

On Squashed Designs

M. Deza and P. Frankl

C.N.R.S., Université de Paris VII, Paris, France

Abstract. Extremal problems and the existence of designs is investigated in a new type of combinatorial structures, called squashed geometries.

1. Introduction

Suppose X is an n -element set $0 < k < n$, $L \subset \{0, 1, \dots, k-1\}$. An (n, k, L) -system is a family \mathcal{A} of k -subsets of X satisfying

$$|A \cap A'| \in L \quad \text{for all distinct } A, A' \in \mathcal{A}. \quad (1.1)$$

The investigation of the maximum size of (n, k, L) -systems has been the subject of a large number of papers. Let us recall:

Theorem 1.1 ([DEF]). *Suppose \mathcal{A} is an (n, k, L) -system, $L = \{l_0, \dots, l_{s-1}\}$, $l_0 < l_1 < \dots < l_{s-1}$. There exists a constant $c = c(k, L)$ such that either $|\mathcal{A}| < cn^{s-1}$ or (i)-(iii) hold:*

- (i) $|\mathcal{A}| \leq \prod_{l \in L} \frac{n-l}{k-l}$;
- (ii) $(l_1 - l_0)(l_2 - l_1) \cdots (l_{s-1} - l_{s-2})(k - l_{s-1})$;
- (iii) $\left| \bigcap_{A \in \mathcal{A}} A \right| = l_0$

Note that the upper bound in (i) has degree s in n , thus it holds for all $n > n_0(k, s)$. In (i) equality holds (for $n > n_0(k, s)$) if and only if \mathcal{A} is the family of flats of rank s of a *perfect matroid design* (a matroid in which all flats of rank i have size l_i , $0 \leq i < s$), cf. [De 1]. Special cases include the Erdős-Ko-Rado theorem [EKR]: $L = \{t, t+1, \dots, k-1\}$, t -designs with $\lambda = 1$: $L = \{0, 1, \dots, t-1\}$, etc.

In the present paper we investigate the problem for restricted (n, k, L) -systems. Let \mathcal{F} be a family of sets closed under intersection $\mathcal{F} \subset 2^X$, $F, G \in \mathcal{F}$ implies $F \cap G \in \mathcal{F}$.

Definition 1.2. An (n, k, L) -system $\mathcal{A} \subset \mathcal{F}$ is called an \mathcal{F} -squashed (or shortly) *squashed family*.

From now on \mathcal{A} will always denote an \mathcal{F} -squashed (n, k, L) -system. The maximum size of \mathcal{A} is denoted by $m_{\mathcal{F}}(n, k, L)$. Set

$$\mathcal{F}^{(r)} = \{F \in \mathcal{F} : |F| = r\}.$$

Let us define the quantity n_r by

$$n_r = \max_{G \in \mathcal{F}^{(r)}} (r) \left| \bigcup_{F \in \mathcal{F}} F - G \right|.$$

Note that $n_r = n - r$ holds for $\mathcal{F} = 2^X$.

Theorem 1.3. Suppose that \mathcal{A} is an \mathcal{F} -squashed (n, k, L) -system. Then either $|\mathcal{A}| < c(k, L)n^{s-1}$ or (i)-(iii) hold.

- (i) $|\mathcal{A}| \leq \prod_{i=0}^{s-1} \frac{n_i}{k - l_i};$
- (ii) $(l_1 - l_0) \cdot \dots \cdot (l_{s-1} - l_{s-2}) (k - l_{s-1});$
- (iii) $\left| \bigcap_{A \in \mathcal{A}} A \right| = l_0.$

The investigation of the case of equality leads to the definition of *squashed designs* and *squashed geometries* which we discuss in Section 3, after the proof of Theorem 1.3 given in Section 2. In the case where $N^{(i)} = \{0, 1, \dots, n^{(i)} - 1\}$ and $X = N^{(1)} \times N^{(2)} \times \dots \times N^{(d)}$ one calls a subset $B \subset X$ *injective* if all elements of B have distinct entries for each $N^{(i)}$, i.e., if $\mathbf{b} = (b_1, \dots, b_d)$ and $\mathbf{b}' = (b'_1, \dots, b'_d)$ are in B then $b_i \neq b'_i$ holds for $1 \leq i \leq d$.

If \mathcal{F} is the family of all injective subsets of $N^{(1)} \times \dots \times N^{(d)}$ then Theorem 1.3 implies

Corollary 1.4 ([DF]). Suppose \mathcal{A} is an (n, k, L) -system of injective subsets of $N^{(1)} \times \dots \times N^{(d)}$ then for $\min_i n^{(i)} \geq n_0(k, L)$

$$|\mathcal{A}| \leq \prod_{i=0}^{s-1} \frac{\prod_{j=1}^d n^{(j)} - l_i}{k - l_i} \text{ holds.} \tag{1.2}$$

In (1.2) equality holds if and only if \mathcal{A} is an *injection design* (cf. [DF]).

Suppose $X = X_1 \cup \dots \cup X_k$ where the X_i are pairwise disjoint \tilde{n} -element sets and \mathcal{F} consists of *partial transversals*, i.e., $\mathcal{F} = \{F \subset X : |F \cap X_i| \leq 1 \text{ for } 1 \leq i \leq k\}$, we have $n_r = (k - r)\tilde{n}$. Then Theorem 1.3 yields

Corollary 1.5. *Suppose \mathcal{A} is an (n, k, L) -system consisting of transversals of $X_1 \cup \dots \cup X_k$, $\tilde{n} > n_0(k, L)$. Then $|\mathcal{A}| \leq \tilde{n}^s$ holds.*

Here equality corresponds to *transversal matroid designs* (cf. [CDF]).

