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BOUQUETS OF MATROIDS, d-INJECTION GEOMETRIES AND DIAGRAMS

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$\mathcal{F}$-squashed geometries, one of the many recent generalizations of matroids, include a wide range of combinatorial structures but still admit a direct extension of many matroidal axiomatizations and also provide a good framework for studying the performance of the greedy algorithm in any independence system. Here, after giving all necessary preliminaries in section 1, we consider in section 2 $\mathcal{F}$-squashed geometries which are exactly the shadow structures coming from the Buekenhout diagram : $\circ\overset{L}{\rule[0.5ex]{2em}{0.4pt}}\circ\ldots\circ\overset{L}{\rule[0.5ex]{2em}{0.4pt}}\circ\overset{\pi}{\rule[0.5ex]{2em}{0.4pt}}\circ$, i.e. bouquets of matroids. We introduce d-injective planes : $\circ\overset{[d]}{\rule[0.5ex]{2em}{0.4pt}}\circ$ (generalizing the case of dual net for d = 1) which provide a diagram representation for high rank d-injective geometries. In section 3, after a brief survey of known constructions for d-injective geometries, we give two new constructions using pointwise and setwise action of a class of mappings. The first one, using some features of permutation geometries (i.e. 2-injection geometries), produces bouquets of pairwise isomorphic matroids. The last section 4 presents briefly some related problems for squashed geometries.

1. PRELIMINARIES FOR SQUASHED GEOMETRIES AND DIAGRAMS

A) Preliminaries for squashed geometries.

DEFINITION 1.1. Let X be a finite set, $\mathcal{F}$ be a clutter of subsets of X and $\mathcal{U} = \bigcup_{F \in \mathcal{F}} 2^F = \{U : \exists F \in \mathcal{F},\ U \subseteq F\}$, called <u>universe</u>. Let $g_0, g_1, \ldots, g_s$ be some pairwise disjoint families of subsets of X and $g = g_0 \cup g_1 \cup \ldots \cup g_s$. g is called a <u>$\mathcal{F}$-squashed geometry of rank s on X if</u> :

(F0) $g \subseteq U$

(F1) g is a meet semi-lattice, i.e. $G \cap G' \in g$ for all $G, G' \in g$

(F2) if $G \in g_i$, $G' \in g_j$ and $G \subsetneq G'$, then $i < j$

(F3) if $G \in g_i$ $(i \leqslant s-1)$, $x \in X-G$ and $G \cup x \in \mathcal{U}$, then there exists $G' \in g_{i+1}$ such that $G' \supseteq G \cup x$.

It is evident that :

(1) in (F3), we have the unicity of G' and, moreover, $H \supseteq G'$ for every

$H \in g$ such that $H \supseteq G \cup x$

(2) $|g_0| = 1$.

(3) from (F2), (F3), g is closed under arbitrary meets

Notice that most of the results of this paper can be extended to the case when X is infinite but the universe $\mathcal{U}$ is finite. Also , without loss of generality, we can suppose that $g_0 = \{\phi\}$ and a $\mathcal{F}$-squashed geometry of rank 0 is just $g = \{\phi\}$. A $\mathcal{F}$-squashed geometry of rank 1 is $g = \{\phi\} \cup g_1$ where g_1 is exactly a partition of X. Examples of those geometries are : 1-designs $S(1,k,v)$ with $k|v$, Latin squares ($X = X_1 \times X_2$ and g_1 consists of $|X_1|$ disjoint permutations of X_1), t-spread ($X = PG(n,q)$ and g_1 is a partition of X into t-dimensional subspaces of X).

A trivial example of $\mathcal{F}$-squashed geometry is the family of flats of a matroid (for basic references on matroids see [W]); moreover, a $\mathcal{F}$-squashed geometry is a matroid if and only if $\mathcal{F} = \{X\}$, i.e. $\mathcal{U} = 2^X$. Hence the word "squashed" refers to the idea that we deal only with elements of the universe.

Let us introduce three other classes of $\mathcal{F}$-squashed geometries that we will study in detail in section 2, each of them being specified by the choice of its universe. Let $X_1,\ldots,X_d$ be d finite sets and $X = X_1 \times \ldots \times X_d$. Let $\alpha \in [1,d]$. A subset A of X is called injective by X_α if, for all distinct elements $a = (a_1,\ldots,a_d)$, $b = (b_1,\ldots b_d)$ of A, $a_\alpha \neq b_\alpha$ holds.

DEFINITION 1.2. (i) Let $X = X_1 \times X_2$. A subset A of X is called 1-injective or transversal if A is injective by X_1.

(ii) Let $X = X_1 \times X_2 \times .. \times X_d$ with $d \geqslant 2$. A subset A of X is called d-injective if A is injective by every X_α for $\alpha \in [1,d]$.

REMARK 1.3. We use the terminology : transversal subset for 1-injective subsets since every 1-injective subset of $X = X_1 \times X_2$ can be viewed as a partial transversal of $\bigcup_{x \in X_1} X_2^x$ where the X_2^x are disjoint copies of X_2.

DEFINITION 1.4. A $\mathcal{F}$-squashed geometry on X is called a d-injection geometry if $\mathcal{F}$ is the set of all maximal d-injective subsets of X where $X = X_1 \times X_2$ when $d = 1$ and $X = X_1 \times \ldots \times X_d$ when $d \geqslant 2$.

REMARK 1.5. A 1-injection geometry is also called a transversal geometry. For results on transversal geometries, see [D], [CDF]. A 2-injection geometry is also called a bijection geometry. Permutation geometries, introduced in [CD], are special cases of bijection geometries when $|X_1| =$

$|X_2| = v$ and all maximal flats have size v. For results on permutation geometries, see [CD], [CDF]. In particular, permutation geometries whose set of maximal flats is a 2-transitive group are known (cf. theorem 5-8 [CD] which is a reformulation of a theorem of Kantor). A d-injection geometry ($d \geqslant 3$) is simply called an *injection geometry*. For results on injection geometries, see [DF1], [DF2]. $\mathcal{F}$-squashed geometries were also independently introduced in [CV].

Let g be a $\mathcal{F}$-squashed geometry. Elements of g (resp. g_i) are called *flats* or *closed sets* (resp. *flats of rank i* or *i-flats*). The maximal flats of g are called *roofs* and the set R of all roofs clearly contains g_s. When $R = g_s$, then g is called *well-cut*. g is called *simple* when all 1-flats have size 1. When all i-flats have the same size ℓ_i for $i \in [0,s]$, then g is called a *$\mathcal{F}$-squashed design with parameters* $(\ell_0, \ell_1, \ldots, \ell_s)$. It is proved in [DF2] that $|g_s| \leqslant \prod_{i=0}^{s-1} \frac{n_{\ell_i}}{\ell_s - \ell_i}$ where $n_r = \mathrm{Max}\{ |\bigcup_{G \subseteq F \in \mathcal{U}, |F| = \ell_S} F - G| : G \in \mathcal{U},\ |G| = r\}$ for any integer r. When equality holds in the above inequality, then g is called a perfect *$\mathcal{F}$-squashed design*. It is easy to see that every d-injection design is indeed perfect (see [DF1]). g is called *stiff* when every (s-1)-flat is contained in at least two distinct roofs, or equivalently, if every (s-1)-flat is the intersection of two roofs. If, moreover, every flat of g is the intersection of two roofs, then g is called *short*. When every subset of a roof is flat of g, then g is called *free*. An *isomorphism* between two squashed geometries g, g' is a bijection from g onto g' that preserves rank and incidence.

Every *interval* $g \cap [G_1, G_2]$, where G_1, G_2 are two flats of g such that $G_1 \subseteq G_2$, is a matroid on G_2. In particular, if $R_1, \ldots, R_m$ denote the distinct roofs of g, then every $\mathcal{M}_i = g \cap [\phi, R_i]$ is the set of flats of a matroid on R_i. This observation yields the following definition.

DEFINITION 1.6. Let g be a $\mathcal{F}$-squashed geometry and $\mathcal{M}$ be the set of flats of a matroid. g is called *$\mathcal{M}$-unisupported* if, for every roof R of g, the matroid $g_R = g \cap [\phi, R]$ is isomorphic to $\mathcal{M}$.

