

## METRIC TRANSFORMS AND EUCLIDEAN EMBEDDINGS

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**ABSTRACT.** It is proved that if  $0 \leq c \leq 0.72/n$  then for any  $n$ -point metric space  $(X, d)$ , the metric space  $(X, d^c)$  is isometrically embeddable into a Euclidean space. For 6-point metric space,  $c = \frac{1}{2} \log_2 \frac{3}{2}$  is the largest exponent that guarantees the existence of isometric embeddings into a Euclidean space. Such largest exponent is also determined for all  $n$ -point graphs with "truncated distance".

### 1. INTRODUCTION

One of the fundamental problems in distance geometry is the isometric embedding problem, that is, to determine conditions for metric spaces to be isometrically embeddable in a given class of spaces, say  $l_p$ -spaces (e.g. [1–5, 8, 11, 12, 18, 23]). For finite metric spaces, there also arises the minimum dimensional embedding problem, that is, to decide the minimum dimension  $m = m_p(n)$  such that any  $n$ -point subset of an  $l_p$ -space can be isometrically embedded in  $l_p^m$ . Except for the case  $p = 2$  (in this case,  $m_2(n) = n - 1$ ), the problem is not trivial. For some bounds on  $m_p(n)$ ,  $p \neq 2$ , see e.g. Ball [6], Witsenhausen [24].

In this paper, we consider deforming the distance function of a metric space so that the resulting space can be embedded in a certain space, say  $l_2$ . This problem has its origin in the 1930s (e.g. [7, 16, 19, 22]).

Throughout this paper,  $X$  stands for a finite metric space and  $X_n$  stands for an  $n$ -point metric space. The distance between two points  $x, y$  of a metric space is denoted by the juxtaposition  $xy$ . If  $X$  is isometric to a subset of a Euclidean space, then  $X$  is said to be *Euclidean*. Let  $F(t)$  be a continuous, monotone increasing, concave function of  $t \geq 0$  with  $F(0) = 0$ . Then, replacing the distance  $xy$  in  $X$  by  $F(xy)$ , we have another metric space with the same point set as  $X$ . This new metric space is called the *metric transform* of  $X$  by  $F(t)$ , and it is denoted by the symbol  $F(X)$ .

Blumenthal [8, p. 131] proved that if  $0 < c \leq \frac{1}{2}$  then the metric transform  $(X_4)^c$  by  $F(t) = t^c$  is Euclidean. Further, he showed that  $c = \frac{1}{2}$  is the largest exponent that guarantees  $(X_4)^c$  to be Euclidean for all  $X_4$ . We are going to

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consider similar problems for  $X_n$ ,  $n > 4$ , in connection with “hypermetric” spaces (e.g. [3, 11, 13]).

We will prove that if  $0 < c \leq 0.7213\dots/n$ , then  $(X_n)^c$  is always Euclidean. In particular,  $c = \frac{1}{2} \log_2 \frac{3}{2} = 0.2924\dots$  is the largest exponent that guarantees  $(X_6)^c$  to be Euclidean.

The vertices of a connected graph  $G$  constitute a metric space with the shortest path distance. We denote this metric space by the same letter  $G$ . Winkler [23] gave a characterization of the connected graphs  $G$  whose metric transforms  $(G)^{1/2}$  are Euclidean.

For a graph  $G$ , define the *truncated distance* between two vertices  $x, y$  of  $G$  by

$$xy = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 2 & \text{otherwise.} \end{cases}$$

The metric space consisting of the vertices of  $G$  with truncated distance is denoted by  $G_T$ . If  $G$  is a graph with diameter  $\leq 2$ , then  $G$  and  $G_T$  are the same space. Isometric embeddings of  $G_T$  into Hamming hypercubes and  $l_1$  were considered in [2, 3].