2. The Proof of Theorem 1.3

We apply induction on k . We distinguish two cases.

(a) $l_0 = 0$

Set $X_0 = \bigcup_{F \in \mathcal{F}} F$. By definition $|X_0| = n_0$. For all $x \in X_0$ the family $\mathcal{A}(x) = \{A - \{x\} : x \in A \in \mathcal{A}\}$ is an $(n - 1, k - 1, \{l_1 - 1, \dots, l_{s-1} - 1\})$ -system which is $\mathcal{F}(x)$ -squashed for $\mathcal{F}(x) = \{F - \{x\} : x \in F \in \mathcal{F}\}$. By induction we infer

$$|\mathcal{A}(x)| \leq \prod_{i=1}^{s-1} \frac{n_{l_i}}{k \times l_i} \quad \text{for } n > n_0(k, L).$$

Moreover, $(l_2 - l_1) \cdot \dots \cdot (k - l_{s-1})$ and $|\bigcap_{B \in \mathcal{A}(x)} B| = l_1 - 1$ follow unless $|\mathcal{A}(x)| < c(k, L)n^{s-2}$. Using $\sum_{x \in X_0} |\mathcal{A}(x)| = k|\mathcal{A}|$, (i) follows.

To prove (ii) we have to show $l_1 = (l_1 - l_0) |l_2 - l_1|$. Define \mathcal{D} as the collection of l_1 -subsets D such that for some $x \in D$ we have $D - \{x\} = \bigcap \mathcal{A}(x)$. By definition \mathcal{D} consists of pairwise disjoint sets only. Omit all $A \in \mathcal{A}$ which contain some x with $|\bigcap \mathcal{A}(x)| \neq l_1 - 1$. By the induction hypotheses we omitted at most $c(k - 1, \{l_1 - 1, \dots, l_{s-1} - 1\})n^{s-1}$ sets. The remaining sets are all the disjoint union of k/l_1 members of \mathcal{D} . If two of them, say A, A' intersect in l_2 elements, then $A \cap A'$ must be the disjoint union of members of \mathcal{D} , yielding $l_1 |l_2$ and thus $l_1 | (l_2 - l_1)$. If not, by induction the number of remaining members of \mathcal{A} is bounded by $m(n, k, L - \{l_2\})$. Thus $|\mathcal{A}| < c(k, L)n^{s-1}$, as desired.

(iii) follows in a trivial way.

(b) $l_0 > 0$

Consider $A_0 = \bigcap \mathcal{A}$. If $|A_0| \neq l_0$, then Theorem 1.1 implies $|\mathcal{A}| < c(k, L)n^{s-1}$. If $|A_0| = l_0$, then (i)-(ii) follow by the induction hypothesis applied to $\mathcal{A}(A_0) = \{A - A_0 : A \in \mathcal{A}\}$. □

Remark 2.1. Looking carefully at the proof of Theorem 1.3, we see that for $|\mathcal{A}| > c(k, L)n^{s-1}$ equality in (i) implies that there is a family \mathcal{D} of pairwise disjoint $(l_1 - l_0)$ -sets D such that $|\bigcup \mathcal{D}| = n_{l_0}$ and every $A \in \mathcal{A}$ is the disjoint union of $(k - l_0)/(l_1 - l_0)$ members of \mathcal{D} along with $A_0 = \bigcap \mathcal{A}$, having cardinality l_0 .

Remark 2.2. Note that for the case \mathcal{F} a perfect matroid design, Theorem 1.3 was proved in [De 1].

Remark 2.3. The proof of Theorem 1.3 shows that if n_{i-1} is sufficiently large and \mathcal{A} attains equality in (i) then \mathcal{A} verifies $|A_1 \cap \dots \cap A_b| \in L$ for all $b \geq 2$, moreover, the meet semilattice generated by \mathcal{A} is *short*, i.e., for all $A_1, \dots, A_b \in \mathcal{A}$ there exist $A, A' \in \mathcal{A}$ with $A_1 \cap \dots \cap A_b = A \cap A'$.

Remark 2.4. It is easy to see that if one strengthens the assumptions of Theorem 1.3 to $|A_1 \cap \dots \cap A_b| \in L$ for all $b \geq 2$, then (i) and (iii) always hold (i.e., for all n). Moreover, equality in (i) implies that \mathcal{A} is a *squashed design* (cf. the definition in the next section).

3. Squashed Designs and Geometries

An \mathcal{F} -squashed geometry of rank s is a family \mathcal{G} partitioned into $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_s$ satisfying:

- (i) $\mathcal{G} \subset \mathcal{F}$, \mathcal{G} is closed under intersection, i.e., $G, G' \in \mathcal{G}$ implies $(G \cap G') \in \mathcal{G}$.
- (ii) If $G \subset H$, $G \in \mathcal{G}_i$, $H \in \mathcal{G}_j$; then $i < j$.
- (iii) If $G \in \mathcal{G}_i$, $i < s$ $x \notin G$, $G \cup \{x\}$ is contained in some member of \mathcal{F} , then there exists an $H \in \mathcal{G}_{i+1}$ containing $G \cup \{x\}$.