Notice that, if g is $\mathcal{M}$-unisupported, then g is well-cut and moreover, all roofs have the same size. A trivial example of unisupported geometry is provided by any free squashed geometry whose roofs have the same size n; then this geometry is $\mathcal{B}$-unisupported, $\mathcal{B}$ being the boolean algebra on [1,n].

Other examples of unisupported squashed geometries are given in section 3. For an example of non unisupported squashed geometry, see remark 2.4 example 1.

Let g be a $\mathcal{F}$-squashed geometry, then g is obviously a $\mathcal{R}$-squashed geometry, $\mathcal{R}$ being the set of roofs of g and, in this case, g is simply called a *squashed geometry* or *bouquet of matroids*. Hence a union of matroids (in the set theoretical sense) is a bouquet of matroids if and only if it is a meet semi-lattice. Notice that this idea of bouquet is often used; for instance, simplicial complexes are bouquets of boolean algebras, buildings are special cases of bouquets of Coxeter complexes, polar spaces are special bouquets of projective spaces.

If g is a $\mathcal{F}$-squashed geometry of rank s, an interesting question is that of *extension* of g, i.e. to find g_{s+1} such that $g \cup g_{s+1}$ is a $\mathcal{F}$-squashed geometry of rank s+1. Dually, if $g = g_0 \cup .. \cup g_s$ is a $\mathcal{F}$-squashed geometry of rank s, then the k-*truncation* $g^{(k)} = g_0 \cup .. \cup g_k$ is a $\mathcal{F}$-squashed geometry of rank k.

A combinatorial structure : M_s-designs, very similar to squashed designs was introduced in [Ne]. Actually, all examples of M_s-designs given in [Ne] are bouquets of matroids. However, the definition of M_s-designs is slightly more general since it allows the intersection of two roofs to be either a flat or the union of flats.

Similarly to the case of matroids, squashed geometries can be defined through their *rank function*, their *closure operator*, their *circuits* (*stigmes* and *critical* subsets) or their *independent subsets*. For axiomatizations and equivalence between the distinct axiomatizations, see [Sch],[La2].In the injective case, see also [DF1] and for a survey of axiomatizations, see also [CL]. For instance, the rank function r and the closure operator σ are defined as follows : for every subset A of X which is contained in some roof, $r(A) = \text{Min}\{i \in [0,s] : \exists G \in g_i \; G \supseteq A\}$ and $\sigma(A) = \bigcap_{A \subseteq G \in g} G$; otherwise $r(A) = \infty$ and $\sigma(A) = X$. Moreover, since g is the bouquet of the matroids $\mathcal{M}_i = g \cap [\phi, R_i]$ on the distinct roofs R_i, r and σ coincide with the rank function and the closure operator respectively of $\mathcal{M}_i$ on subsets of R_i. Hence this structure of bouquet preserves the compatibility between the different matroids composing g. The set of independent subsets of g is simply the set of all independent subsets of some of the matroids $\mathcal{M}_i$. Conversely, any independence system $\mathcal{I}$, that is $\mathcal{I} \subseteq 2^X$ satisfying $J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$, can be entitled with a structure of squashed geometry; this point of view permitting to obtain sharp bounds for the performance of the greedy algorithm in $\mathcal{I}$ (see [CL]).

B) Preliminaries for diagrams.

For all definitions and terminology for diagrams and geometries, we follow [B2] and give them here for the sake of completeness.

DEFINITION 1.7. Let Δ be a finite set, $\Delta = [0,n-1]$. A geometry P over Δ is a triple P = (I,S,t) where S is a set (the elements or varieties of P), I is a symmetric and reflexive relation on S (the incidence relation of P) and t is a mapping of S onto Δ (the type function of P) satisfying : (TP) (Transversality property) the restriction of t to every maximal set of pairwise incident elements of S is a bijection onto Δ.

We shall always assume to deal with finite geometries (that is, S will be finite). See for example [BP] for an infinite analogue of theorem 1.15.

The rank of P is the cardinality of Δ. Elements of Δ can be seen as names such as points, lines, planes, etc. given to the elements of P with the name or type of each variety determined by the mapping t. So, 0-varieties are called points, 1-varieties are called lines etc. A flag F of P is a set of pairwise incident elements of P. The residue of F is the geometry $P_F = (S_F, I_F, t_F)$ over $\Delta - t(F)$ defined by : S_F is the set of all elements of P not in F, incident with all elements of F, I_F (resp. t_F) is the restriction of I (resp. t) to S_F. An isomorphism of P is a permutation of the elements of P leaving incidence invariant as well as types. Most known geometries are obtained as sets of points equipped with distinguished subsets, i.e. their elements are identified with sets of points. The shadow structure develops this point of view for any geometry. For any flag F of P and any $i \in \Delta$, the i-shadow or shadow $\sigma_i(F)$ of F in $P_i = t^{-1}(i)$ is the set of all elements of P_i incident with F. Let $0 \in \Delta$, then 0-shadows will be simply called shadows and $\sigma_0(v)$ denoted by $\sigma(v)$ for $v \in P$.

The following properties are often required for geometries :

(F) (Firmness) a geometry P over Δ is called firm if every non maximal flag of P is contained in at least two distinct maximal flags.

(SC) (Strong connectivity) a geometry P = (S,I,t) over Δ is called strongly connected if, for all distinct $i,j \in \Delta$, $t^{-1}(i) \cup t^{-1}(j)$ is a connected graph for the incidence relation and the same holds in every residue of a flag of P.

(IP) (Intersection property) a geometry P over Δ is said to have the intersection property if, for each $i \in \Delta$, $x \in S$ and each flag F of P, either $\sigma_i(x) \cap \sigma_i(F)$ is empty or there exists a flag F' incident with x and F such that $\sigma_i(x) \cap \sigma_i(F) = \sigma_i(F')$ and, moreover, the same property holds in every residue of a flag of P.

An important motivation for introducing diagrams for geometries is the fact that a geometry is often determined by all its residues of rank 2. Let us first introduce two geometries of rank 2 : generalized digons and partial planes (or partial linear spaces). Let P be a geometry of rank 2 over $\Delta = \{0,1\}$, then P is determined by axioms on its points and lines.

DEFINITION 1.8. A partial plane is a geometry of rank 2 characterized by the following axiom : any two distinct points (resp. lines) are incident with at most one line (resp. point).

DEFINITION 1.9. A generalized digon is a geometry of rank 2 characterized by the following axiom : every line is incident with every point.

DEFINITION 1.10. A geometry P over Δ is called pure if, for any distinct $i,j \in \Delta$ such that there exists a flag of type $\Delta-\{i,j\}$ in P whose residue is not a generalized digon, then no residue of type $\{i,j\}$ is a generalized digon.

We now define special diagrams that are a specialization of basic diagrams, this notion being sufficient for our treatment.

DEFINITION 1.11. A special diagram (Δ,f) on a set Δ is a mapping f which assigns to every ordered pair of distinct elements $(i,j) \in \Delta$ some class $\Delta_{ij} = f(i,j)$ of rank 2 geometries over $\{i,j\}$ such that :
(i) either all members of Δ_{ij} are generalized digons, or all members of Δ_{ij} are partial planes
(ii) $\Delta_{ij} = \Delta^{\star}_{ji}$ where $\Delta^{\star}_{ij}$ is the dual class of Δ_{ij} and the dual of a rank 2 geometry $P = (S,I,t)$ over $\{i,j\}$ is the geometry $P^{\star} = (S,I,t^{\star})$ where $t^{\star}$ is defined by $t^{\star}(v)=j$ (resp. i) if $t(v)=i$ (resp. j) for all $v \in S$.
(iii) Δ_{ij} is closed under isomorphisms.

DEFINITION 1.12. A geometry P over Δ belongs to the diagram (Δ,f) if, for every ordered pair of distinct elements i,j of Δ and every flag F of P of

type $\Delta-\{i,j\}$, the residue P_F is a member of Δ_{ij}.
Notice that, if P is a geometry over Δ belonging to the special diagram (Δ,f), then P is indeed pure.

If (Δ,f) is a special diagram, a structure of graph is defined with Δ as set of vertices, two distinct elements $i,j \in \Delta$ being joined if Δ_{ij} is not a class of generalized digons and, in this case, Δ_{ij} is the weight of the edge (i,j).

PROPOSITION 1.13 [B1]. If $P = (S,I,t)$ is a geometry belonging to the special diagram (Δ,f), then, for every flag F of P, the residue P_F belongs to the subdiagram $(\Delta-t(F), f|_{\Delta-t(F)})$.

