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Clin d'oeil on L_1-embeddable planar graphs

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Abstract

In this note we present some properties of L_1-embeddable planar graphs. We present a characterization of graphs isometrically embeddable into half-cubes. This result implies that every planar L_1-graph G has a scale 2 embedding into a hypercube. Further, under some additional conditions we show that for a simple circuit C of a planar L_1-graph G the subgraph H of G bounded by C is also L_1-embeddable. In many important cases, the length of C is the dimension of the smallest cube in which H has a scale embedding. Using these facts we establish the L_1-embeddability of a list of planar graphs.

1. Introduction

Graphs with their shortest-path metrics are particular instances of discrete metric spaces, and may be investigated from the metric point of view. The L_1-embeddability question for metric spaces leads to a characterization of L_1-graphs [28, 9]. A particular class of L_1-graphs, possessing special features and applications [10, 13, 24], is formed by planar L_1-embeddable graphs. It is the purpose of this note to present some properties of this class of graphs, which can be applied for testing whether a given planar graph is L_1-embeddable or not. For other results on L_1-embeddable planar graphs we refer to [1, 2, 10, 14–16, 26].

An *l_1-metric d* on a finite set X is any positive linear combination of cut metrics

$$d=\sum_{C\in\mathcal{C}} \lambda_C \cdot \delta_C,$$

where the "cut" metric δ_C associated with the cut (i.e., bipartition) $C=\{A,B\}$ of X is defined as follows:

$$\delta_C(x,y)=\delta_{\{A,B\}}=\begin{cases}0 & \text{if } x,y\in A \text{ or } x,y\in B,\\ 1 & \text{otherwise.}\end{cases}$$

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More generally, a metric space (X,d) is said to be L_1-*embeddable* if there is a measurable space $(\Omega,\mathscr{A})$, a nonnegative measure μ on it and an application λ of X into the set of measurable functions F (i.e. with $\|f\|_1=\int_\Omega |f(w)|\mu(dw)<\infty$) such that

$$d(x,y)=\|\lambda(x)-\lambda(y)\|_1$$

for all $x,y\in X$ [13]. The path-metric of the infinite rooted binary tree is perhaps the simplest L_1-embeddable metric which cannot be embedded into an l_1-space of finite dimension. On the other hand, a well–known compactness result of [6] implies that L_1-embeddability of a metric space is equivalent to l_1-embeddability of its finite subspaces.

Let $G=(V,E)$ be a connected (not necessarily finite) graph endowed with the distance $d_G(u,v)$ that is equal to the length of a shortest path joining the vertices u and v. Given two connected graphs G and H and a positive integer λ, we say that G is *scale* λ *embeddable* into H if there exists a mapping

$$\phi: V(G)\to V(H)$$

such that

$$d_H(\phi(u),\phi(v))=\lambda d_G(u,v)$$

for all vertices $u,v\in V(G)$ [1, 2]. In the particular case $\lambda=1$ we obtain the usual notion of *isometric embedding.* In what follows we consider scale or isometric embeddings of graphs into hypercubes, half-cubes, cocktail-party graphs and their Cartesian products. The *half-cube* $\frac{1}{2}H_n$ is the graph whose vertex set is the collection of all vertices in one part of the bipartite representation of the n-cube H_n and two vertices are adjacent in $\frac{1}{2}H_n$ if and only if they are at distance 2 in H_n. Recall also that the *cocktail-party graph* $K_{m\times 2}$ is the complete multipartite graph with m parts, each of size 2. Both notions can be extended in an evident fashion to infinite graphs, too (it suffices to let n and m be cardinal numbers). Finally, note that a 2-fold n-way covering of a polyhedron due to Pedersen [25] corresponds to a scale 2 embedding of its skeleton into a half-cube H_n.

