. every t-subset of $\{1, 2, \dots, n\}$ is contained in at most one of the s-sets. We infer $|V| \leqslant \binom{n}{t} \Big/ \binom{s}{t}$, as desired. For the case $d_i^+ d_i^-$ is non-zero we shall prove that $d_i^+ + d_i^-$ is at most $c(s)n^{t-2}$ only. Let us omit all these vertices – denote by q their number. What remains can be viewed as a family of s-subsets of an $(n-q)$-element set, no two of which intersect in more than $t-1$ elements. By the above argument the number of such s-sets is at most $\binom{n-q}{t} \Big/ \binom{s}{t}$. We infer

$$|V| \leqslant \binom{n-q}{t} \Big/ \binom{s}{t} + qc(s)n^{t-2} < \binom{n}{t} \Big/ \binom{s}{t}, \qquad \text{for} \quad n > n_0(s).$$

Thus to conclude the proof of the theorem it is sufficient to prove the next proposition.

PROPOSITION 3.5. *If both* $\deg_i^+(V)$ *and* $\deg_i^-(V)$ *are non-zero, then*

$$\deg_i^+ + \deg_i^- \leqslant \left(\binom{n}{t-2} \Big/ \binom{s-2}{t-2} \right) 2^{t-1}(s-t+1)(s-1)$$
$$\leqslant c(s)n^{t-2}.$$

Proof. By symmetry we assume that $\deg_i^+ \geqslant \deg_i^-$. Let u_0 be a vector in V which has -1 in the ith position, and let $u_1, \dots, u_r \in V$, be all those having $+1$ in the ith position. Let us set $T_j = S(u_j) - \{i\}$. As $(u_0, u_j) \geqslant 0$, we have $T_0 \cap T_j \neq \varnothing$ for $1 \leqslant j \leqslant r = \deg_i^+$. Moreover, in at least one position of $T_0 \cap T_j$ the two vectors u_0, u_j have the same sign. Thus there is one position, say the kth, of T_0 where at least $r/(s-1)$ of the u_j have the same non-zero entry. Omitting from these vectors the ith and kth entry we obtain at least $r/(s-1)$ vectors forming $V \in V(n-2,\ s-2,\ \{-2,\ -1,\ 0, \dots, t-3\})$. Thus applying Corollary 3.2 with $j = t-2$, and taking into account Theorem

1.4(i), we deduce

$$\deg_i^+/(s-1) \leqslant 2^{t-2}\left(\binom{n}{t-2}\Big/\binom{s-2}{t-2}\right)(s-t+1).$$

As $\deg_i^- \leqslant \deg_i^+$, the statement of the proposition follows. □

4. The proof of theorem 2.2

Let $V = \{v_1, \dots, v_m\}$ be in $V(n, s, D)$, and let N be the matrix which has m rows, n columns, the ith row being just v_i. Set $M = NN^T$, where N^T is the transpose of N. Of course, M is $m \times m$, with general element $m_{i,j} = (v_i, v_j)$. Since rank M = rank $N \leqslant n$, it will be sufficient to show that rank $M \geqslant m - 1$. To do so add one more row, consisting only of 1's, to M. Let M' be the new matrix. Obviously rank $M' \leqslant$ rank $M + 1$. Thus it is sufficient to show rank $M' \geqslant m$. Let us subtract a times the last row from all the others. Then the first m rows of the new matrix are congruent modulo b to $(s-a)I_m$, where I_m is the $m \times m$ identity matrix, proving rank $M' \geqslant m$. □

5. The proof of theorem 2.3

Let V be in $V(n, s, \{-d, d\})$ with $|V| = m$. First note that if d does not divide s, then an application of Theorem 2.2 yields $m(n, s, \{-d, d\}) \leqslant n + 1$. So assume $s = bd$, with $b \geqslant 2$, integer. In view of Theorem 1.4(i) there are no $b + 2$ vectors in V having pairwise scalar products equal to $-d$. Thus for $m \geqslant R(b + 2, s^2)$ Theorem 2.5 yields an independent set $W = \{w_1, \dots, w_r\} \subset V$, $r \geqslant s^2$, i.e. $(w_i, w_j) = d$ for $1 \leqslant i < j \leqslant r$.

Now Theorem 1.4(iii) and (iv) yields that W is a sunflower.