THEOREM 1.14 [B1]. Suppose (Δ,f) is a special diagram such that the graph induced on Δ is a tree. Let 0 be an endpoint of Δ. Let P be a geometry over Δ belonging to the diagram (Δ,f) and satisfying (IP). For all varieties v,w of P of respective types, i,j, if $\sigma(v) \subseteq \sigma(w)$, then v and w are incident and i belongs to the path joining 0 to j in Δ.

THEOREM 1.15 [B2]. Let P be a firm, strongly connected geometry over Δ that satisfies the intersection property. Then, for every $i \in \Delta$, any intersection of i-shadows of flags of P is an i-shadow of a flag or is empty.

C) Some basic rank 2 diagrams

All rank 2 diagrams introduced here, except the diagram $\circ\!\!\overset{[d]}{-\!\!-\!\!-}\!\!\circ$ for d-transversal planes, are taken from [B1]. We first recall the main rank 2 diagrams used for the representation of buildings and sporadic groups.

a) Partial plane : $\circ\!\!\overset{\pi}{-\!\!-\!\!-}\!\!\circ$
For axioms, see definition 1.8.

b) Linear space : $\circ\!\!\overset{L}{-\!\!-\!\!-}\!\!\circ$
This diagram is contained in $\circ\!\!\overset{\pi}{-\!\!-\!\!-}\!\!\circ$ and characterized by the following axiom : any two distinct points are incident with a unique line.

c) Generalized projective plane : $\circ\!\!-\!\!-\!\!-\!\!\circ$
This is the intersection of $\circ\!\!\overset{L}{-\!\!-\!\!-}\!\!\circ$ and $\circ\!\!\overset{L*}{-\!\!-\!\!-}\!\!\circ$

d) Affine plane : $\circ\!\!\overset{AF}{-\!\!-\!\!-}\!\!\circ$
This diagram is contained in $\circ\!\!\overset{L}{-\!\!-\!\!-}\!\!\circ$ and characterized by the axiom : if ℓ is a line and p is a point non incident with ℓ, then there exists a unique line ℓ' such that p is incident with ℓ' and ℓ, ℓ' have no incident point in common.

e) Circles : $\circ\overset{c}{\text{———}}\circ$

This is contained in $\circ\overset{L}{\text{———}}\circ$ and characterized by the axiom : every line is incident with exactly two points.

f) Generalized n-gon (n integer, n ⩾ 2) : $\circ\overset{(n)}{\text{———}}\circ$

For n = 2, this is the generalized digon (definition 1.9) with diagram $\circ\quad\quad\circ$. For n ⩾ 3, this is contained in $\circ\overset{\pi}{\text{———}}\circ$ and characterized by : any two varieties are joined by at least one chain of length $\ell \leqslant n+1$ and at most one chain of length $\ell < n+1$.

We now introduce the diagram $\circ\overset{[d]}{\text{———}}\circ$ that we will use for the representation of d-injection geometries.

g) d-transversal plane (d integer, d ⩾ 1) : $\circ\overset{[d]}{\text{———}}\circ$

This is contained in $\circ\overset{\pi}{\text{———}}\circ$ and characterized by the following property : there exists d resolutions of the set P of points, i.e. d partitions of P such that (i) every line is incident with exactly one point of each class of every partition of P; and satisfying, moreover, (ii) any two points which belong to distinct classes in every partition of P are incident with a common line. The dual diagram is $\circ\overset{[d]*}{\text{———}}\circ$ and characterized by d partitions of the set L of lines satisfying dually (i) and (ii).

REMARK 1.16. The diagram $\circ\overset{[1]*}{\text{———}}\circ$ coincides with the diagram $\circ\overset{N}{\text{———}}\circ$ for net structures considered in detail in [Spl]. Notice that a rank 2 geometry with d parallelisms is a special case of semi-net structures.

2. DIAGRAM REPRESENTATION FOR SQUASHED GEOMETRIES AND d-INJECTION GEOMETRIES

A) Diagrams for squashed geometries

Many interesting examples of diagrams, for instance diagrams for sporadic groups and most of the buildings, are linear, that is of type : $\circ\overset{\pi}{\text{———}}\circ\ \dots\ \circ\overset{\pi}{\text{———}}\circ\overset{\pi}{\text{———}}\circ$. Hence efforts were made for characterizing geometries belonging to linear diagrams or to special cases of linear diagrams where one or more stroke $\circ\overset{\pi}{\text{———}}\circ$ is replaced by a more restricted rank 2 diagram such as $\circ\overset{L}{\text{———}}\circ$, $\circ\text{———}\circ$, $\circ\overset{AF}{\text{———}}\circ$, $\circ\overset{c}{\text{———}}\circ$. The following theorem is a reformulation of theorem 7 in [B1].

THEOREM 2.1. (i) Let P be a firm, strongly connected geometry over $\Delta = [0,n-1]$ that satisfies the intersection property and belongs to the diagram

$\underset{0}{\circ}\overset{L}{\text{———}}\underset{1}{\circ}\ \ldots\ \circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\underset{n-1}{\circ}$. Then the set of all shadows of varieties of P is the set of flats of a simple matroid of rank n on P_0 whose i-flats are exactly the shadows of (i-1)-varieties of P for all $i \in [1,n]$.

(ii) Conversely, any simple matroid of rank n belongs to the diagram $\circ\overset{L}{\text{———}}\circ\ \ldots\ \circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ$.

We now characterize geometries belonging to the Diagram 2.2 : $\circ\overset{L}{\text{———}}\circ\ \ldots\ \circ\overset{L}{\text{———}}\circ\overset{\pi}{\text{———}}\circ$.

THEOREM 2.3. (i) Let P be firm, strongly connected geometry over $\Delta = [0,n-1]$ that satisfies the intersection property and belongs to the diagram 2.2. Then the set of shadows of varieties of P is the set of flats of a simple and well-cut squashed geometry of rank n on P_0, the i-flats being exactly the shadows of (i-1)-varieties of P for all $i \in [1,n]$.

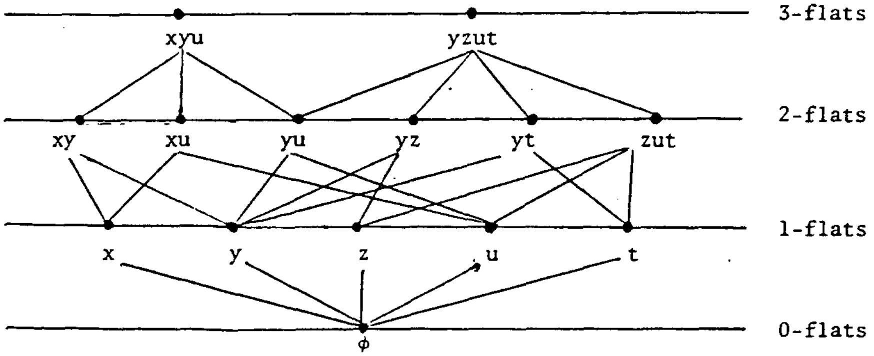
(ii) Conversely, any simple and well-cut squashed geometry of rank n belongs to the diagram 2.2.

Proof. Let us first prove (i). We define : $g_0 = \{\phi\}$, $g_i = \{\sigma(v) : v \in P_{i-1}\}$ for $i \in [1,n]$ and $g = g_0 \cup \ldots \cup g_n$. Let us show that g is a squashed geometry on P_0, i.e. satisfies axioms (F1),(F2),(F3). Notice first that, if v is a i-variety, w a j-variety, $i < j$ and vIw, then $\sigma(v) \subseteq \sigma(w)$; hence, for every flag F of P, there exists a variety v of P such that $\sigma(F) = \sigma(v)$. Thus, theorem 1.15 implies clearly (F1). We infer (F2) from theorem 1.14. Let v be a i-variety; $i \leq n-3$, $p \in P_0$ be a point non-incident to v and u be a (n-1)-variety such that $\sigma(v) \cup p \subseteq \sigma(u)$. From theorem 1.14, v and u are incident. The residue of u is a geometry P containing the varieties v, p and belonging to the diagram $\underset{0}{\circ}\overset{L}{\text{———}}\underset{1}{\circ}\ \ldots\ \circ\overset{L}{\text{———}}\underset{n-2}{\circ}$. Thus, theorem 2.1 yields the existence of a (i+1)-variety w such that : $\sigma(w) \supseteq \sigma(v) \cup p$ which achieves the proof of (F3). It is clear that g is simple and well-cut.