We will prove that  $c_n$ , the largest  $c$  such that  $(G_T)^c$  are Euclidean for all  $n$ -point graphs  $G$ , is given by

$$c_n = \begin{cases} \frac{1}{2} \log_2(k/(k-1)) & \text{if } n = 2k, \\ \frac{1}{2} \log_2(2k(k+1)/(2k^2-1)) & \text{if } n = 2k+1. \end{cases}$$

## 2. PRELIMINARIES

A space  $X$  is said to be *hypermetric* if

$$\sum_{x \in X} a_x = 1 \Rightarrow \sum_{x, y \in X} a_x a_y (xy) \leq 0,$$

where  $a_x$  are integers. A space  $X$  is of *negative type* if the above holds with 1 replaced by 0. Schoenberg [18] proved that  $(X)^{1/2}$  is Euclidean if and only if  $X$  is of negative type. This characterization goes back to Cayley, but was first stated in this form by Schoenberg in the 1930s. A space  $X$  is said to be  $l_1$ -embeddable if there is an isometric embedding of  $X$  into  $R^n$  with the  $l_1$ -metric, i.e. the metric

$$d(\bar{x}, \bar{y}) = |x_1 - y_1| + \dots + |x_n - y_n|.$$

A metric space is called *ultrametric* (see e.g. [1]) if it satisfies  $xz \leq \max(xy, yz)$  for any three points  $x, y, z$ . In Maehara [15], an ultrametric space  $X$  is called an *irreducible space*, because any metric transform of  $X$  requires  $(|X| - 1)$ -dimensional Euclidean space for an isometric embedding. In particular, [15, Theorem 7] implies that ultrametric spaces are Euclidean (see also [1, Theorem 6.7]).

There is the following hierarchy in finite metric spaces (see, e.g. [2–4, 10–13, 15, 17, 23]):

- $X$  is ultrametric
- ⇒  $X$  is Euclidean
- ⇒  $X$  is  $l_1$ -embeddable
- ⇒  $X$  is hypermetric
- ⇒  $X$  is of negative type
- ⇒ The distance matrix of  $X$  has only one positive eigenvalue.

The conditions for hypermetricity and for negative type can be described by the polygonal inequalities. A space  $X$  is said to be  $(2N + \varepsilon)$ -gonal if the  $(2N + \varepsilon)$ -gonal inequality

$$(1) \quad \sum_{i < j} x_i x_j + \sum_{i < j} y_i y_j \leq \sum_{i, j} x_i y_j$$

holds for any  $(2N + \varepsilon)$ -sequence of points of  $X$ ,

$$(2) \quad x_1, \dots, x_N, y_1, \dots, y_{N+\varepsilon},$$

where  $\varepsilon = 0$  or  $1$ . These  $2N + \varepsilon$  points are not necessarily different. The 3-gonal inequality is just the triangle inequality. Then a space  $X$  is hypermetric if and only if it is  $(2N + 1)$ -gonal for all  $N \geq 1$ , while  $X$  is of negative type if and only if it is  $2N$ -gonal for all  $N \geq 1$  (e.g. [3]).

In this paper the function  $\log_2(1 + 1/s)$  appears frequently, so we denote this function by  $\gamma$ :

$$\gamma(s) = \log_2(1 + 1/s).$$

First we prove two lemmas.

**Lemma 1.** *Suppose that  $0 < c \leq \gamma(s)$ ,  $s > 1$ . Then, in the metric transform  $(X)^c$ ,  $xy \leq yz$  implies that*

$$xz \leq (xy)/s + yz.$$

*Proof.* Denote the distance in the space  $X$  by  $\overline{xy}$ . Then  $xy = (\overline{xy})^c$  by definition. Since  $\overline{xy} + \overline{yz} \geq \overline{xz}$ , it is enough to show that

$$(\overline{xy})^c/s + (\overline{yz})^c - (\overline{xy} + \overline{yz})^c \geq 0.$$

Dividing the left-hand side by  $(\overline{xy})^c$  and letting  $t = (\overline{yz})/(\overline{xy})$ , we have

$$f(t) := 1/s + t^c - (1 + t)^c.$$

Thus it remains only to show that  $f(t) \geq 0$  for  $t \geq 1$ . Now, since the condition  $0 < c \leq \gamma(s)$  implies that  $(1 + 1/s) \geq 2^c$ , we have  $f(1) \geq 0$ . And since  $c < 1$ , we have

$$f'(t) = ct^{c-1} - c(1 + t)^{c-1} > 0 \quad \text{for } t \geq 1.$$

Therefore  $f(t) \geq 0$  for  $t \geq 1$ . □

Let (2) be a  $(2N + 1)$ -sequence of points of  $(X)^c$ . By changing suffixes if necessary, we may assume that

$$(3) \quad x_k y_k \leq x_i y_j \quad \text{for } i \geq k, \quad j \geq k, \quad k = 1, \dots, N$$

holds in the sequence (2). If  $i < k$  then by (3) and Lemma 1,

$$\begin{aligned} x_i x_k &\leq (x_i y_i)/s + x_k y_i \leq (x_k y_k)/s + x_k y_i, \\ y_i y_k &\leq (x_i y_i)/s + x_i y_k \leq (x_k y_k)/s + x_i y_k. \end{aligned}$$

Hence we have the following lemma.