If $\mathcal{F} = 2^X$, then a squashed geometry is just a matroid. For $G \in \mathcal{G}_i$ the family $\mathcal{G}_G = \{G \cap G' : G' \in \mathcal{G}\}$ is the family of flats of a matroid of rank i on G . An \mathcal{F} -squashed design is an \mathcal{F} -squashed geometry in which flats of equal rank have equal size. The sizes l_0, l_1, \dots, l_s along with $l_{s+1} = n = |\bigcup \mathcal{G}|$ are called the *parameters* of the squashed design. Clearly, setting $l_s = k$, \mathcal{G}_s is an $(n, k, \{l_0, \dots, l_{s-1}\})$ -system. For $G \in \mathcal{G}_s$, $\mathcal{G}_G = \{G \cap G' : G' \in \mathcal{G}\}$ is the family of flats of a perfect matroid design.

Proposition 3.1. *If \mathcal{G} is a squashed design of rank s with parameters $l_0, \dots, l_{s-1}, k, n$ then*

$$|\mathcal{G}_s| \leq \prod_{i=0}^{s-1} \frac{n_{l_i}}{k - l_i} \tag{3.1}$$

with equality holding if and only if

$$|\bigcup (H - G) : G \subset H \in \mathcal{F}| = n_{l_i} \text{ holds for all } G \in \mathcal{G}_i, \quad 1 \leq i < s. \tag{3.2}$$

Proof. Let us note first that in view of (i) and (ii) \mathcal{G}_0 consists of a single set, say G_0 , $|G_0| = l_0$. Consequently every $G \in \mathcal{G}$ contains G_0 . Defining $\mathcal{G}(G_0) = \{G - G_0 : G \in \mathcal{G}\}$, $\mathcal{F}(G_0) = \{F - G_0 : G_0 \subset F \in \mathcal{F}\}$, $\mathcal{G}(G_0)$ is an $\mathcal{F}(G_0)$ -squashed design with parameters $(0, l_1 - l_0, \dots, l_{s-1} - l_0, k - l_0, n - l_0)$, thus it is sufficient to consider the case $l_0 = 0$, i.e., $\mathcal{G}_0 = \{\emptyset\}$. Now (i) and (iii) imply that $\bigcup \mathcal{G} = \bigcup \mathcal{F}$.

We apply induction on s . If $s = 1$, then $\mathcal{G}_s = \mathcal{G}_1$ is a family of pairwise disjoint $l_1 - l_0 = l_1$ -element sets. In view of (i) and (iii) we have $\bigcup \mathcal{G}_1 = \bigcup \mathcal{F}$. Consequently $|\mathcal{G}_1| = n_0/l_1$ holds. Next consider the general case, $s \geq 2$. For an arbitrary $x \in \bigcup \mathcal{F}$

$\mathcal{G}(x) = \{G - x : x \in G \in \mathcal{G}\}$ is an $\mathcal{F}(x)$ -squashed design with parameters $(l_1 - 1, \dots, l_{s-1}, k - 1, n(\mathcal{F}(x)))$, where

$$n(\mathcal{F}(x)) = \left| \bigcup_{H \in \mathcal{F}(x)} H \right| = l_1 - 1 + |\bigcup (F - G_0(x)) : G_0(x) \subseteq F \in \mathcal{F}| \leq l_1 - 1 + n_l.$$

Consequently

$$|\mathcal{G}_s(x)| \leq \prod_{i=1}^{s-1} \frac{n_{l_i}}{k - l_i}$$

holds with equality iff $n(\mathcal{F}(x)) = l_1 - 1 + n_l$ and $\mathcal{F}(x)$ satisfies the last conclusion of Proposition 3.1. Now (3.1) follows from $\sum_x |\mathcal{G}_s(x)| = k|\mathcal{G}_s|$, and the condition for the equality is derived via the above observation. \square

If (3.2) holds for all $i, 1 \leq i \leq s$, then \mathcal{G} is called a *perfect squashed design*. Let us note that every \mathcal{F} -squashed design \mathcal{G} is a \mathcal{G} -squashed design.

Proposition 3.2. *Suppose $\mathcal{G} = \mathcal{F}$ is a \mathcal{F} -squashed design. Then the following conditions are equivalent:*

- (i) \mathcal{G} is a perfect \mathcal{F} -squashed design of rank s ;
- (ii) there exist integers r_0, \dots, r_s so that each $H \in \mathcal{G}_i$ is contained in exactly r_i members of \mathcal{G}_s .