We now prove (ii). Let $g = g_0 \cup g_1 \cup \ldots \cup g_n$ be a simple and well-cut squashed geometry. Then $P = (g-\{\phi\},I,t)$ is a geometry over $\Delta = [0,n-1]$ where I is the inclusion relation and t is the mapping from $g-\{\phi\}$ onto Δ defined by : $t(G) = i-1$ for every $G \in g_i$, $i \in [1,n]$. Let F be a flag of P of type $\Delta-\{i,j\}$ where $0 \leq i < j \leq n-1$. If $i+1 < j$, then the residue P_F is clearly a generalized digon. Consider now $j = i+1$. If $j \leq n-2$, there exists $G_1 \in g_i$, $G_2 \in g_{i+3}$ such that $P_F = \{G \in g : G_1 \subseteq G \subseteq G_2\}$. Since $g \cap [\phi,G_2]$ is a matroid on G_2, it is clear that P_F is a linear space $\circ\overset{L}{\text{———}}\circ$. If $j = n-1$, there exists $G_1 \in g_{n-2}$ such that $P_F = \{G \in g : G_1 \subseteq G\}$ and therefore we can only claim that P_F is a partial plane.□

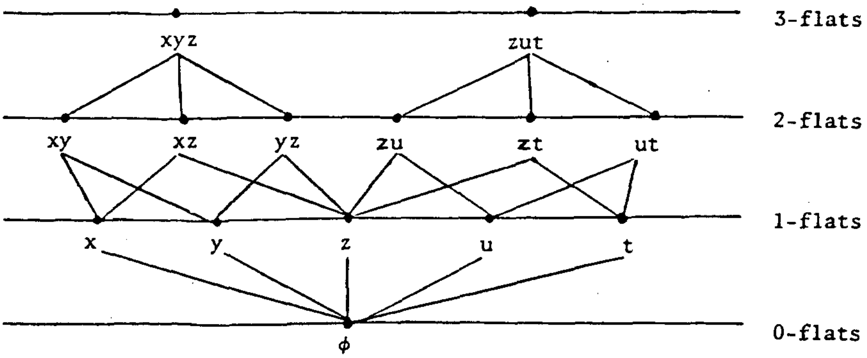
REMARK 2.4. A squashed geometry may not be well-cut, or firm, or pure, or strongly connected as shown by the following examples 1, 2.

Example 1. Let g be the squashed geometry of rank 3 on {x,y,z,u,t} whose flats have the following configuration :



then g is neither firm, nor pure (P_y is not a generalized digon, P_x is a generalized digon) and g−{x,y,u} is not well-cut.

Example 2. Let g be the squashed geometry of rank 3 on {x,y,z,u,t} whose flats have the following configuration:



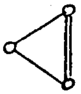
then g is not strongly connected (xyz and zut cannot be connected by any chain in $g_2 \cup g_3$). Notice that for any well-cut squashed geometry g, g is strongly connected if and only if $g_{n-1} \cup g_n$ is connected.

REMARK 2.5. Simple and well-cut squashed geometries of rank 2 are just partial linear spaces. So, examples of such geometries include all rank 2 diagrams given before except generalized digons.

Many examples of squashed geometries are provided by known diagrams for spherical buildings and sporadic groups :

(i) Spherical diagrams (see [Ne]). Those diagrams involve only strokes of type $\circ\overset{(n)}{\text{——}}\circ$ (n ⩾ 2) and all except F_4 : $\circ\text{——}\circ\text{══}\circ\text{——}\circ$ provide examples of squashed geometries. (In fact, for E_6, E_7, E_8, they come from appropriated subdiagrams). For instance, projective spaces whose diagrams are A_n : $\circ\text{——}\circ\ \ldots\ \circ\text{——}\circ\text{——}\circ$ are examples of matroids; polar spaces whose diagrams are C_n : $\circ\text{——}\circ\ \ldots\ \circ\text{——}\circ\text{══}\circ$ are special bouquets of projective spaces.

(ii) Diagrams for sporadic groups (see [T], [RS]). These diagrams involve usually strokes of type $\circ\overset{(n)}{\text{——}}\circ$ or $\circ\overset{c}{\text{——}}\circ$. For instance, the Matthieu group M_{11} admits the following diagrams : $\circ\overset{c}{\text{——}}\circ$, $\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ$, $\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ$ each of them yielding a squashed geometry. Examples of diagrams of sporadic groups which do not yield a squashed geometry are

H_e :  , Co_1 : $\circ\text{══}\circ\text{——}\circ\text{══}\circ\text{——}\circ$, Co_2 : $\circ\text{——}\circ\text{══}\circ\text{——}\circ$,

BM : $\circ\text{——}\circ\text{——}\circ\text{══}\circ\text{——}\circ$.

We now study diagrams 2.2 in which we specialize the last stroke $\circ\overset{\pi}{\text{——}}\circ$ to be $\circ\overset{[d]}{\text{——}}\circ$.

B) Diagrams for d-injection geometries

From theorem 2.3, any d-injection geometry g (see definition 1.4) which is simple and well-cut belongs to the diagram 2.2; but, with some more assumptions on g, we can precise the last stroke $\circ\overset{\pi}{\text{——}}\circ$.

THEOREM 2.6. Let g be a simple d-injection geometry (d ⩾ 1) of rank n. Suppose that, for all $\alpha \in [1,d]$, there exists a matroid M^{α} on X_{α} such that p_{α} is an isomorphism between M^{α} and $g_R = g \cap [\phi,R]$ for every roof R of g. Then g belongs to the diagram : $\underset{0}{\circ}\overset{L}{\text{——}}\underset{1}{\circ}\overset{L}{\text{——}}\circ\ \ldots\ \circ\text{——}\underset{n-2}{\circ}\overset{[d]}{\text{——}}\underset{n-1}{\circ}$.
Notice that, under the conditions of theorem 2.6, all roofs of g have the same size $v = |X_1| = \ldots = |X_d|$.

Proof. We have only to prove that the last stroke is $\circ\overset{[d]}{\text{——}}\circ$. Let $F_0 \in g_{n-2}$. Define : $P = \{F \in g_{n-1} : F \supseteq F_0\}$ and $L = \{G \in g_n : G \supseteq F_0\}$. We show that (P,L) belongs to the diagram $\circ\overset{[d]}{\text{——}}\circ$. We first define d resolutions on P as follows : for $\alpha \in [1,d]$ and $I \in M^{\alpha}_{n-1}$ such that $I \supseteq p_{\alpha}(F_0)$, let $P^{\alpha}_{I} = \{F \in P : p_{\alpha}(F) = I\}$. Then we have clearly d partitions :

$P = \cup \{P^{\alpha}_{I} : I \in M^{\alpha}_{n-1}$ and $I \supseteq p_{\alpha}(F_0)\}$ for every $\alpha \in [1,d]$.

Let $G \in L$, we prove that for all $\alpha \in [1,d]$ and $I \in M^{\alpha}_{n-1}$ such that $I \supseteq p_{\alpha}(F_0)$, there exists a unique flat of P^{α}_{I} such that $F \subseteq G$. Since p_{α} is an isomorphism between g_G and M^{α}, there exists a unique flat F of g_G such that $p_{\alpha}(F) = I$. We have only to verify that $F \supseteq F_0$. Choose $x \in F_0$, then $x_{\alpha} = p_{\alpha}(x) \in p_{\alpha}(F)$, since $x_{\alpha} \in p_{\alpha}(F_0)$, $p_{\alpha}(F_0) \subseteq I$. Hence there exists $x' \in F$ such that $p_{\alpha}(x') = x_{\alpha}$. Since $F \subseteq G$, $x' \in G$. Thus x, x' are two elements of G having the same α^{th} coordinate which yields $x = x'$ and therefore $x \in F$.