**Lemma 2.** *Suppose that  $0 < c \leq \gamma(s)$  and  $k \leq N$ . If  $i < k < j$ , then*

$$\begin{aligned} x_i x_k &\leq (x_k y_k)/s + x_k y_i, \\ x_j x_k &\leq (x_k y_k)/s + x_j y_k, \\ y_i y_k &\leq (x_k y_k)/s + x_i y_k, \\ y_j y_k &\leq (x_k y_k)/s + x_k y_j. \end{aligned}$$

Therefore, for any  $i, k \leq N$ ,

$$(4) \quad x_i x_k + y_i y_k \leq 2(x_k y_k)/s + x_i y_k + x_k y_i. \quad \square$$

### 3. THE POLYGONAL INEQUALITIES

**Theorem 1.** *If  $0 < c \leq \gamma(N)$ , then the metric transform  $(X)^c$  is  $M$ -gonal for all  $M \leq 2N + 2$ , and if  $c > \gamma(N)$  then there is a space  $X$  such that  $(X)^c$  is not  $(2N + 2)$ -gonal.*

*Proof.* Suppose that  $0 < c \leq \gamma(N)$ . It is known that if a metric space is  $(2N + 1)$ -gonal, then it is  $M$ -gonal for all  $M \leq 2N + 2$  (see e.g. [11]). Hence we show that  $(X)^c$  is  $(2N + 1)$ -gonal.

Let (2) be a  $(2N + 1)$ -sequence taken from  $(X)^c$ . We may suppose that (3) holds for this sequence. Fix  $k < N + 1$  and sum (4) over  $i \neq k$  to obtain

$$\begin{aligned} \sum_{i=1}^N (x_i x_k + y_i y_k) &\leq \sum_{i=1}^N (x_i y_k + x_k y_i) \\ &\quad - 2(x_k y_k) + 2(N - 1)(x_k y_k)/N. \end{aligned}$$

Summing over  $k < N + 1$  gives

$$\sum_{k=1}^N \sum_{i=1}^N (x_i x_k + y_i y_k) \leq \sum_{k=1}^N \sum_{i=1}^N (x_i y_k + x_k y_i) - 2 \sum_{k=1}^N (x_k y_k)/N.$$

Dividing this inequality by 2, and adding the inequality

$$\sum_{k=1}^N y_k y_{N+1} \leq \sum_{k=1}^N \{ (x_k y_k)/N + x_k y_{N+1} \},$$

we have

$$\sum_{i < j} x_i x_j + \sum_{i < j} y_i y_j \leq \sum_{i=1}^N \sum_{k=1}^N x_i y_k + \sum_{k=1}^N x_k y_{N+1} = \sum_{i,j} x_i y_j.$$

This is (1), proving the first part of the theorem. For the second part, consider the complete bipartite graph  $K(N+1, N+1)$  with vertex classes  $\{x_1, \dots, x_{N+1}\}$  and  $\{y_1, \dots, y_{N+1}\}$ . Then all the vertices of  $K(N+1, N+1)$  constitute a metric space  $X$  by the usual graph distance. Suppose  $c > \gamma(N) = \log_2(1 + 1/N)$ . Then  $2^c > (N + 1)/N$ . Hence, in  $(X)^c$ ,

$$\sum_{i < j} x_i x_j + \sum_{i < j} y_i y_j = N(N + 1)2^c > (N + 1)^2;$$

on the other hand

$$\sum_{i,j} x_i y_j = (N + 1)^2.$$

Thus the  $(2N + 2)$ -gonal inequality fails in  $(X)^c$ .  $\square$

Deza [11] proved that  $X_5$  is  $l_1$ -embeddable if and only if it is pentagonal (=5-gonal). Hence we have the following corollary.

