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The Classification of Finite Connected Hypermetric Spaces

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Abstract. A finite distance space X, d $d: X^2 \to \mathbb{Z}$ is hypermetric (of negative type) if $\sum a_x a_y d(x, y) \leq 0$ for all integral sequences $\{a_x | x \in X\}$ that sum to 1 (sum to 0). X, d is connected if the set $\{(x, y) | d(x, y) = 1, x, y \in X\}$ is the edge set for a connected graph on X, and graphical if d is the path length distance for this graph. Then we prove

Theorem 1. *A connected space X, d has negative type if and only if X may be realised as a subset of a Euclidean space E, $\| \ \|$, such that*
(i) *X contains 0 and spans E*
(ii) $d(x, y) = 1/2 \|x - y\|^2$ $(x, y \in X)$
(iii) *$L = \mathbb{Z}X$ is a root lattice, i.e. an orthogonal direct sum of lattices of type A_n, D_n, E_6, E_7, and E_8.*
Call a hypermetric space X, d complete if for each triple $x, y, z \in X$ with $d(y, z) = 1$ and $d(x, y) + 1 = d(x, z)$, there is a unique element $w \in X$ with $d(w, x) = 1$, $d(w, y) = d(x, z)$, and $d(w, z) = d(x, y)$. Then we also prove

Theorem 2. (i) *A connected distance space is hypermetric if and only if it is isomorphic to a subspace of a complete connected hypermetric spcae.* (ii) *The complete connected hypermetric spaces are graphical, and are precisely the Cartesian products of Johnson graphs, half cubes, Cocktail Party graphs, the Schlafli graph on 27 vertices, and the Gosset graph on 56 vertices.*

We finish by describing how a given connected hypermetric space may be canonically embedded in a complete one, and give some open problems.

Theorem 1 is an extention of a result of Schoenberg. Theorem 2 is obtained by applying a result of Assouad to show any connected hypermetric space may be identified with a subset of a minimal saturated set induced by a coset of some root lattice in its lattice of weights.

1. Introduction

A *distance space* X, d consists of a finite set X (of *vertices*) and a symmetric integer valued *distance function* d satisfying $d(x, y) = 0$ if and only if $x = y (x, y \in X)$. Vertices in X at distance 1 are called *adjacent*. X, d is *connected* if no proper nonempty subset Y of X contains all vertices adjacent to some vertex in Y, and in this case X possesses

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a second distance function ∂, the *path length distance*, with

$$\partial(x, y) = \min\{t \mid \exists x = x_0, x_1, \ldots, x_t = y, x_i \in X, d(x_{i-1}, x_i) = 1, 1 \le i \le t,\ x, y \in X\}.$$

In general $d \ne \partial$; if d and ∂ coincide we call X, d *graphical.* To generalize the notion of a metric space, we say X, d is *hypermetric*, or an *h-space*, (is of *negative type*) if

$$\sum_{x,y \in X} a_x a_y d(x, y) \le 0 \text{ for all } \{a_x \mid a_x \in \mathbb{Z}, x \in X\} \text{ with } \sum_{x \in X} a_x = 1 \left(\sum_{x \in X} a_x = 0 \right) \quad (1)$$

H-spaces are metric, but the converse is not true in general (Deza [10]). Spaces of negative type were studied by Schoenberg [15], who used them to characterize subspaces of L^2. Interest in h-spaces centers around their use in characterizing subspaces of L^1 (Assouad [1, 2, 3], Avis [4], Deza [10, 11], Kelly [13]), although they also appear in graph theory (Neumaier [14], Terwilliger [16, 17], Winkler [18]). In this paper we apply the theory of root lattices to characterize connected h-spaces and spaces of negative type. Our main idea for the first characterization is that the vertex set of any connected h-space may be identified with a subset of a so called minimal saturated set induced by a coset of some root lattice in its weight lattice. Our result is then obtained through the well known decomposition of root lattices into orthogonal direct sums of lattices of type A_n, D_n, E_6, E_7, E_8, and the corresponding decomposition of their minimal saturated sets into Cartesian products of known h-spaces. Theorems 1 and 2 below are our main results, which we prove after reviewing root lattices and their associated h-spaces.