According to [2] *the L_1-embeddable graphs are exactly those graphs which admit a scale embedding into a hypercube.* Evidently, every graph scale 2 embeddable into a hypercube, is an isometric subgraph of a half-cube. This analogy is much deeper: according to [28, 9] *a graph G is an L_1-graph if and only if it is an isometric subgraph of the* (*weak*) *Cartesian product of half-cubes and cocktail-party graphs.*

In Section 2 we characterize all L_1-graphs which are scale 2 embeddable into hypercubes. As a consequence we obtain that every planar L_1-graph is scale 2 embeddable into a hypercube. For planar graphs in Section 3 we introduce the notion of alternating cut and show that under some natural conditions the collection of alternating cuts of a given planar graph G defines an isometric embedding of G into a half-cube. In particular, we deduce that every outerplanar graph is an L_1-graph. Finally, in Section 4 we establish an efficient formula for computing the Wiener index of an L_1-graph, which extends a similar result of [24] for benzenoid systems.

2. Characterization of isometric subgraphs of half-cubes

Let H be the Cartesian product of two connected graphs H_1 and H_2. For any two vertices $\hat{u}=(u',u'')$ and $\hat{v}=(v',v'')$ of H we have $d_H(\hat{u},\hat{v})=d_{H_1}(u',v')+d_{H_2}(u'',v'')$. For a vertex v of H_1 we will say that $\{v\}\times H_2$ is the *fibre* (or H_1-*fibre*) of v in $H=H_1\times H_2$. For an isometric subgraph G of $H=H_1\times H_2$, its *projection* to H_1 is the subgraph G_1 of H_1 induced by all vertices $v\in H_1$ such that the fibre $\{v\}\times H_2$ intersects G. The following auxilliary result, being of independent interest, expresses that if the projection of G to H_1 contains a clique, then G has a clique of the same size.

Lemma 1. *If G is an isometric subgraph of $H=H_1\times H_2$ and the projection G_1 of G to H_1 contains a clique C then G contains a clique $\hat{C}$ bijectively projecting onto C.*

Proof. Consider a clique $\hat{C}$ of G of maximal size, such that $\hat{C}$ bijectively projects onto a subclique of C. Suppose that $\hat{C}$ does not intersect the fibre $\{w\}\times H_2$ of some vertex $w\in C$. Since w is in the projection of G, its fibre shares a vertex $\hat{w}$ with G. We may assume that $\hat{w}$ is chosen as close as possible to $\hat{C}$. The clique $\hat{C}$ is contained in a single H_2-fibre, say, in the fibre of the vertex $v\in H_2$. Pick an arbitrary vertex $\hat{x}$ of $\hat{C}$ and suppose that $\hat{x}=(x,v)$ for a vertex $x\in C$. Let $\hat{w}=(w,y)$ for $y\in H_2$. Since

$$d_G(\hat{w},\hat{x})=d_H(\hat{w},\hat{x})=d_{H_1}(w,x)+d_{H_2}(y,v)=1+d_{H_2}(y,v),$$

we conclude that $\hat{w}$ is equidistant to all vertices of the clique $\hat{C}$. In particular, $y\neq v$; otherwise, $\hat{w}\cup\hat{C}$ is a clique, contrary to the maximality of $\hat{C}$. Hence $d_G(\hat{w},\hat{x})\geqslant 2$. Since G is an isometric subgraph of H, the vertices $\hat{w}$ and $\hat{x}$ can be connected in G by a shortest path. Let $\hat{u}=(u,z)$ be the neighbour of $\hat{w}$ in this path. If $\hat{u}$ belongs to the fibre $\{w\}\times H_2$ (i.e., $u=w$) then we obtain a contradiction to the choice of $\hat{w}$. Therefore, $u\neq w$. In this case the fact that G is isometric in $H_1\times H_2$ yields that $u=x$, i.e., $\hat{u}$ and $\hat{x}$ belong to the fibre of x. On the other hand, since $\hat{u}$ and $\hat{w}$ are adjacent, we conclude that $y=z$. Therefore, for every vertex $\hat{x}=(x,v)\in\hat{C}$ the vertex $\hat{u}=(x,y)$ is in G. We obtain that $\hat{w}$ together with all these new vertices $\hat{u}$ constitute a clique. This shows that G contains a larger clique projecting onto a subclique of C, contrary to the choice of $\hat{C}$. □