**Corollary 1.** *The metric transform  $(X_5)^c$  is  $l_1$ -embeddable for  $0 < c \leq \gamma(2)$ , and  $\gamma(2)$  is the largest exponent that guarantees  $(X_5)^c$  to be  $l_1$ -embeddable for all  $X_5$ .  $\square$*

#### 4. REPEATING NUMBERS

In a sequence of points, some points may appear repeatedly. Define the maximum *repeating number* of a sequence as the maximum value of the number of times a point appears in that sequence. For example, the maximum repeating number of the sequence

$$a, b, c, a, b, b, c, b, c$$

is 4 provided that  $a, b, c$  are all different.

**Lemma 3.** *Suppose that  $0 < c \leq \gamma(s)$  and  $(X)^c$  is  $(2N - 1)$ -gonal. Then, in  $(X)^c$ , the  $(2N + 1)$ -gonal inequality holds for any  $(2N + 1)$ -sequence with maximum repeating number at least  $2N/(s + 1)$ .*

*Proof.* Suppose that (2) is a  $(2N + 1)$ -sequence with maximum repeating number  $r \geq 2N/(s + 1)$ .

First, we consider the case  $\{x_1, \dots, x_N\} \cap \{y_1, \dots, y_{N+1}\} \neq \emptyset$ . Suppose, say,  $x_2 = y_3$ . Then since the  $(2N - 1)$ -gonal inequality is valid for

$$x_1, x_3, x_4, \dots, x_N, y_1, y_2, y_4, \dots, y_{N+1}$$

and since

$$\sum_{i \neq 2} x_i x_2 + \sum_{j \neq 3} y_j y_3 = \sum_{i \neq 2} x_i y_3 + \sum_{j \neq 3} x_2 y_j,$$

the  $(2N + 1)$ -gonal inequality (1) is valid for the  $(2N + 1)$ -sequence (2).

Now, suppose that  $\{x_1, \dots, x_N\} \cap \{y_1, \dots, y_{N+1}\} = \emptyset$ . Changing suffixes of  $x_i, y_j$  if necessary, we may assume that (3) holds in the sequence (2). Let  $k$  be the smallest suffix  $i$  such that the maximum repeating number  $r$  equals the repeating number of  $z_i$ , where  $z_i = x_i$  or  $y_i$ . In the following we assume that  $z_k = x_k$ . (The case  $z_k = y_k$  is similar, and is omitted.) Since  $(2N - 1)$ -gonal inequality is valid in  $(X)^c$ , we can apply it to the sequence left when  $x_k$  and  $y_k$  are removed, to wit

$$\sum_{\substack{i < j \\ i, j \neq k}} x_i x_j + \sum_{\substack{i < j \\ i, j \neq k}} y_i y_j \leq \sum_{i, j \neq k} x_i y_j.$$

Hence, in order to prove the  $(2N + 1)$ -gonal inequality (1) for the sequence (2), it is enough to show that

$$(5) \quad \sum_{i \in I+J} x_i x_k + \sum_{j \neq k} y_j y_k \leq \sum_i x_i y_k + \sum_{j \neq k} x_k y_j,$$

where

$$I = \{i : i < k, x_i \neq x_k\}, \quad J = \{i : i > k, x_i \neq x_k\}.$$

Note that since  $\{x_1, \dots, x_N\} \cap \{y_1, \dots, y_{N+1}\} = \emptyset$ , it follows that  $|I| + |J| = N - r$ . Now applying Lemma 2, we have

$$\begin{aligned} \sum_{i \in I+J} x_i x_k + \sum_{j=1}^{N+1} y_j y_k &\leq \sum_{i \in I} \{(x_k y_k)/s + x_k y_i\} + \sum_{i \in J} \{(x_k y_k)/s + x_i y_k\} \\ &\quad + \sum_{j=1}^{k-1} \{(x_k y_k)/s + x_j y_k\} + \sum_{j=k+1}^{N+1} \{(x_k y_k)/s + x_k y_j\}. \end{aligned}$$