Theorem 1. *A connected space X, d has negative type if and only if X may be realised as a subset of a Euclidean space E, $\| \ \|$, such that*

(i) *X contains 0 and spans E* (2)

(ii) $d(x, y) = 1/2 \|x - y\|^2 \quad (x, y \in E)$ (3)

(iii) *$L = \mathbb{Z}X$ is a root lattice, i.e. an orthogonal direct sum of lattices of type A_n, D_n, E_6, E_7, and E_8.* (4)

We note the nonstandard 1/2 in (ii) is convenient to our purposes, but can be eliminated throughout this paper by scaling all vectors by $1/\sqrt{2}$.

A *subspace* of an h-space X, d is a subset of X with the induced distance. Subspaces of h-spaces are again h-spaces, so we consider those that are in a sense maximal or in our terminology *complete*, as defined below. Call a sequence $p = \{x, y, z\}$ of vertices in X *linear* if $d(y, z) = 1$ and $d(x, y) + 1 = d(x, z)$. Call $w \in X$ a *completion* of p if $d(w, x) = 1$, $d(w, y) = d(x, z)$, and $d(w, z) = d(x, y)$. X, d is *complete* if each linear triple possesses a unique completion.

Theorem 2. (i) *A connected distance space is hypermetric if and only if it is isomorphic to a subspace of a complete connected h-space.* (5)

(ii) *The complete connected h-spaces are graphical, and are precisely the Cartesian products of Johnson graphs $J(d, n)$ $(2 \le 2d \le n)$, half cubes $1/2H_n$ $(n \ge 4)$, Cocktail Party graphs CP_n $(n \ge 4)$, the Schlafli graph S on 27 vertices, and the Gosset graph Q on 56 vertices.* (6)

2. Root Lattices and Their Hypermetric Spaces

We now define root lattices and their associated h-spaces. Let $E, \| \ \|$ be a Euclidean space, with inner product $\langle u,v\rangle = 1/2(\|u\|^2 + \|v\|^2 - \|u-v\|^2)$ $(u, v \in E)$. An *integral lattice* L is a subgroup spanning E, where $\langle u,v\rangle \in \mathbb{Z}$ $(u, v\in L)$. We suppress E if its identity is clear. If L equals the set $\mathbb{Z}X$ of all integral linear combinations from X we say X *generates* L. L is *irreducible* if it cannot be expressed as the orthogonal direct sum $H \oplus K$ of nonzero integral lattices H, K. A *root lattice* is any integral lattice L generated by a set of norm $\sqrt{2}$ vectors. $\Phi = \{r | r\in L, \|r\|^2 = 2\}$ is the *root system* of L. The *dual* or *weight* lattice L^* of a root lattice L is the group $\{u | u\in E, \langle u,v\rangle \in \mathbb{Z} \text{ for all } v\in L\}$, which we note contains L as a subgroup. To each coset H of L in L^* we assign a distance space, called a *minimal saturated space* or *ms-space*, consisting of the set $X = \{x | x\in H, \|x\|^2 = t\}$, where $t = \min\{\|x\|^2 | x\in H\}$, and distance $d(x,y) = 1/2\|x-y\|^2$ $(x, y\in X)$. We write $t(X) = t$, $L(X) = L$. It is immediate that

$$\langle x,r\rangle = -1, 0, 1 \quad (x\in X, r\in\Phi), \tag{7}$$

since if the integer $|\langle x,r\rangle|$ exceeds 1 then either $\|x+r\|^2 < t$ or $\|x-r\|^2 < t$. We also note distinct cosets of L may yield isomorphic distance spaces, since the orthogonal transformation $v \to -v$ $(v\in E)$ fixes L and L^* but permutes the cosets. If *ms*-spaces X, $-X$ are distinct we call them *opposites*. Call $X = 0$ the *trivial ms*-space. To prove Theorem 2 we will show

(i) an *ms*-space X, d is a complete graphical h-space (8)
(ii) the *ms*-spaces are precisely the Cartesian products of the spaces listed in (4) (9)
(iii) any connected h-space X, d may be identified with a subspace of an *ms*-space, and if complete, with the whole *ms*-space. (10)

It will be most convenient if we prove (8) now, then review the classification of root lattices and their *ms*-spaces to show (9), then prove Theorem 1, and finish with the proof of (10).