Proof. (i) \Rightarrow (ii): $H \in \mathcal{G}_i$, it is easily seen that $\mathcal{G}(H) = \{G - H : H \in G \in \mathcal{G}\}$ is a perfect $\mathcal{F}(H)$ -squashed design. Thus (ii) holds with

$$r_i = \prod_{i \leq j < s} \frac{n_{l_j}}{k - l_j}.$$

(ii) \Rightarrow (i): For $H \in \mathcal{G}_i$ define $X(H) = \bigcup_{H \in F \in \mathcal{F}} F - H$. By definition $n_{l_i} = \max_{H \in \mathcal{G}_i} |X(H)|$. For $i = s - 1$ we obtain $r_{s-1} = |X(H)| / (k - l_{s-1})$, i.e., $|X(H)|$ is constant: $n_{l_{s-1}} = r_{s-1}(k - l_{s-1})$ proving (3.2) for $i = s - 1$. Applying induction backwards we see that r_i , the number of $G \in \mathcal{G}_s$ containing $H \in \mathcal{G}_i$, is

$$r_i = \frac{|X(H)| \prod_{i < j < s} n_{l_j}}{\prod_{i \leq j < s} (k - l_j)}.$$

This yields that $|X(H)|$ is the same for all $H \in \mathcal{G}_i$ as desired. \square

In view of Proposition 3.2 a meet semilattice \mathcal{F} is a perfect \mathcal{F} -squashed design if and only if it is an M_s -design in the sense of Neumaier [Ne]. We refer the reader for some interesting results on M_s -designs to [Ne].

Definition 3.3. An \mathcal{F} -squashed geometry \mathcal{G} of rank s is called *regular* if it enjoys the following two properties:

- (i) Given $G \in \mathcal{G}_s, H \in \mathcal{G}_t, H \subset G$, the number of sets $E \in \mathcal{G}_r$ with $H \subset E \subset G$ is a constant $m(t, r)$.
- (ii) Given $H \in \mathcal{G}_t$, the number of sets $E \in \mathcal{G}_r$ with $H \subset E$ is a constant $c(t, r)$.

Note that conditions (i) and (ii) are in complete analogy with those of [D 1], however, Delsarte in his definition of regular semilattices puts a third, stronger condition which ensures that \mathcal{G}_s is an association scheme where $G, G' \in \mathcal{G}_s$ are i -associated if $G \cap G' \in \mathcal{G}_i$.

4. Examples of Squashed Designs

The simplest case is $\mathcal{F} = 2^X$. Then $\mathcal{G} = \mathcal{F}$ is a perfect squashed design. So are the truncations of \mathcal{F} : $\mathcal{F}^{(s)}: \{F \subset X: |F| \leq s\} \cup \{X\}$. In general, given an \mathcal{F} -squashed design \mathcal{G} with parameters $(l_0, l_1, \dots, l_{s-1}, k, n)$ one may define its t th *truncation* $\mathcal{G}^{(t)} = \mathcal{G}_0 \cup \dots \cup \mathcal{G}_t, 1 \leq t < s$. It is an \mathcal{F} -squashed design with parameters $(l_0, l_1, \dots, l_t, n)$. Analogously one can define *derived designs*: $\mathcal{G}(G) = \{H - G: G \subseteq H \in \mathcal{G}\}$ is an $\mathcal{F}(G)$ -squashed design for all $G \in \mathcal{G}_i, i < s$.

Instead of 2^X one can consider \mathcal{F} as all subspaces of V , an n -dimensional vector space, projective or affine space over $GF(q)$. By analogy we denote it by 2^V . In general, for $\mathcal{F} = 2^X$, a perfect squashed design is simply a perfect matroid design. In most cases \mathcal{G} is uniquely determined by \mathcal{G}_s , therefore in most examples we define \mathcal{G}_s only.

Example 4.1. (a) Suppose X is the Cartesian product $X_1 \times \dots \times X_d$ where $|X_i| = n_i$ and \mathcal{F} is the family of all *injective* subsets ($F \subset X$ is injective if for all distinct elements $(f_1, \dots, f_d), (f'_1, \dots, f'_d) \in F, f_i \neq f'_i$ for $1 \leq i \leq d$ (cf. [DF])). We set $\mathcal{F} = \mathcal{F}(X_1, \dots, X_d)$.

(b) Suppose now $V = V_1 \times V_2 \times \dots \times V_d$ where V_i is a vector space of dimension n_i . If W is an affine subspace of V , one can define $\pi_i(W)$ as its natural projection on V_i . Define $\mathcal{F}_A(V_1, \dots, V_d)$ as the set of all injective affine subspaces, $\mathcal{F}_A(V_1, \dots, V_d) = \{A \leq V: \dim(\pi_i(A)) = \dim(A) \text{ for } i = 1, \dots, d\}$. Similarly, for vector subspaces

$$\mathcal{F}_V(V_1, \dots, V_d) = \{W \leq V: \dim(\pi_i(W)) = \dim(W) \text{ for } i = 1, \dots, d\}.$$

Example 4.2. Suppose Y and Z are both either finite sets or finite-dimensional vector spaces over $GF(q)$. Let $\mathcal{F}(Y, Z)$ denote the set of all partial transversals (transverse subspaces), i.e., all $W < Y \times Z$ such that $|W \cap (Y \times \{z\})| \leq 1$ holds for all $z \in Z$. Note that the definition is not symmetric in Y and Z . (This example is due to Delsarte [D 1].)

Let us note that Example 4.2 is a regular semilattice in the sense of Delsarte [D 1]), but Example 4.1(a), (b) is not.

Example 4.3. Suppose V is an n -dimensional vector space and $f(x, y)$ is a sesquilinear form on V (cf. [Di]). Let $\mathcal{F} = \mathcal{V}_f$ be the collection of singular subspaces, i.e.,

$$\mathcal{V}_f = \{W \leq V : f(w, w') = 0 \text{ for all } w, w' \in W\}.$$

Note that $2^V = \mathcal{V}_f$ for f the identically zero form. Note also that if f is nondegenerate then \mathcal{V}_f is a perfect squashed design. This is true for the other examples and their truncations as well.