Since  $k$  is the minimum value of  $i$  such that  $x_i$  appears  $r$  times in the sequence (2), it follows that

$$I = \{1, 2, \dots, k - 1\}.$$

Hence the right-hand side of the above inequality is

$$\begin{aligned} &(|I| + |J| + N)(x_k y_k)/s + \sum_{i \in I+J} x_i y_k + \sum_{j \neq k} x_k y_j \\ &= (2N - r)(x_k y_k)/s + \sum_{i \in I+J} x_i y_k + \sum_{j \neq k} x_k y_j. \end{aligned}$$

Since

$$\sum_i x_i y_k = r(x_k y_k) + \sum_{i \in I+J} x_i y_k,$$

inequality (5) follows if we show  $(2N - r)/s \leq r$ . But this inequality is equivalent to the assumption  $r \geq 2N/(s + 1)$ .  $\square$

**Theorem 2.** *The metric transform  $(X_{2n})^c$  is  $(2n + 1)$ -gonal for  $0 < c \leq \gamma(n - 1)$ .*

*Proof.* Suppose  $0 < c \leq \gamma(n - 1)$ . Then by Theorem 1,  $(X_{2n})^c$  is  $(2n - 1)$ -gonal. To prove that  $(X_{2n})^c$  is  $(2n + 1)$ -gonal, let  $s = n - 1$ ,  $N = n$  in Lemma

3. Since the maximum repeating number of any  $(2n + 1)$ -sequence from  $(X_{2n})^c$  is at least  $\lceil (2n + 1)/(2n) \rceil = 2 = 2n/(s + 1)$ , the  $(2n + 1)$ -gonal inequality holds in  $(X_{2n})^c$  by Lemma 3.  $\square$

Thus  $(X_6)^c$  is heptagonal (7-gonal) for  $0 < c \leq \gamma(2)$ .

Recently, Avis [5] proved that  $X_6$  is  $l_1$ -embeddable if and only if it is heptagonal. Hence we have the following corollary.

**Corollary 2.** *The metric transform  $(X_6)^c$  is  $l_1$ -embeddable for  $0 < c \leq \gamma(2)$ , and  $\gamma(2)$  is the largest exponent that guarantees  $(X_6)^c$  to be  $l_1$ -embeddable for all  $X_6$ .  $\square$*

**Theorem 3.** *The metric transform  $(X_n)^c$  is hypermetric for  $0 < c \leq \gamma(n - 1)$ .*

*Proof.* We show that  $(X_n)^c$  is  $(2N + 1)$ -gonal for all  $N \geq 1$ . This is done by induction on  $N$ . Since  $(X_n)^c$  is a metric space, it is 3-gonal by definition. Suppose that  $(X_n)^c$  is  $(2N - 1)$ -gonal,  $N \geq 2$ . Then since the maximum repeating number of any  $(2N + 1)$ -sequence from  $(X_n)^c$  is at least  $(2N + 1)/n$ , the  $(2N + 1)$ -gonal inequality holds in  $(X_n)^c$  by Lemma 3.  $\square$

### 5. EUCLIDEAN EMBEDDINGS

Since a hypermetric space is of negative type, its “square root” is Euclidean. Hence, from Theorem 3, we have the following.

**Corollary 3.** *The metric transform  $(X_n)^c$  is Euclidean for  $0 < c \leq \gamma(n - 1)/2$ .  $\square$*

Let  $e(n)$  be the supremum of  $c$  such that  $(X_n)^c$  is Euclidean for every  $X_n$ . Since any  $n$ -point metric space can be isometrically embedded into an  $(n + 1)$ -point metric space, we have  $e(n + 1) \leq e(n)$ . By Blumenthal,

$$e(4) = 1/2 = \gamma(1)/2.$$

Using the inequality

$$\log_e(1 + 1/s) > 1/s - (1/s)^2/2 > 1/(s + 1),$$

we have

$$\frac{1}{2} \log_2(1 + 1/(n - 1)) > \frac{1}{2}(\log_2 e)(1/n) = 0.7213\dots/n.$$

Thus

$$e(n) \geq 0.7213\dots/n.$$

By Corollary 1,  $e(5) \geq \gamma(2)/2$ . But this is the exact value of  $e(6)$ .