Let $F, F' \in P$ such that $p_{\alpha}(F) \neq p_{\alpha}(F')$ for all $\alpha \in [1,d]$. We prove that there exists $G \in L$ such that $F \cup F' \subseteq G$. Since $p_{\alpha}(F)$, $p_{\alpha}(F')$ are distinct (n-1)-flats of M^{α} containing the (n-2)-flat $p_{\alpha}(F_0)$, we have $p_{\alpha}(F) \cap p_{\alpha}(F') = p_{\alpha}(F_0)$. Choose an element a_1 of $p_1(F') - p_1(F_0)$, thus $a_1 \notin p_1(F)$. Since $a_1 \in p_1(F')$ there exists $a \in F'$, $a = (a_1, a_2, \ldots, a_d)$ such that $p_1(a) = a_1$. We show that $a_{\alpha} \notin p_{\alpha}(F)$ for every $\alpha \in [1,d]$. Suppose by contradiction that, for instance, $a_2 \in p_2(F)$, then $a_2 \in p_2(F_0)$. Thus there exists $x \in F_0$ such that $p_2(x) = a_2$. Since both elements a, x belong to F' and $p_2(x) = p_2(a)$, we deduce that $x = a$ and therefore $p_1(x) = a_1 \in p_1(F_0)$ which yields a contradiction. Thus, $F \cup a$ is a d-injective subset and axiom (F3) for squashed geometries yields the existence of $G \in g_n$ such that $G \supseteq F \cup a$. Since F' is the unique (n-1)-flat containing $F_0 \cup a$, we have also $F' \subseteq G$ (see remark (1) following definition 1.1) and therefore $F \cup F' \subseteq G$. □

DEFINITION 2.7. A d-injection geometry g is called <u>concentrated in</u> X_{α} for some $\alpha \in [1,d]$ if, for all $G, G' \in g$, there exists $G'' \in g$ such that $p_{\alpha}(G) \cap p_{\alpha}(G') = p_{\alpha}(G \cap G'')$.

PROPOSITION 2.8. Let g be a well-cut d-injection design with parameters $(l_0, l_1, \ldots, l_s)$. Suppose that for some $\alpha \in [1,d]$, $l_s = |X_{\alpha}|$ and g is concentrated in X_{α}. Then, there exists a matroid M^{α} on X such that p is an isomorphism between M^{α} and g_R for every roof R of g.

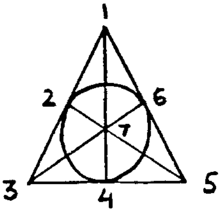
<u>Proof</u>. Let R be a roof of g. We define $M^{\alpha}(R) = \{p_{\alpha}(G) : G \in g_R\}$. We prove first that $M^{\alpha}(R)$ is a matroid on X_{α}. Since g is concentrated in X_{α}, $M^{\alpha}(R)$ is a meet semi-lattice. Take G a i-flat of g_R $(i < s)$ and an element

$x_\alpha \in X_\alpha - p_\alpha(G)$. Since $p_\alpha(R) = X_\alpha$, there exists an element x of R such that $p_\alpha(x) = x_\alpha$. Since $G \cup x \subseteq R$, $G \cup x$ is a d-injective subset, thus there exists a (i+1)-flat G' of g_R such that $G \cup x \subseteq G'$. Hence $p_\alpha(G')$ is a (i+1)-flat of $M^\alpha(R)$ containing $p_\alpha(G) \cup x_\alpha$ which achieves the proof that $M^\alpha(R)$ is a matroid on X_α. We now show that $M^\alpha(R) = M^\alpha(R')$ for all distinct roofs R, R' of g. Let $G \in g_R$ and $G' \subseteq R'$ be the unique subset of R' such that $p_\alpha(G) = p_\alpha(G')$. We want to prove that in fact $G' \in g_{R'}$. Suppose by contradiction that G is a flat of minimal rank r (r > 0) such that $G' \notin g$. Let F be a (r-1)-flat contained in G, then the unique subset F' of R' such that $p_\alpha(F) = p_\alpha(F')$ is indeed a (r-1)-flat of $g_{R'}$. Choose $a \in G' \setminus F'$. Since $F' \cup a \subseteq R'$, there exists a unique r-flat $G'' \in g_{R'}$ such that $F' \cup a \subseteq G''$. Since g is concentrated in X_α, $p_\alpha(G) \cap p_\alpha(G'')$ is a flat of $M^\alpha(R)$. We have : $p_\alpha(F) \cup p_\alpha(a) \subseteq p_\alpha(G) \cap p_\alpha(G'') \subseteq p_\alpha(G)$, hence $p_\alpha(G) \cap p_\alpha(G'') = p_\alpha(G)$ holds, which yields : $p_\alpha(G') = p_\alpha(G) \subseteq p_\alpha(G'')$. Since $p_\alpha(G')$, $p_\alpha(G'')$ have the same size ℓ_r, equality $p_\alpha(G') = p_\alpha(G'')$ holds, from which we deduce : $G' = G''$ contradicting $G' \notin g$.□

COROLLARY 2.9. Let g be a simple well-cut d-injection design with parameters $(\ell_0, \ell_1, \ldots, \ell_s = |X_1| = \ldots = |X_d|)$ and that is concentrated in every X_α for $\alpha \in [1,d]$. Then g belongs to the diagram : $\circ \overset{L}{—} \circ \; \ldots \; \circ \overset{L}{—} \circ \overset{[d]}{—} \circ$.

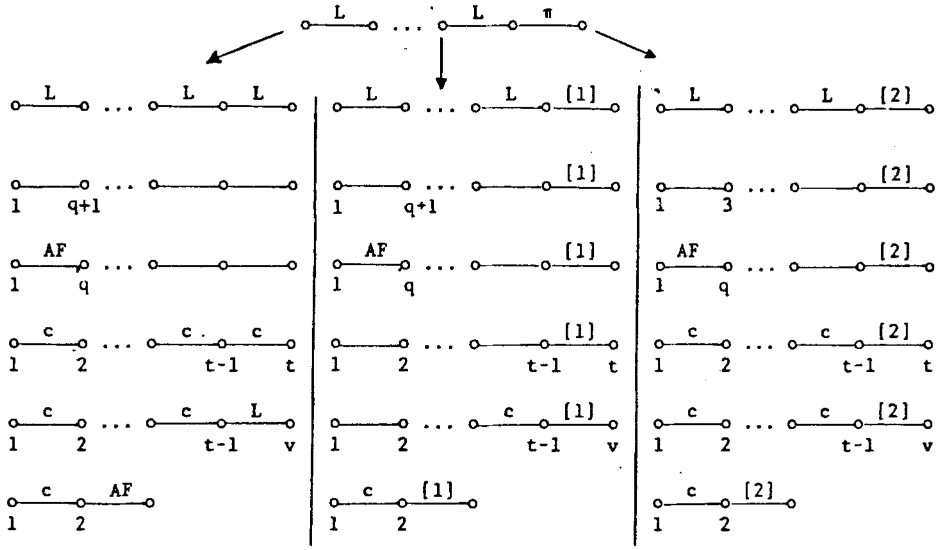
Proof. It follows trivially from theorem 2.6 and Proposition 2.8.□

REMARK 2.10. The converse of theorem 2.6 is false as shown by the following example (introduced as example 4.1 in [Sp 2]). Consider the Fano plane having the following configuration :



We introduce the geometry P of rank 3 whose points are : 1, 2, 3, 4, 5, 6, 7; whose lines are all pairs of points; whose planes are the complements of the lines of Fano plane, i.e. 4567, 2356, 2347, 1346, 1357, 1245, 1267 (here, for example, 4567 stands for {4,5,6,7} for the sake of shortness). It is easy to see that P is not a transversal geometry since the only convenient partition of the points in order to have all planes to be transversal subsets is the partition of [1,7] by the singletons, see now that $12 \cup 3$ is transversal but not contained in any flat. However, it can be verified that P belongs to the diagram $\circ\overset{c}{\text{——}}\circ\overset{[1]}{\text{——}}\circ$. For instance, in the residue P_1, axioms for [1] are satisfied if we choose the following resolution for lines of P_1 : $\{12,13\} \cup \{14,17\} \cup \{15,16\}$.

Let us now see some examples of the diagram : $\circ\overset{L}{\text{——}}\circ \ldots \circ\overset{L}{\text{——}}\circ\overset{[d]}{\text{——}}\circ$ for $0 \leq d \leq 2$ (By definition : $\circ\overset{[0]}{\text{——}}\circ$ is $\circ\overset{L}{\text{——}}\circ$). In the following diagrams, a weight is sometimes assigned to a node i that represents the size of all i-varieties.