By symmetry we may assume that all the vectors in W have 1 in the first d coordinates. Let us denote by e_d the vector which has 1 in the first d positions and zero elsewhere. Define $u_i = w_i - e_d$ for $i = 1, \dots, r$.

PROPOSITION 5.1. *For every $v \in V$ we have $(v, e_d) = \pm d$.*

Proof. Suppose the contrary. This means that for some $v \in V$ all the scalar products (v, u_i), $i = 1, \dots, r$ have absolute values at least 1. Let us fix $\varepsilon_i = \pm 1$ in such a way that $(v, \varepsilon_i u_i) \geqslant 1$ for $i = 1, \dots, r$. Expanding the inequality $(sv - \Sigma_{1 \leqslant i \leqslant s^2} \varepsilon_i u_i, sv - \Sigma_{1 \leqslant i \leqslant s^2} \varepsilon_i u_i) \geqslant 0$, we obtain $-ds^2 = s^3 - 2ss^2 + s^2(s-d) \geqslant 0$; a contradiction since $d > 0$. □

Thus we have shown that every $v \in V$ has either only $+1$ or only -1 in the first d positions. If necessary we replace some of the v by $-v$, thus we may assume: each $v \in V$ coincides with e_d in the first d positions.

Let us denote by V' the set of $(n-d)$-dimensional vectors, which we obtain by deleting from each $v \in V$ the first d positions. Then V' is in $V(n-d, s-d, \{-2d, 0\})$. Again Theorem 2.2 yields $|V'| = |V| \leqslant n+1$ unless $s-d$ is divisible by $2d$, i.e. $b = 2b_0 + 1$ is an odd integer ($s = bd$). This proves the first part of the statement of Theorem 2.3.

Now consider the set of supports $\mathscr{F} = \{S(v) : v \in V'\}$. Then $\mathscr{F}$ is a family of $(s-d)$-subsets of the $(n-d)$-set $\{d+1, \ldots, n\}$. Hence we can find $i, d < i \leqslant n$, such that i is contained in at least $m(s-d)/(n-d)$ of the sets. Thus by symmetry, we may assume that there are at least $q \geqslant m(s-d)/2(n-d)$ vectors $u_1, \ldots, u_q$ in V' having 1 in the ith position. Omitting the ith position of these vectors we are left with $V \in V(n-d-1, s-d-1, \{-2d-1, -1\})$. Thus Theorem 1.4(i) yields $q \leqslant s-d$ and, consequently, $|V| = |V'| \leqslant 2(n-d)$. □

6. Concluding remarks

I. First we give some examples for the use of $(0, \pm 1)$-matrices.

(a) Signed set-systems and, especially, oriented matroids (cf. [1]).

(b) Weighing matrices (a weighing matrix is just $V \in V(n, s, \{0\})$ with $|V| = n$).

(c) Generalized balanced matrices (cf. [6]), i.e. $V \in V(n, s, \{d\})$ converting (by changing all -1's to 1's) into a BIB design with parameters $v = |V|$, $b = n$, $r = s$, k, λ; more general, Bhascar Rao designs (cf. [14]).

(d) Circulant (v, k, μ) designs (cf. [8]), i.e. circulant $V \in V(n, s, \{d\})$ with $|V| = n = v$, $s = k$, $d = \mu$.

II. On some related research connected with scalar products of unit vectors, cf. [2] and [10].

III. Let us mention the following related

CONJECTURE 6.1 (Neumaier [11]). *Suppose that $m \geqslant 2n+1$ equiangular lines at* $\cos \varphi = 1/p$ *span R^n (so the product of unit vectors along different lines is $1/p$). Then m is of the order p^2.*

IV. One of the ways to estimate $m(n, s, D)$ is to try to use linear programming bounds in the following association scheme on the set X of all n-dimensional $(0, \pm 1)$-vectors with s non-zero positions: two vectors are (i, j)-related if they have Hamming distance i and agree in j non-zero positions. The association classes are the orbits of the group Z_2^n wreath S_n. This association scheme is not metric. It is studied in [7], [13], in a more general setting. There X consists of all the q-ary sequences of length n and weight s (our case is $q = 3$). Let us mention that in this case the association classes are the orbits of S_{q-1}^n wreath S_n on X^2.

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(Received August 3, 1982; revised version, November 23, 1982)