In the examples considered so far \mathcal{G} was simply a truncation of \mathcal{F} . There are several other examples of this type, related to polar spaces and buildings (cf. [Ne]).

Example 4.4. Suppose $\mathcal{F} = 2^X, 2^A, 2^V, 2^P$, or more generally a squashed geometry. A family $\mathcal{G} \subset \mathcal{F}_s$ is called a t -design with repetition λ in \mathcal{F}_s if every $F \in \mathcal{F}_t$ is contained in exactly λ members of \mathcal{G} . For $\lambda = 1$ we call \mathcal{G} shortly a t -design in \mathcal{F}_s . If \mathcal{F}_s is a perfect squashed design then a t -design \mathcal{G} defines a perfect squashed design.

Note that for regular semilattices our definition of t -designs coincides with the notion of combinatorial t -design in [D 1] or combinatorial relative t -designs in [D 2]. If $\mathcal{F}_0 \cup \dots \cup \mathcal{F}_s$ is a regular squashed geometry then a t -design with repetition λ is also a t' -design with some repetition λ' for $0 < t' < t$.

Let us remark that a t -design \mathcal{T} in a regular squashed geometry is a packing of maximal size of members of \mathcal{F}_s such that any two of them have intersection of rank less than t . Similarly, it is a minimal covering of the members of \mathcal{F}_t .

Let us note that $|\mathcal{T}| = |\mathcal{F}_t|/m(0, t)$. It follows from a much more general result of [FR] that for an arbitrary positive ϵ and n_{t-1} sufficiently large with respect to $m(0, t)$ and ϵ there exists a packing of size at least $(1 - \epsilon)|\mathcal{F}_t|/m(0, t)$.

Note that if $d = 2$ and $|X_1| = |X_2| = n$ then there is a natural correspondence between the symmetric group S_n and the flats of maximal rank (i.e., of rank n) in $\mathcal{J}(X_1, X_2)$. Namely, a permutation (i_1, i_2, \dots, i_n) is associated with $\{(1, i_1), \dots, (n, i_n)\}$. In this case any t -fold transitive permutation group Γ is a t -design with repetition $\lambda = |\Gamma|/n(n-1) \cdots (n-t+1)$. A general t -design with repetition λ in $\mathcal{J}(X_1, X_2)$ is called a λ -uniform t -transitive permutation set (cf. [De 2]) or orthogonal permutation array (cf. [It]).

The existence problem of t -designs is a hopelessly difficult one in general, even for $\mathcal{F} = 2^X$. In this case 1-designs are just partitions of X into subsets of size s . In the case $\mathcal{F} = 2^X, t = 2$ Wilson [Wi] proved that t -designs with repetition λ exist in $\binom{X}{2}$ whenever the trivial necessary conditions and $n > n_0(\mathcal{F}, \lambda)$ are fulfilled. The existence problem for t -designs in 2^V was raised in [Ray].

Example 4.5. Suppose A is an n -dimensional affine space over $\text{GF}(q)$ and \mathcal{G}_1 is the class of subspaces of dimension d , parallel to a fixed d -dimensional subspace. Then \mathcal{G}_1 is a 2^A -squashed design.

Example 4.6. Suppose P is an n -dimensional projective space, $(d + 1) \mid (n + 1)$ and \mathcal{G}_1 is a spread of d -dimensional subspaces (i.e., the members of \mathcal{G}_1 are pairwise disjoint and their union is P , cf. [Dem]). Then \mathcal{G}_1 is a 2^P -squashed design.

Example 4.7. Let V be an $(n + 1)$ -dimensional vector space corresponding to P from the preceding example, and A the corresponding $(n + 1)$ -dimensional affine space. Let \mathcal{G}_2 consist of all affine translates of all $(d + 1)$ -dimensional subspaces whose projection in P is in the spread. Then \mathcal{G}_2 is a 2^A -squashed design with parameters $(0, 1, q^{d+1}, q^{n+1})$.

Example 4.8 (Laurent [La]). Suppose $k \mid n_1 n_2 \dots n_d$, $k \leq \min_i n_i$. Then there exists a 1-design in $\mathcal{F}^{(k)}(X_1, X_2, \dots, X_d)$.

Example 4.9. In $\mathcal{T}(Y, Z)$ and Y, Z finite sets t -designs with repetition λ correspond to orthogonal arrays of strength t (cf. [Rao]). In the vector space case t -designs always exist (cf. [D 1]).

Example 4.10. If a family \mathcal{A} attains equality in Theorem 1.3(i) for $\mathcal{F} = \mathcal{F}(Y, Z)$, it is called a transversal matroid design (cf. [CDF] for some examples).

Example 4.11. A family \mathcal{A} attaining equality in Theorem 1.3(i) for $\mathcal{F} = \mathcal{F}(X_1, \dots, X_d)$ is called an injection design (cf. [DF]). The special case $d = 2$, $|X_1| = |X_2| = k$ corresponds to permutation geometries (cf. [CD]). In [CD], [DF], and [DCF] several examples are given and we describe several new constructions in Sections 5 and 6.