We now give examples of geometries belonging to the above diagrams. In the first row, we have respectively matroids, transversal geometries, permutation geometries (with some additional assumptions, see theorem 2.6). In the design case, the matroid is called PMD (for details, see [D2], only 4 types of examples are known) and all design cases for d = 2 are classified when the additionnal condition that all roofs form a group is imposed (see remark 1.5). For the second row, we have projective spaces, transversal geometries from [Spl] and remark after theorem 3.2 in [CDF], permutation geometries of theorem 5.8 (i) in [CD] (geometric group GL(r,2)). For the third row, we have affine spaces, transversal geometries of theorem 3.2 in [CDF], permutation geometries of theorem 5.8 (ii) in [CD] (geometric group AGL(r-1,g)). For the fourth row, we have truncations of boolean algebras, full transversal geometries (i.e. set of all transversal mappings, see definition 3.5), full permutation geometries (i.e. set of all permutations). For the fifth line, we have t-designs, transversal t-designs (sharply t-transitif set of transversal mappings, see definition 3.6), sharply t-transitif set of permutations. For the sixth row, we have Möbius plane, Laguerre plane, Minkowski plane, which are d-transversal geometries (d = 0,1,2). In [DF] (example 5.4) there is a d-injective analogue of these planes with diagram $\circ\overset{c}{\text{———}}\circ\overset{[d]}{\text{———}}\circ$ for any $d \geq 2$. In [Sp2], examples of geometries having diagram $\circ\overset{L}{\text{———}}\circ\overset{[1]}{\text{———}}\circ$ are given. Notice that none of them are transversal geometries.

3. SOME CONSTRUCTIONS OF SQUASHED GEOMETRIES

Several operations : interval, extension, truncation, defined in section 1, and also (see [CV]) : restriction, direct sum, already yield new squashed geometries from old ones.

In this section, we give a construction of d-injection geometries by direct product and a construction of permutation geometries by blow-up, extending slightly the corresponding constructions given in [DF2], [CDF]. Then we introduce a construction of *M*-unisupported squashed geometries by pointwise action of a set of mappings on a matroid and a construction of a class of squashed designs by setwise action.

A) Construction of d-injective designs by direct product.

THEOREM 3.1. Suppose g' (resp. g'') is a d'-injection design (resp. d''-injection design) with parameters $(\ell_0,\ell_1,\ldots\ell_s)$ on $X' = X'_1\times\ldots\times X'_{d'}$ (resp. $X'' = X''_1\times\ldots\times X''_{d''}$) and suppose also that g' (resp. g'') is concentrated in X'_1 (resp. X''_1). Suppose further that $X'_1 = X''_1$ and that $p_1(g') = p_1(g'')$. Then there exists a $(d' + d'' - 1)$-injection design g with parameters $(\ell_0,\ell_1,\ldots,\ell_s)$ on $X = X'_1 \times X'_2 \times\ldots\times X'_{d'} \times X''_2 \times\ldots\times X''_{d''}$.

Proof. In [DF2], a proof is given in injective case (i.e. $d \geq 2$) which extends easily to the case of 1-injection designs.

Let us just recall the idea of construction by direct product. With $a = (a_1,..,a_{d'}) \in X'$ and $b = (b_1,\ldots,b_{d''}) \in X''$ such that $a_1 = b_1$, we associate the element $ab = (a_1,\ldots,a_{d'}, b_2,\ldots,b_{d''})$ of X. Similarly, if $A \in g'$, $B \in g''$ and $p_1(A) = p_1(B)$, we can define $AB = \{ab : a \in A,\ b \in B \text{ and } p_1(a) = p_1(b)\}$. This leads to the definition of $g = g'g'' = \{AB : A \in g',\ B \in g'' \text{ and } p_1(A) = p_1(B)\}$. The proof consists now to show that g is the desired $(d'+d''-1)$-injection design.□

B) Construction of permutation geometries by blow-up (inflation)

THEOREM 3.2. Let $X_1 = [1,n]$ and $X_2 = [1,m]$. Let $\mathcal{M}$ be a matroid on X_1. Let φ be a transversal geometry on $X_1 \times X_2$ such that p_1 is an isomorphism between $\mathcal{M}$ and φ_R for every roof R of φ. Let $\mathcal{P}$ be a permutation geometry on $X_1 \times X_1$ such that p_1 is an isomorphism between $\mathcal{M}$ and φ_R for every roof R of φ. Hence φ and $\mathcal{P}$ have the same rank s. Then there exists a permutation geometry g of rank s on $[1,nm]^2$. Moreover g is $\mathcal{M}$-unisupported, i.e. $\mathcal{M}$ and g_R are isomorphic matroids for every roof R of g.

Proof. This theorem is proved in [CDF] (theorem 3.1) when we assume that φ and $\mathcal{P}$ are squashed designs having the same parameters. For the proof in general case, see [La2].

Let us just give the idea of the construction by blow-up. Consider the square $[1,n]^2$. Replace each of its points (a,b) by a $m \times m$ square $K(a,b)$. So, we obtain a $nm \times nm$ square divided into n^2 small squares. Suppose we have a latin square on every $K(a,b)$, i.e., if we denote by $G_1(a,b),\ldots,G_m(a,b)$ the rows of $K(a,b)$, then $\{G_1(a,b),\ldots,G_m(a,b)\}$ is a set of pairwise disjoint 2-injective subsets of size m of $R_a \times C_b$ where R_a is the set of indexes of the rows of $K(a,b)$ and C_b is the set of indexes of the columns of $K(a,b)$, R_a, C_b being subsets of $[1,nm]$. Let $A \in \varphi_s$, $A = \{(i,a_i) : i \in [1,n]\}$ and $P \in \mathcal{P}_s$, $P = \{(i,p_i) : i \in [1,n]\}$. We construct the following permutation of $[1,nm]^2$: $G(A,P) = \bigcup_{i=1}^{n} G_{a_i}(i,p_i)$. Define $g = \{G(A,P) : A \in \varphi_s,\ P \in \mathcal{P}_s\}$ and $g^\star$ the meet semi-lattice generated by g. Then $g^\star$ is the desired permutation geometry.□

REMARK 3.3. This construction by blow-up can be extended :
- to the infinite case using an infinite latin square whenever we have an infinite analogue of transversal geometry (on $X_1 \times X_2$ with $|X_2|$ infinite) (section 5 [CDF] gives examples of infinite permutation geometries)
- to the case of d-injection geometries ($d \geqslant 2$) (see [La2]).

C) Construction of $\mathcal{M}$-unisupported squashed geometries.

Any unisupported squashed geometry (cf. definition 1.6) is clearly well-cut.

The following classes a), b), c) of $\mathcal{F}$-squashed geometries are unisupported.
a) d-injection design with parameters $(l_0,\ldots,l_s = |X_1|)$ and concentrated in X_1 (see Proposition 2.8), including in particular a)', a)" :
a)' sharply s-transitive set of d-injective mappings ($l_i = i$ for $i \in [0,s-1]$) (see Proposition 3.8)
a)" permutation geometries whose set of roofs is a group (Proposition 3.5 in [CD]).
b) strongly connected $\mathcal{F}$-squashed designs with parameters $(l_0,\ldots,l_{s-1}$, $l_s = l_{s-1} + l_1 - l_0)$ (see [La2])
c) permutation geometries obtained by blow-up (Theorem 3.2).

We now introduce a method of construction of unisupported squashed geometries by the pointwise action of mappings. Let X_1, Z be two finite sets. Let $\mathcal{M}_i$ $(i\in[1,s-1])$ be some pairwise disjoint collections of subsets of X_1 and $\mathcal{M}_0 = \{\phi\}$, $\mathcal{M}_s = \{X_1\}$, $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_s$. Let E be a set of mappings from X_1 into Z. For every $I \in \mathcal{M}$ and $f \in E$, we define the subset of $X_1 \times Z$: $I_f = \{(x,f(x)),\ x \in I\}$. Let us define : $g_i = \{I_f : I \in \mathcal{M}_i,\ f \in E\}$ for $i \in [0,s]$ and $g = g(\mathcal{M},E) = g_0 \cup g_1 \cup \ldots \cup g_s$. Notice that, for $I = X_1$, $I_f = \{(x,f(x)), x \in X_1\}$ is indeed the graph P_f of the mapping f.