A subset S of vertices of a graph G is *convex* if for any vertices $u,v\in S$ all vertices on shortest (u,v)-paths belong to S. If G is an L_1-graph then for every cut $\{A,B\}$ occurring in the L_1-decomposition of d_G both sets A and B are convex (we call such cuts *convex*). As was established in [3, 16] *a graph G is scale λ embeddable into a hypercube if and only if there exists a collection $\mathscr{C}(G)$ of (not necessarily distinct) convex cuts of G, such that every edge of G is cut by exactly λ cuts from $\mathscr{C}(G)$* (a cut $\{A,B\}$ cuts an edge (u,v) if $u\in A$ and $v\in B$ or $u\in B$ and $v\in A$). For $\lambda=1$ we obtain the well-known Djokovic characterization [18] of graphs isometrically embeddable into hypercubes. In fact, a similar characterization is valid for weighted graphs. Namely, let each edge (u,v) has a positive integer length $l(u,v)$. Define the distance between two vertices be the length of a shortest (weighted) path connecting the given pair of vertices. Assume in addition that the distance between any adjacent vertices u and v is $l(u,v)$. Then just repeating the proof from [3] we can show that the obtained metric space with integer-valued distances is scale λ embeddable into a hypercube if and only if there is a collection $\mathscr{C}$ of convex cuts, such that every edge (u,v) is cut by exactly $\lambda l(u,v)$ cuts from $\mathscr{C}$.

For a given nonnegative integer k let T_k denotes the following metric space defined on the set $X=\{a_0,a_1,a_2,a_3,a_4,b_0,b_1,b_2,b_3,b_4\}$:

$$d(a_i,a_j)=d(b_i,b_j)=1 \quad (i,j\in\{0,1,2,3,4\}),$$

$$d(a_0,b_i)=d(b_0,a_i)=k+1 \quad (i\in\{1,2,3,4\}), \qquad d(a_0,b_0)=k+2,$$

while $d(a_i,b_i)=k$ and $d(a_i,b_j)=k+1$ if $i\neq j$, $i,j\neq 0$. Note that T_0 can be seen as the metric of the graph K_6-e, (i.e., a complete graph on 6 vertices minus an edge). Actually, K_6-e is the unique L_1-graph with at most 6 vertices having scale larger than 2. T_k can be seen as the sum of T_0 and k cut semimetrics with all $d(a_i,b_j)=1$, all other distances are 0.

Using the above-mentioned results from [3, 16] and [28, 9] and Lemma 1 we can state the following characterization of scale 2 embeddable graphs (alias isometric subgraphs of half-cubes).

Proposition 1. *An L_1-graph G is an isometric subgraph of a half-cube if and only if it does not contain any T_k $(k\geqslant 0)$ as an isometric subspace.*

Proof. We start by showing that T_k $(k\geqslant 0)$ is not scale 2 embeddable into a hypercube; its scale is 4. This can be verified in a straightforward way for $T_0=K_6-e$. Suppose by way of contradiction that we can select the smallest T_k which has a scale 2 embedding into a hypercube. Equivalently, there is a collection $\mathscr{C}(T_k)$ of convex cuts of T_k such that every edge (u,v) is cut by $2l(u,v)$ cuts from this collection. Consider an arbitrary edge (a_i,b_i) $(i\in\{1,2,3,4\})$. Then

$$d(b_i,a_j)=d(b_i,a_i)+d(a_i,a_j),$$

$$d(a_i,b_j)=d(a_i,b_i)+d(b_i,b_j)$$

for any $j\in\{0,1,2,3,4\}$. The unique convex cut $\{A,B\}$ with the property that $a_i\in A$ and $b_i\in B$ has the form $A=\{a_0,a_1,a_2,a_3,a_4\}$ and $B=\{b_0,b_1,b_2,b_3,b_4\}$. Since $d(a_i,b_i)=k$ the cut $\{A,B\}$ is included in $\mathscr{C}(T_k)$ $2k$ times. Removing two occurrences of $\{A,B\}$ in $\mathscr{C}(T_k)$ we obtain a family of convex cuts which define a scale 2 embedding of T_{k-1} into a hypercube, contrary to the choice of T_k.