5. Some Examples of t -Designs

Proposition 5.1. Suppose $\mathcal{F}_0 = \mathcal{F}_0 \cup \dots \cup \mathcal{F}_s$ is a (regular) squashed design and $\mathcal{T} = \mathcal{F}_s$ is a t -design in \mathcal{F} . Suppose that for some $k \leq s$ and all $T \in \mathcal{T}$ there exists a t -design with repetition λ in $(\mathcal{F}_0 \cup \dots \cup \mathcal{F}_k)_T = \{F \in \mathcal{F}_i : F \subset T, i \leq k\}$. That is, for each $T \in \mathcal{T}$ there exists $\mathcal{J}(T) \subset (\mathcal{F}_k \cap 2^T)$ with the property that each $F \in \mathcal{F}_i$ with $F \subset T$ is contained in exactly λ members of $\mathcal{J}(T)$. Then $\bigcup_{T \in \mathcal{T}} \mathcal{J}(T)$ is a t -design with repetition λ in the k th truncation, $\mathcal{F}_0 \cup \dots \cup \mathcal{F}_k$ of \mathcal{F} .

Proof. By inspection, it is left to the reader. □

Example 5.2. Suppose $|X| = p^\alpha$, p a prime and consider the elements of X as the elements of the finite field of order p^α . Let $A = \text{AL}(1, p^\alpha)$ be the group of all affine linear transformations $x \rightarrow ax + b$. Then $|A| = p^\alpha(p^\alpha - 1)$ and A is sharply 2-transitive. Thus (see the remark after Example 4.4) A corresponds to a 2-design $\mathcal{T}(A)$ in $\mathcal{F}(X, X)$. If for some $2 < k < p^\alpha$ there exists a 2-design in $\binom{X}{k}$ with repetition λ then there exists a 2-design with repetition λ in the k th truncation of $\mathcal{F}(X, X)$. In view of Example 1.6 in [DF] the same is true in $\mathcal{F}(X, X, \dots, X)$.

Corollary 5.3. *Suppose $p^\alpha \equiv 1$ or $3 \pmod{6}$. Then there exists a Steiner triple-system in $\mathcal{F}(X_1, \dots, X_d)$ for $|X_1| = \dots = |X_d| = p^\alpha$.*

Proof. Use Example 5.2 and the fact that Steiner triple-systems exist in $\binom{X}{3}$ whenever $p^\alpha \equiv 1$ or $3 \pmod{6}$ (cf. [Ki]) □

Example 5.4. Using the general projective group $\text{PGL}(2, p^\alpha)$ which acts sharply 3-transitively on the $p^\alpha + 1$ points of the projective line, it was shown in [DF] that there exists a 3-design in $\mathcal{F}(X_1, X_2, \dots, X_d)$ for $|X_1| = \dots = |X_d| = p^\alpha + 1$, $d \geq 2$. Since there exist 3-designs in $\binom{X}{p^\beta + 1}$ for $|X| = p^{2\beta} + 1$, in the above way, we infer the existence of 3-designs in the $(p^\beta + 1)$ th truncation of $\mathcal{F}(X_1, \dots, X_d)$ for $|X_1| = \dots = |X_d| = p^{2\beta} + 1$.

Let us introduce the notation $\mathcal{F}(n_1, \dots, n_d)$ for $\mathcal{F}(\{1, \dots, n_1\}, \dots, \{1, \dots, n_d\})$. For given $k \geq 3$ and $d \geq 1$ define $M(k, d)$ as the set of those values of (n_1, \dots, n_d) for which there exists a 2-design of k -element injective sets in $\mathcal{F}(n_1, \dots, n_d)$.

Proposition 5.5. *If $(n_1, \dots, n_d) \in M(k, d)$ then $\min\{n_1, \dots, n_d\} \geq k$; moreover (i) and (ii) hold:*

$$(i) \quad k(k-1) \mid \prod_{1 \leq j \leq d} (n_j(n_j-1)),$$

$$(ii) \quad (k-1) \mid \prod_{1 \leq j \leq d} (n_j-1).$$

Proof. Clearly the number of 2-element injective subsets in $\mathcal{F}(n_1, \dots, n_d)$ must be a multiple of $\binom{k}{2}$, this implies (i). To prove (ii), one considers the derived designs, e.g., the $(k-1)$ -element injective sets left over after removing the point (n_1, \dots, n_d) from all sets containing it. These sets partition $\mathcal{F}(n_1-1, \dots, n_d-1)$ yielding (ii).

Conjecture 5.6. *If $\min_{1 \leq j \leq d} n_j$ is sufficiently large with respect to k and (i) and (ii) are fulfilled, then $(n_1, \dots, n_d) \in M(k, d)$.*

Let us note that for $d = 1$, Conjecture 5.6 follows from the existence results of Wilson [Wi].

Recall that a pairwise balanced design $\mathcal{B} \subset 2^X$ is just a collection of subsets of X so that $|B| \geq 2$ for all $B \in \mathcal{B}$ and every 2-subset of X is contained in exactly one member of \mathcal{B} .