THEOREM 3.4. $g(\mathcal{M},E)$ is a squashed geometry (i.e. bouquet of matroids) if and only if the two following conditions are satisfied :
(i) $\mathcal{M}$ is the set of flats of a matroid on X_1
(ii) for all $f,h \in E$, $\{x \in X_1 : f(x) = h(x)\} \in \mathcal{M}$
Furthermore, if (i), (ii) hold, then g is $\mathcal{M}$-unisupported.

Proof. Suppose first that (i), (ii) are satisfied. Let us show that (F1), (F2), (F3) hold. Let $I,J \in \mathcal{M}$ and $f,h \in E$. Then $I_f \cap J_h = (I \cap J \cap K)_f$, $K = \{x \in X_1 : f(x) = h(x)\}$, thus $I_f \cap J_h \in g$. Suppose $I_f \in g_i$, $J_h \in g_j$ and $I_f \subset J_h$, then $I \subset J$ and therefore $i < j$. Take now $I \in \mathcal{M}_i$ $(i \leqslant s-1)$, $f \in E$ and $(x_0,z_0) \in X_1 \times Z$ such that $(x_0,z_0) \notin I_f$ and $I_f \cup (x_0,z_0)$ is contained in J_h for some $J \in \mathcal{M}$ and $h \in E$. Thus $I \cup x_0 \subseteq J$ and $h|_I = f|_I$, $h(x_0) = z_0$. Let K be the unique i+1-flat of $\mathcal{M}$ containing $I \cup x_0$, then K_h is a flat of g_{i+1} containing $I_f \cup (x_0,z_0)$ which achieves the proof of (F3).
Conversely, suppose g is a squashed geometry. Since g is a meet semi-lattice, for all $f,h \in E$, $P_F \cap P_h$ is a flat of g which implies clearly (ii). (i) is trivially satisfied.
When (i), (ii) hold, for every $f \in E$, the mapping that associates whith every

flat $I \in M$ the flat $I_f \in g$ provides clearly an isomorphism between M and $g \cap [\phi, P_f]$. □

In order to get by this process F-squashed geometries, we need some additionnal information on the set of mappings E. More exactly, in the case of d-injective mappings, theorem 3.4 provides a constructive method for d-injection geometries.

DEFINITION 3.5. Let $X_1, \ldots, X_d$, Z be finite sets and f be a mapping from X_1 into Z.

(i) when $Z = X_2$, f is called a <u>1-injective mapping</u> or <u>transversal mapping</u> (or application)

(ii) when $Z = X_2 \times \ldots \times X_d$ $(d \geq 2)$ and $P_f = \{(x, f(x)),\ x \in X_1\}$ is a d-injective subset of $X_1 \times \ldots \times X_d$, then f is called a <u>d-injective mapping</u>.

From now on, in this section, we keep the notation of definition 3.5 for $X_1, \ldots, X_d$, Z.

DEFINITION 3.6. Let E be a set of d-injective mappings from X_1 into Z. E is called <u>t-transitive</u> (resp. <u>sharply t-transitive</u>) if, for all distinct $(x_i, z_i) \in X_1 \times Z$, $i \in [1,t]$, if $\{(x_i, z_i) : i \in [1,t]\}$ is a d-injective subset of $X_1 \times Z$, then there exists $f \in E$ (resp. a unique $f \in E$) such that $f(x_i) = z_i$ for all $i \in [1,t]$.

Notice that this definition generalizes the notion of t-transitive and sharply t-transitive set of permutations (for case d = 2).

THEOREM 3.7. Let M be a matroid on X_1 and $t = \mathrm{Max}(|I| : I \in M - \{X_1\})$. Let E be a set of d-injective mappings from X_1 into Z that is (t+1)-transitive and satisfies : (ii) for all $f, h \in E$, $\{x \in X_1,\ f(x) = h(x)\} \in M$. Then $g(M,E)$ is a d-injection geometry.

<u>Proof</u>. Theorem 3.4 already yields that $g(M,E)$ is a bouquet of matroids. All flats of $g(M,E)$ are trivially d-injective subsets of $X_1 \times Z$. Let $f \in E$, $I \in M_i$ $(i \leq s-1)$ and $(x_0, z_0) \in X_1 \times Z - I_f$ such that $I_f \cup (x_0, z_0)$ is a d-injective subset of $X_1 \times Z$. Since $|I \cup x_0| \leq t+1$ and E is (t+1)-transitive, there exists $h \in E$ such that $h(x) = f(x)$ for all $x \in I$ and $h(x_0) = z_0$. Let now J be the unique (i+1)-flat of M containing $I \cup x_0$. Therefore, J_h is a (i+1)-flat of M containing $I_f \cup (x_0, z_0)$ which achieves the proof that g is a d-injection geometry. □

PROPOSITION 3.8. Let E be a set of d-injective mappings from X_1 into Z which

is sharply (t+1)-transitive and B_t denote the t-truncation of the boolean algebra on X_1. Then the meet semi-lattice generated by $\{P_f,\ f \in E\}$ is a B_t-unisupported d-injection geometry of rank t+1.

Proof. The proof is straightforward. □

A natural question arises : among given examples of unisupported squashed geometries, which ones do come as results of a construction $g(M,E)$ by pointwise action of mappings as described in theorem 3.4. The following theorem gives a partial answer.

THEOREM 3.9. Let g be a d-injection geometry on $X = X_1 \times Z$. Suppose there exists a matroid M on X_1 such that p_1 is an isomorphism between g_R and M for every roof R of g. Then there exists a set E of d-injective mappings from X_1 into Z such that g and $g(M,E)$ are isomorphic.

Proof. Let $R = \{R_1,\dots,R_m\}$ be the set of roofs of g. By assumption, $p_1(R_i) = X_1$ for all $i \in [1,m]$. Thus, there exists d-injective mappings : $f_1,\dots f_m$ from X_1 into Z such that $R_i = \{(x,f_i\ (x))\ :\ x \in X_1\}$. We define a mapping φ from g into $g(M,E)$ as follows : if G is a flat of g_{R_i}, then $\varphi(G) = (p_1(G))_{f_i}$. It is easy to see that φ is an isomorphism between the squashed geometries g, $g(M,E)$. □

REMARK 3.10. Example a) satisfies theorem 3.9.

D) Construction of squashed designs by setwise action

Let $P = (S,I,t)$ be a geometry over the set $\Delta = [0,n-1]$. Let E be a set of isomorphisms of P. For every $f \in E$ and every maximal flag F of P, we define the following subset of $S \times S$: $R(F,f) = \{(v,f(v)),\ v \in F\}$ and $R = \{R(F,f)\ :\ f \in E$ and F is a maximal flag of P$\}$. Let $g = g(P,E)$ be the meet semi-lattice generated by R and $g_i = \{G \in g\ :\ |G| = i\}$ for $i \in [0,n]$, hence $g_n = R$.

THEOREM 3.11. If P is a firm geometry, then $g(P,E)$ is a short squashed design with parameters $(0,1,\dots,n)$ on $S \times S$.

Proof. We only have to prove that $g(P,E)$ satisfies axiom (F3). Let $G \in g_i$ $(i \leqslant n-1)$ and $(v_0,w_0) \in S \times S - G$ such that $G \cup (v_0,w_0)$ is contained in $R(F,f)$ for some $f \in E$ and some maximal flag F. Thus $G = \{(v_p,f(v_p))\ :\ p \in [1,i]\}$, $(v_0,w_0) = (v_{i+1},f(v_{i+1}))$ with $\{v_1,\dots,v_{i+1}\} \subseteq F$. Since P is firm, it is easy to see that every non maximal flag of P is the intersection of two maximal flags of P. Hence, let F' be a maximal flag such that $\{v_1,\dots,v_{i+1}\} = F \cap F'$, then $G \cup (v_0,w_0) = R(F,f) \cap R(F',f)$ and therefore $G \cup (v_0,w_0) \in g_{i+1}$. By the same argument, we obtain that g is short. □

4. SOME REMARKS AND RELATED PROBLEMS

A) Links with greedoids, supermatroids, independence systems

Looking at the structure of their collections of independent subsets enables us to compare squashed geometries with the above combinatorial structures. For any collection I of subsets of X, we consider the following three properties :

(P1) $\phi \in I$

(P2) $\forall J \in I$, if $I \subseteq J$, then $I \in I$

(P3) $\forall I,J \in I$, if $|J| > |I|$, then there exists $x \in J-I$ such that $I \cup x \in I$.