The proof of (8). To show X, d is hypermetric, pick an integral sequence $\{a_x | x\in X\}$ that sums to 1, pick a base vertex $z\in X$, and set $u = \sum_{x\in X} a_x x - z \in L(X)$, $t = t(X)$. Then

$$\sum_{x,y\in X} a_x a_y d(x,y) = 1/2\left(\sum_{x,y\in X} a_x a_y \|x-y\|^2\right) = t - \left\|\sum_{x\in X} a_x x\right\|^2 = t - \|u+z\|^2,$$

which is nonpositive by the construction of X. X, d is complete, since if x, y, $z\in X$ is linear any completion must equal $x - y + z$, which is in X. To show X, d is graphical we fix $x, y\in X$ $(x \neq y)$ and produce some $z\in X$ with $d(x,z) = 1$ and $d(y,z) = d(x,y) - 1$. (This will show X is connected with $\partial \le d$, but then $\partial \ge d$ by the triangle inequality). Now $z = x - r$ is such a vertex if $r\in\Phi$ with $\langle x,r\rangle = -\langle y,r\rangle = 1$, so assume no such r exists. Now write $x - y = r_1 + \cdots + r_e$ $(r_i\in\Phi, 1\le i\le e)$ with e minimal, and note $\langle r_i, r_j\rangle \ge 0$ $(1 \le i,j\le e)$ and (7) imply

$$2e \le \left\|\sum_{i=1}^{e} r_i\right\|^2 = \|x-y\|^2 = \sum_{i=1}^{e} \langle r_i, x-y\rangle \le e,$$

a contradiction. Hence X, d is graphical.

The Classification of Root Lattices and Their *ms*-Spaces

We now describe the irreducible root lattices $L = A_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7, and E_8, with indices $[L^* : L] = n + 1$, 4, 3, 2, and 1, respectively, along with their nontrivial *ms*-spaces. For more information see Bourbaki [6, Appendix] and Humphreys [12, p72]. Let $B^n = \{e_1, \ldots, e_n\}$ be the standard basis for the Euclidean space $\mathbb{R}^n$ and let $\mathbb{Z}^n = \mathbb{Z}B^n$.

(1) $A_n = \{x | x \in \mathbb{Z}^{n+1}, x_1 + \cdots + x_{n+1} = 0\}$ has *ms*-spaces $J(d, n+1)$ $(1 \leq d \leq n)$, called the *Johnson graphs*, with $J(d, n+1)$ consisting of all vectors in $\mathbb{R}^{n+1}$ possessing d coordinates equal $e = (n+1-d)/(n+1)$, and $n+1-d$ coordinates equal $e - 1$. $J(d, n+1)$ and $J(n+1-d, n+1)$ are opposites $(1 \leq d \leq n)$.

(2) The *ms*-spaces for $D_n = \{x | x \in \mathbb{Z}^n, x_1 + \cdots + x_n \text{ even}\}$ are the *Cocktail Party graph* $CP_n = \{\mp e_i | e_i \in B^n, 1 \leq i \leq n\}$, the *half cube* $1/2H_n = \{x | x \in \mathbb{R}^n, x_i = \mp 1/2, 1 \leq i \leq n, x_1 x_2 \ldots x_n > 0\}$, and its opposite.

(3) E_8, consisting of all integral linear combinations of D_8 and the vector $1/2(1,1,1,1,1,1,1,1)$, is self-dual and hence has no nontrivial *ms*-spaces.

(4) $E_7 = \{x | x \in E_8, x_7 + x_8 = 0\}$ has one *ms*-space, called the *Gosset graph Q* (Coxeter [9], Neumaier [14]), consisting of the 56 vectors of the form $1/2(a_1, a_2, \ldots, a_6, 0, 0)(a_i = \mp 1, 1 \leq i \leq 6, a_1 a_2 \ldots a_6 = 1)$, and $\mp e_i \mp 1/2(e_7 - e_8)$ $(1 \leq i \leq 6)$ $(e_j \in B^8, 1 \leq j \leq 8)$.

(5) The two *ms*-spaces for $E_6 = \{x | x \in E_8, x_6 - x_7 = x_7 + x_8 = 0\}$ are opposites, so we describe one. The *Schlafli graph S* (Cameron [8]) has 27 vertices consisting of $2/3(e_6 + e_7 - e_8)$ and all vectors of the form $1/2(a_1, a_2, a_3, a_4, a_5, 1/3, 1/3, -1/3)$ $(a_i = \mp 1, 1 \leq i \leq 5, a_1 a_2 \ldots a_5 = 1)$ and $\mp e_i - 1/3(e_6 + e_7 - e_8)$ $(1 \leq i \leq 5)$ $(e_j \in B^8, 1 \leq j \leq 8)$.