Proposition 5.7. *Suppose there exist pairwise balanced designs $\mathcal{B}_i \subset 2^{\{1, 2, \dots, n_i\}}$, $i = 1, \dots, d$ so that $(|B_1|, \dots, |B_d|) \in M(k, d)$ holds for all $B_i \in \mathcal{B}_i$, $i = 1, \dots, d$. Then $(n_1, \dots, n_d) \in M(k, d)$.*

Proof. For each choice of $B_1 \in \mathcal{B}_1, \dots, B_d \in \mathcal{B}_d$ replace the brick $B_1 \times \dots \times B_d$ by an injection 2-design. It is left to the reader to verify that in this way we obtain an injection 2-design in $\mathcal{F}(n_1, \dots, n_d)$. □

Finally, let us mention that for $k = 3, d = 2$ an injection 2-design in $\mathcal{F}(3, n)$ is equivalent to a latin square with $(1, 2, \dots, n)$ as main diagonal (i.e., multiplication table of idempotent quasigroup). This latter exists for all n except $n = 2$. To see this equivalence, first assume \mathcal{G} is an injection 2-design in $\mathcal{F}(3, n)$. Then $|\mathcal{G}| = n(n - 1)$. If $G \in \mathcal{G}, G = \{(1, a), (2, b), (3, c)\}$ then put in the latin square a in position (b, c) . This way the diagonal will remain unfilled. Write a in position (a, a) . This leads to a latin square since \mathcal{G} is an injection design and clearly, one can reverse the steps. Note that, in view of Proposition 5.7, the existence of an injection 2-design in $\mathcal{F}(3, n)$ implies that it exists in $\mathcal{F}(m, n)$ whenever $m \equiv 1$ or $3 \pmod{6}$. The first case for which we could not decide whether an injection 2-design (with $k = 3$) exists is in $\mathcal{F}(4, 5)$.

6. A Recurrent Construction for Injection Designs

Let us recall that an injection design \mathcal{D} with parameters $(l_0, \dots, l_{s-1}, k, n)$ and of dimension d is a perfect $\mathcal{F}(X_1, \dots, X_d)$ -squashed design of rank s in which the size of the flats are l_0, \dots, l_{s-1}, k . For a set $D = \{x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(d)}, \dots, (x_k^{(1)}, \dots, x_k^{(d)})\}$ we define the first projection of D by $\pi(D) = \{x_1^{(1)}, \dots, x_k^{(1)}\}$. We know (cf. [DF]) that $\pi(D, \mathcal{D}) = \{\pi(D \cap D') : D' \in \mathcal{D}\}$ defines a perfect matroid design for all $D \in \mathcal{D}$, in particular for $D \in \mathcal{D}_s$. \mathcal{D} is called *concentrated* if for all $D, D' \in \mathcal{D}, \pi(D) \cap \pi(D') \in \pi(D, \mathcal{D})$ holds.

Proposition 6.1. *Suppose \mathcal{D} is a concentrated injection design with parameters $(l_0, \dots, l_{s-1}, k, n)$ then for all $i, 0 < i < s$ (and even for $i = s$ if $k < n$) the family $\pi(\mathcal{D}_i) = \{\pi(D) : D \in \mathcal{D}_i\}$ is a perfect matroid design with parameters $(l_0, \dots, l_{i-1}, l_i, n)$.*

Proof. Clearly $\pi(\mathcal{D}_i)$ is an $(n, l_i, \{l_0, \dots, l_{i-1}\})$ -system which verifies the assumptions of Remark 2.4, i.e., the intersection of any number of distinct members of $\pi(\mathcal{D}_i)$ has size belonging to L . The proof can be concluded as that of Theorem 1.3. □

For $A \subset X_1$ clearly $A = \pi(D)$ implies that D is an injective subset in $A \times X_2 \times \dots \times X_d$. Thus, if $|A| = l_i, |X_j| = n^{(j)}$, then the number of $D \in \mathcal{D}_i$ with $A = \pi(D)$ is at most $\prod_{0 \leq a \leq i-1} \prod_{j=1}^d (n^{(j)} - l_a)$.

Proposition 6.2. *Suppose $\mathcal{D}^{(\nu)}$ is a concentrated injection design with parameters $(l_0, l_1, \dots, l_{s-1}, k, n)$ and of dimension $d^{(\nu)}$ in $\mathcal{F}(X_1^{(\nu)}, \dots, X_{d^{(\nu)}}^{(\nu)})$, $\nu = 1, 2$. Suppose further $X_1^{(1)} = X_1^{(2)}$ and for all $i, 0 \leq i \leq s, \pi(\mathcal{D}_i^{(1)}) = \pi(\mathcal{D}_i^{(2)})$ holds. Then there exists a concentrated injection design with parameters $(l_0, \dots, l_{s-1}, k, n)$ and of dimension $d^{(1)} + d^{(2)} - 1$ in $\mathcal{F}(X_1^{(1)}, \dots, X_{d_1}^{(1)}, X_2^{(2)}, \dots, X_{d_2}^{(2)})$.*

Proof. With $\mathbf{x} = (a, x_2^{(1)}, \dots, x_{d_1}^{(1)}) \in X_1^{(1)} \times \dots \times X_{d_1}^{(1)}$ and $\mathbf{y} = (a, y_2^{(2)}, \dots, y_{d_2}^{(2)}) \in X_1^{(2)} \times \dots \times X_{d_2}^{(2)}$ we associate $\mathbf{xy} = (a, x_2^{(1)}, \dots, x_{d_1}^{(1)}, y_2^{(2)}, \dots, y_{d_2}^{(2)})$. Similarly, if $A \in \mathcal{D}^{(1)}, B \in \mathcal{D}^{(2)}$ and $\pi(A) = \pi(B)$ then we can define $AB = \{\mathbf{xy} : \mathbf{x} \in A, \mathbf{y} \in B, \mathbf{x}$ and \mathbf{y} have the same first coordinate $\}$. This leads to the definition of $\mathcal{D} = \mathcal{D}^{(1)} \mathcal{D}^{(2)} = \{AB : A \in \mathcal{D}^{(1)}, B \in \mathcal{D}^{(2)}, \pi(A) = \pi(B)\}$.