Recall that I is an independence system if I satisfies (P1), (P2); that I is the collection of independent subsets of a greedoid (resp. matroid) if I satisfies (P1), (P3) (resp. (P1), (P2), (P3)). Hence the class of matroids is exactly the intersection of the classes of greedoids and independence systems and more exactly the intersection of the classes of greedoids and squashed geometries.

Greedoids (introduced in [KL]) are exchange languages; this concept of language provides a common framework for two apparently unrelated structures but both defined by exchange properties : matroids and Coxeter groups (see [Bj]).

Squashed geometries are also a particular case of supermatroids, a concept introduced in [DIW] that generalizes both matroids and polymatroids (see [W]). More precisely, if $(P,\leq)$ is a poset, a supermatroid on P in a subset Q of P satisfying :

(i) $0 \in Q$

(ii) $\forall x \in Q$, if $y \leq x$, then $y \in Q$

(iii) $\forall a \in P$, all maximal elements of $\{x \in Q : x \leq a\}$ have the same height in P denoted by p(a).

Hence a matroid on X is a supermatroid on $P = 2^X$, a squashed geometry with roofs $R_1,\ldots,R_m$ is a supermatroid on $P = \bigcup_{i=1}^{m} 2^{R_i}$. Notice that a squashed geometry is indeed a strong supermatroid, i.e. satisfies the following two properties :

(iv) $\forall a,b \in P$, if b covers a, then $p(a) \leq p(b) < p(a)+1$

(v) $\forall a,b,b',c \in P$, if b,b' cover a and c covers b,b', then $p(a) = p(b) = p(b')$ implies $p(a) = p(c)$.

However, nice properties for supermatroids hold when P is a lattice. Notice that P is never a lattice in case of squashed geometries.

B) Duality

A natural way for defining a dual structure to a squashed geometry g would be to consider the union of the duals of the matroids composing g. If it is a meet semi-lattice, then it can be seen as a dual of g, but in general it is not the case. Example of such structure is the bouquet of self dual matroids, giving a self dual squashed geometry.

C) The Dilworth truncation

In a similar way as for matroids (see [Ma]), a Dilworth-truncation g^D can be defined for every squashed geometry g. More precisely, if g is a squashed geometry on X, then g^D is a squashed geometry on $\overline{X}$, set of all flats of g with non-zero rank. In a similar way, the k^{th} Dilworth truncation $g^{(k)}$ of g is a squashed geometry on the set $\overline{X}^k$ of all k-flats of g. For details, see [Sch].

D) Other squashed geometries with diagram $\circ\overset{L}{\text{———}}\circ \dots \circ\overset{L}{\text{———}}\circ\overset{[d]}{\text{———}}\circ$

Let X be a finite set. Consider d distinct partitions of X with classes : $X_1^\alpha, X_2^\alpha, \dots, X_{n_\alpha}^\alpha$ for $\alpha \in [1,d]$; so $X = \bigcup_{i=1}^{n_\alpha} X_i$ for all $\alpha \in [1,d]$. A subset T of X is called partial transversal if $|T \cap X_i^\alpha| \leq 1$ for all $i \in [1,n_\alpha]$ and all $\alpha \in [1,d]$. Let F be the set of all maximal partial transversals. Then a F-squashed geometry is called a d-transversal geometry. Notice that, for $d = 1$, a 1-transversal geometry is exactly a transversal geometry as considered in definition 1.4.

Suppose moreover that $|X_{i_1}^1 \cap X_{i_2}^2 \cap \dots \cap X_{i_d}^d| \leq 1$ for all $(i_1,i_2,\dots,i_d) \in [1,n_1] \times [1,n_2] \times \dots \times [1,n_d]$. Then every point $x \in X$ can be represented by the d-uple $(i_1,\dots,i_d) \in [1,n_1] \times \dots \times [1,n_d]$ such that $x \in X_{i_\alpha}^\alpha$ for each $\alpha \in [1,d]$; each partial transversal is indeed a d-injective subset of $[1,n_1] \times \dots \times [1,n_d]$ and each element X_i^α of each partition is projective, that is, all its elements have the same value i as α^{th}-coordinate. If we suppose furthermore that $|X_{i_1}^1 \cap \dots \cap X_{i_d}^d| = 1$ for all $(i_1,\dots,i_d) \in [1,n_1] \times \dots \times [1,n_d]$, then a d-transversal geometry is exactly a d-injection geometry as given in definition 1.4. (Actually the more general case $|X^1_{i_1} \cap \ldots \cap X^d_{i_d}| \leqslant 1$ corresponds to the larger concept of injection geometries as introduced in [DF1]. See also in [D1] some examples of such structures of rank 2 with any $|X^\alpha_i \cap X^\beta_j| \leqslant 1$, one of them corresponding to complete sets of pairwise orthogonal Latin rectangles).

Theorem 2.6 can be generalized as follows : instead of the projection p_α, consider the mapping q_α that associates with every point $x \in X$ the index $i \in [1,n_\alpha]$ such that $x \in X^\alpha_i$. Suppose that g is a d-transversal geometry such that there exists a matroid M^α on $q_\alpha(X)$ such that q_α is an isomorphism between M^α and g_R for every roof R of g. Then g belongs to the diagram $\circ\overset{L}{\text{———}}\circ \ldots \circ\overset{L}{\text{———}}\circ\overset{[d]}{\text{———}}\circ$. It would be interesting to find other (if any) squashed geometries having the same diagram $\circ\overset{L}{\text{———}}\circ \ldots \circ\overset{L}{\text{———}}\circ\overset{[d]}{\text{———}}\circ$, also at least to classify all geometries with diagram $\circ\overset{c}{\text{———}}\circ\overset{[d]}{\text{———}}\circ$ (this classification for d = 1 follows from [Sp2]).

E) Embedding and representability

It would be interesting to study embedding (i.e. mapping into - preserving flats, rank, incidence) of squashed geometries into matroids and, in particular, into projective spaces PG(n,q). This problem of embedding in PG(n,q) was already considered for example for matroids (Kantor, Percsy), semi modular lattices of rank greater or equal than 5 (Percsy), polar spaces (Lefevre - Percsy). The relevant problem of representability of $\mathcal{F}$-squashed geometries over vector spaces can be handled as follows (cf. [DF1]) : Let V be a M-dimensional vector space over a commutative field K and W be a squashed geometry in the family of all subspaces of V. A $\mathcal{F}$-squashed geometry on X is called W-representable if there exists a function from X onto V preserving the rank.

F) Homotopy problems

Recall that a topological space can be attached to any poset by means of the simplicial complex of finite chains (see for example [BjW]). It would be interesting to study the homotopy groups for bouquets of matroids using a discrete analogue of Van Kampen's theorem for bouquets of pointed topological spaces. The Möbius function and characteristic polynomials for bouquets can be easily calculated.

G) Extremal problems and association schemes

The following problems were studied for $\mathcal{F}$-squashed designs : extremal intersection properties of roofs system (cf. [DF1], [DF2]), the case when $\mathcal{F}$ carries an association scheme (cf. [Ne]).

H) d-injection geometries - groups (d ⩾ 2)

An interesting particular case of 2-injection geometries, as mentionned in remark 1-5, is the case of permutation geometries on $[1,n]^2$ whose set of roofs is a subgroup of S_n. More generally, any d-injective subset of $[1,n]^d$ of size n can be seen as an element of $(S_n)^{d-1}$; therefore, one can ask about the existence of d-injection geometries whose set of roofs is a subgroup of $(S_n)^{d-1}$; such a group will be called a d-geometric group. Results for 2-geometric groups are given in [CD], [CDF]; some of them can be easily extended to the general case d ⩾ 2; for instance, any d-injection geometry whose set of roofs is a group is unisupported. A subgroup G of $(S_n)^{d-1}$ is said to be of type (L,n) if $L = \{\vartheta(a) : a \in G, a \neq 1\}$ where $\vartheta(a) = |\{i \in [1,n] : a(i) = i\}|$ for all $a \in (S_n)^{d-1}$. Cameron ([Ca]) proved that there is equivalence between the following assertions (i), (ii) :

(i) the existence of a d-geometric group for some d ⩾ 2

(ii) the existence of a d-geometric group for all d ⩾ 2

It is not true in general that a d-geometric group is the direct product of 2-geometric groups ; however, Cameron made the following conjecture : For large values of d, any d-geometric group is the direct product of a e-geometric group and of a (d-e+1)-geometric group for some integer e.

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