**Theorem 4.**  $e(6) = \gamma(2)/2 = 0.2924\dots$

*Proof.* By Corollary 2,  $(X_6)^c$  is  $l_1$ -embeddable for  $0 < c \leq \gamma(2)$ . Hence  $(X_6)^c$  is of negative type for that range of  $c$ , and hence, applying the result of Schoenberg, we have  $e(6) \geq \gamma(2)/2$ . On the other hand, if  $c > \gamma(2)$  then, by Theorem 1, there exists a metric space  $X_6$  whose metric transform  $(X_6)^c$  is not hexagonal. Therefore,  $e(6) \leq \gamma(2)/2$ .  $\square$

TABLE 1. Values of  $e(n)$  for small  $n$

$n$	2	3	4	5	6	7
$e(n)$	$\infty$	1	1/2	?	$\gamma(2)/2$	?

**Conjecture.**

$$e(2n) = \gamma(n - 1)/2,$$

$$e(2n + 1) = \gamma(2n(n + 1)/(2n + 1) - 1)/2.$$

These are true for the known values of  $e(n)$ . Further, as will be seen in the next section (Theorem 6), if we restrict ourselves to graphs with “truncated distance”, the conjecture is also true.

Denote by  $i(n)$  the sup of  $c$  such that  $(X_n)^c$  has property (i) for every  $X_n$ . Then it may be also an interesting problem to determine  $i(n)$  for such properties as:

- (a)  $\leq n$ -gonal,
- (b) hypermetric,
- (c) negative type,
- (d)  $l_1$ -embeddable.

6. GRAPHS WITH TRUNCATED DISTANCE

Here we consider the following problem: Given a graph  $G$ , up to what value of  $c$  is the metric transform  $(G_T)^c$  Euclidean?

Let  $\phi(G; x)$  be the characteristic polynomial of a graph  $G$ , that is  $\phi(G; x) = \det(xI - A(G))$ , where  $A(G)$  is the adjacency matrix of  $G$ . Define the polynomial  $P(G; x)$  by

$$P(G; x) = \phi(G; -x) - (-1)^{|G|} \phi(\overline{G}; x - 1),$$

where  $\overline{G}$  denote the complement of the graph  $G$ . Then the following result was proved in Maehara [14]:

A graph  $G$  can be embedded in Euclidean space in such a way that adjacent vertices have distance 1 and nonadjacent vertices have distance  $t > 1$  if and only if

$$(6) \quad 1 - 1/t^2 \leq 1/z_{\max},$$

where  $z_{\max}$  is the maximum root of the polynomial  $P(G; x)$ .

From this result the next follows easily.

**Theorem 5.** For a graph  $G$ , the metric transform  $(G_T)^c$  is Euclidean if and only if  $c \leq \frac{1}{2}\gamma(z_{\max} - 1)$ .

*Proof.* The inequality (6) is equivalent to

$$t \leq (1 + 1/(z_{\max} - 1))^{1/2}.$$

Since the truncated distances between nonadjacent vertices of  $G$  are 2, letting  $t = 2^c$  and applying the above result, we have that  $(G_T)^c$  is Euclidean if and only if

$$c \leq \frac{1}{2} \log_2(1 + 1/(z_{\max} - 1)) = \gamma(z_{\max} - 1)/2. \quad \square$$

**Corollary 4.** *For a graph  $G$ , the truncated metric space  $G_T$  is of negative type if and only if  $z_{\max} \leq 2$ .  $\square$*

If  $G$  is a regular graph of degree  $d$ , then the polynomial  $P(G)$  is given by

$$P(G; x) = (|G|/(x + d))\phi(G; -x)$$

(see [14]). Hence, for a regular graph  $G$ ,

$$z_{\max} = -(\text{minimum eigenvalue of } G).$$

**Corollary 5.** *For a regular graph  $G$ ,  $G_T$  is of negative type if and only if the minimum eigenvalue of  $G$  is  $\geq -2$ .  $\square$*

The graphs with least eigenvalue  $\geq -2$  were characterized by Cameron, Goethals, Seidel and Shult [9] in terms of root systems. In the same terms, Terwillinger and Deza [21] gave a characterization of more general class of connected finite distance spaces of negative type [21, Theorem 1] and of hypermetric spaces between them [21, Theorem 2].