We want to show that \mathcal{D} is the desired injection design. For $C \subset X_1^{(1)} \subset X_2^{(1)}$ we define $\pi_\nu^{-1}(C) = \{D \in \mathcal{D}^{(\nu)} : \pi(D) = C\}$, $\nu = 1, 2$. Since $\pi(\mathcal{D}^{(1)}) = \pi(\mathcal{D}^{(2)})$ is a PMD on $X_1^{(1)}$, in view of our remarks after Proposition 6.1, $\pi_\nu^{-1}(C)$ is an injection design on $\mathcal{F}(C, X_2^{(\nu)}, \dots, X_d^{(\nu)})$ for $\nu = 1, 2$ and $C \in \pi(\mathcal{D}^{(1)})$. Define $\pi^{-1}(C) = \{D \in \mathcal{D} : \pi(D) = C\}$. Clearly, $|\pi^{-1}(C)| = |\pi_1^{-1}(C)| \cdot |\pi_2^{-1}(C)|$ holds, i.e., $\pi^{-1}(C)$ has the size of an injection design in $\mathcal{F}(C, X_2^{(1)}, \dots, X_{d_1}^{(1)}, X_2^{(2)}, \dots, X_{d_2}^{(2)})$. Thus $\mathcal{D}_i = \{D \in \mathcal{D} : |D| = l_i\}$ satisfies

$$|\mathcal{D}_i| = |\pi(\mathcal{D}_i^{(1)})| \prod_{\nu=1}^2 \prod_{j=2}^{d^{(\nu)}} \prod_{b=0}^{i-1} (n_j^{(\nu)} - l_b).$$

Using $|\pi(D_i^{(1)})| = \prod_{0 \leq b < i} (n_1^{(1)} - l_b) / (l_i - l_b)$, we see that \mathcal{D}_i has the desired size.

To conclude the proof, in view of our remarks after the proof of Theorem 1.3, we only need to show that \mathcal{D} is closed under intersection. Let $AB, A'B' \in \mathcal{D}$. Since $\mathcal{D}^{(\nu)}$ is concentrated for all $C \in \pi(\mathcal{D}^{(\nu)})$, $D^{(\nu)} \in \mathcal{D}^{(\nu)}$, the set $\{x \in D^{(\nu)} : \pi(x) \in C\}$ is in $\mathcal{D}^{(\nu)}$. Consequently if $\pi(D^{(1)}) = \pi(D^{(2)})$ then $\{x \in D^{(1)}D^{(2)} : \pi(x) \in C\} \in \mathcal{D}$ holds. Since $\pi(\mathcal{D}^{(\nu)})$ is closed intersection, $C = \pi(A \cap A') \cap \pi(B \cap B') \in \pi(\mathcal{D}^{(\nu)})$. Now $AB \cap A'B' = \{x \in AB : \pi(x) \in C\}$, thus $(AB \cap A'B') \in \mathcal{D}$, concluding the proof. □

Proposition 6.3. *Suppose $\mathcal{D}^{(\nu)}$ is an injection design with parameters $(0, 1, \dots, s-1, k^{(\nu)}, n^{(\nu)})$ in $\mathcal{F}(X_1^{(\nu)}, \dots, X_{d_A}^{(\nu)})$, $\nu = 1, 2$. Moreover $X_1^{(1)} = X_1^{(2)}$, $|X_1^{(2)}| = k^{(2)}$. Then there exists an injection design with parameters $(0, 1, \dots, s-1, k^{(2)}, n^{(1)}n^{(2)}/n_1^{(2)})$ in $\mathcal{F}(X_1^{(1)}, \dots, X_{d_1}^{(1)}, X_2^{(2)}, \dots, X_{d_2}^{(2)})$.*

Proof. For $0 \leq i < s$ the flats of rank i in the product design \mathcal{D} are all injective subsets of size i in $\mathcal{F}(X_1^{(1)}, \dots, X_{d_2}^{(2)})$.

As to \mathcal{D}_s , for arbitrary $A \in \mathcal{D}_s^{(1)}$ and $B \in \mathcal{D}_s^{(2)}$ we first define B_A as the unique subset of B satisfying $\pi(A) = \pi(B_A)$. Then the corresponding element in \mathcal{D}_s is AB_A . From here on the proof practically coincides with that of the preceding proposition. □

Note that the reason why we do not need concentratedness explicitly in this proposition is that $\pi(\mathcal{D}_i^{(1)}) = \pi(\mathcal{D}_i^{(2)})$ is automatically satisfied for $0 \leq i < s$. Also $k^{(2)} = n_1^{(2)}$ implies that $\pi(D) = X_1^{(2)}$ holds for all $D \in \mathcal{D}_s^{(2)}$, i.e., $\mathcal{D}^{(2)}$ is concentrated in a very strong sense.

For more constructions see [DL], [La].

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