We note the above graphs are distance-transitive (Bannai and Ito [5]). The *ms*-spaces for reducible root lattices are readily constructed from the above examples. If subsets $Y_1, Y_2, \ldots, Y_s$ of E lie in mutually orthogonal subspaces, their *Cartesian Product* $Y_1 \times Y_2 \times \cdots \times Y_s$ is the set $\{y_1 + y_2 + \cdots + y_s | y_i \in Y_i, 1 \leq i \leq s\}$. Now Y is an *ms*-space for $L_1 \oplus L_2 \cdots \oplus L_s$ if and only if $Y = Y_1 \times Y_2 \times \cdots \times Y_s$, with each Y_i an *ms*-space for L_i $(1 \leq i \leq s)$ (Humphreys [12, p63]).

The proof of Theorem 1. First let X, d have negative type. By Schoenberg [15] X may be realised as a subset of a Euclidean space E satisfying (2) and (3). Now the lattice $L = \mathbb{Z}X$ is integral, so it suffices to show L is generated by the set of norm $\sqrt{2}$ vectors

$$F(X) = \{x - y | d(x, y) = 1, x, y \in X\} \tag{11}$$

But L is generated by X, and if $x \in X$, by connectivity there is a sequence $0 = x_0$, $x_1, \ldots, x_s = x$ $(x_i \in X, d(x_{i-1}, x_i) = 1, 1 \leq i \leq s)$, so $x = x_1 - x_0 + x_2 - x_1 + \cdots + x_s - x_{s-1}$ is a sum of elements in $F(X)$. Thus (4) holds. The converse is immediate from Schoenberg [15].

The proof of (10). Let X, d be a connected h-space. Then by Deza [10] X, d has negative type, so we view X as a subset of a Euclidean space $E, \| \ \|$, satisfying (2)–(4). By Assouad [1, Lemmas 1.12, 1.13] there is a vector $a \in E$ with $t = \|a - x\|^2$ independant of $x \in X$, where $\|a - v\|^2 \geq t$ for all $v \in L = \mathbb{Z}X$. Now $a \in L^*$, for if $x \in X$ then $\langle a, x\rangle = 1/2(\|a - 0\|^2 - \|a - x\|^2 + \|x - 0\|^2) = t - t + d(x, 0)$, an integer. But now the translation $X - a$ is contained in the *ms*-space for the coset $-a + L$ of L^*.

Now assume X, d is a complete connected h-space, presented as a subspace of an *ms*-space Y for a root lattice L. Let $F = F(X)$ be as in (11), which we may assume generates L. We assume $X \neq Y$ and get a contradiction. The completeness of X implies

$$x - \langle x, r\rangle r \in X \quad \text{for all } x \in X \text{ and } r \in F. \tag{12}$$

For each pair x, y with $x \in X$ and $y \in Y \backslash X$, fix an integral sequence $\{a_r | r \in F, a_r \geq 0, a_r = a_r(x, y)\}$ with $s(x, y) = \sum_{r \in F} a_r$ minimal subject to $x - y = \sum_{r \in F} a_r r$. Pick a pair x, y with $s(x, y)$ minimal. Now $\langle x, r\rangle \leq 0$ for all $r \in F$ with $a_r > 0$, otherwise $x - r \in X$ by (12) and $s(x - r, y) < s(x, y)$. But then $x \neq y$ implies $0 < \|x - y\|^2 = 2\langle x, x - y\rangle \leq 0$, a contradiction. Hence $X = Y$.

We finish by describing how a given connected h-space X, d may be canonically embedded in a complete connected h-space. The construction is based on the observation that if w is the completion of a linear triple x, y, $z \in X$, then

$$d(p, x) - d(p, y) + d(p, z) - d(p, w) = 0 \quad (p \in X). \tag{13}$$

This can be seen by applying (1) twice, with $(a_x, a_y, a_z, a_w, a_p) = (1, -1, 1, -1, 1)$, and $(-1, 1, -1, 1, 1)$. If X, d is not complete, pick any linear triple x, y, $z \in X$ with no completion. Now enlarge X by adding the missing completion w, extending the distance function so that $d(w, p)$ $(p \in X)$ satisfies (13). The resulting (connected) space is still hypermetric by (5), and if it is not complete, we repeat the process, obtaining a complete space X^*, d^* in a finite number of steps. That the process does end is also a consequence of (5). We call X^*, d^* the *completion* of X, d.

Open Problems. (1) Call a connected h-space X, d *minimal* if the completion of Y is properly contained in the completion of X for any connected subspace Y properly contained in X. What are the minimal connected h-spaces?
(2) Call a connected distance space X, d *minimal forbidden* if it is not hypermetric, but all its proper connected subspaces are. Find the minimal forbidden connected distance spaces.

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