**Example 1.** Let  $G$  be the Petersen graph. Then

$$\phi(G; x) = (x - 3)(x - 1)^5(x + 2)^4$$

(see e.g. [20]), and hence  $G_T$  is of negative type. Further, since the diameter of  $G$  is 2, we have  $G = G_T$ , and hence the Petersen graph is of negative type.

**Example 2.** Let  $G = K(m, n)$ , the complete bipartite graph. Then

$$\begin{aligned} \phi(G; x) &= (x^2 - mn)x^{m+n-2}, \\ \phi(\overline{G}; x) &= (x - m + 1)(x + 1)^{m-1}(x - n + 1)(x + 1)^{n-1} \end{aligned}$$

(see e.g. [20]), and hence

$$P(G; x) = (-x)^{m+n-2}\{(m + n)x - 2mn\}.$$

Therefore,  $z_{\max} = 2mn/(m + n)$ . Thus  $(G)^c = (G_T)^c$  is Euclidean if and only if  $c \leq \gamma(2mn/(m + n) - 1)/2$ .

**Theorem 6.** *The sup  $c_n$  of  $c$  such that  $(G_T)^c$  are Euclidean for all  $n$ -point graphs  $G$  is given by*

$$(7) \quad c_n = \begin{cases} \gamma(k - 1)/2 & \text{if } n = 2k, \\ \gamma(2k(k + 1)/(2k + 1) - 1)/2 & \text{if } n = 2k + 1. \end{cases}$$

*Proof.* Consider the  $2N$ -gonal inequality (1) in the metric transform  $(G_T)^c$ . The left-hand side of (1) does not decrease when the nonzero terms are replaced by  $2^c$ 's, and the right-hand side of (1) does not increase when the nonzero

terms are replaced by 1's. Hence the worst case (i.e. the most difficult case for  $2N$ -gonal inequality to hold) will be attained by a complete bipartite graph. However, by Example 2, it will be easy to see that the sup of  $c$  for  $n$ -point complete bipartite graphs is given by (7). Hence the theorem follows.  $\square$

Winkler [23] proved that the graph  $G = K_2 + \overline{K}_n$  (= the complement of the disjoint union of  $\overline{K}_2$  and  $K_n$ ) is not of negative type for  $n > 4$  (though  $G$  has only one positive eigenvalue). Applying Corollary 4, let us present a shorter proof of this result.

**Example 3** (cf. Winkler [23]). Let  $G = K_2 + \overline{K}_n$ . Applying Cvetkovic's theorem [10, p. 57], we have

$$\phi(K_2 + \overline{K}_n; x) = x^{n-1}(x^3 - (2n + 1)x - 2n),$$

and hence

$$P(G; x) = (-x)^{n-1}\{-(n + 2)x^2 + 2(2n + 1)x - 3n\}.$$

Hence the maximum root is  $z_{\max} = 3n/(n+2)$ . Thus, if  $n > 4$ , then  $z_{\max} > 2$ , and hence  $G$  (=  $G_T$ ) is not of negative type.

The distance matrix  $D$  of  $G$  is

$$\left. \begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ \hline 1 & 1 & 0 & 2 & 2 & \cdots & 2 \\ 1 & 1 & 2 & 0 & 2 & \cdots & 2 \\ \cdot & & & & \cdots & & \\ \cdot & & & & \cdots & & \\ 1 & 1 & 2 & 2 & \cdots & 2 & 0 \end{array} \right\} n$$

Using the spectral resolution

$$\begin{pmatrix} -x & k & k & \cdots & k \\ k & -x & k & \cdots & k \\ \cdots & & & & \\ \cdots & & & & \\ k & k & \cdots & k & -x \end{pmatrix} = (kn - k - x)(1/n)J - (x + k)(I - (1/n)J)$$

(where  $J$  is the matrix with all entries 1 and  $I$  is the identity matrix), and the formula

$$\det \left( \begin{array}{c|c} M & N \\ \hline P & Q \end{array} \right) = \det(M) \det(Q - PM^{-1}N),$$

the characteristic polynomial of  $D$  is calculated as

$$(-1)^n(x + 1)(x + 2)^{n-1}(x^2 - (2n - 1)x - 2).$$

Thus  $G$  has one positive eigenvalue but is not of negative type if  $n > 4$ .

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