Conversely, assume that G is an L_1-graph which does not contain any T_k $(k\geqslant 0)$ as an isometric subspace. Since the Cartesian product of half-cubes is isometrically embeddable into a larger half-cube, by the result of [28] we can assume that G is isometrically embeddable into a graph $\Gamma=K_{m\times 2}\times H$, where $m\geqslant 5$ and H is a Cartesian product of some half-cubes and cocktail-party graphs. (Recall that the cocktail-party graph $K_{4\times 2}$ coincides with $\frac{1}{2}H_4$.) Moreover, the projection of G to $K_{m\times 2}$ must contain a subgraph $K = K_6 - e$, otherwise we can reduce $K_{m\times 2}$ to a smaller cocktail-party graph. The subgraph K is a union of two 5-cliques C_1 and C_2, sharing a common 4-clique C. Let $x_1 \in C_1 - C$, $x_2 \in C_2 - C$ and $C = \{y_1, y_2, y_3, y_4\}$. By Lemma 1 G contains two 5-cliques $\hat{C}_1$ and $\hat{C}_2$ bijectively projecting onto C_1 and C_2, respectively. Each of these cliques is contained in a single H-fibre, say, $\hat{C}_1 \subset \{u\} \times K_{m\times 2}$ and $\hat{C}_2 \subset \{v\} \times K_{m\times 2}$. Then $\hat{C}_1 = \{\hat{x}_1, \hat{y}'_1, \hat{y}'_2, \hat{y}'_3, \hat{y}'_4\}$ and $\hat{C}_2 = \{\hat{x}_2, \hat{y}''_1, \hat{y}''_2, \hat{y}''_3, \hat{y}''_4\}$, where $\hat{x}_1 = (x_1, u)$, $\hat{y}'_i = (y_i, u)$ and $\hat{x}_2 = (x_2, v)$, $\hat{y}''_i = (y_i, v)$ for $i = 1,2,3,4$. Let $d_H(u,v) = k$. Since G is an isometric subgraph of $\Gamma = K_{m\times 2} \times H$, we obtain that

$$d_G(\hat{y}'_i, \hat{y}''_i) = k, \quad d_G(\hat{y}'_i, \hat{y}''_j) = d_G(\hat{x}_1, \hat{y}''_j) = d_G(\hat{x}_2, \hat{y}'_i) = k + 1,$$

and $d_G(\hat{x}_1, \hat{x}_2) = k+2$ for all $i \neq j$, $\quad i,j \neq 1$. We see that $\hat{C}_1$ union $\hat{C}_2$ form the forbidden configuration T_k. This leads us to a contradiction, because all selected vertices belong to G. □

A characterization of isometric subgraphs of hypercubes similar to Proposition 1 has been given in [27]. We already mentioned the result of [6] that a metric space is L_1-embeddable if and only if all its finite subspaces are L_1-embeddable. Proposition 1 can be reformulated in the same vein: *an L_1-graph is isometrically embeddable into a half-cube if and only all its subspaces with at most 10 vertices are isometrically embeddable into a half-cube.*

Corollary 1. *Every planar L_1-graph is isometrically embeddable into a half-cube.*

Indeed, a planar graph does not contain K_5 and hence does not contain T_k, so we can apply Proposition 1. For finite planar graphs Corollary 1 has another proof via Delaunay polytopes (for notions and results in this direction the reader can consult [11]). If a graph G is an l_1-graph, then it generates a Delaunay polytope $P(G)$, and contains an affine basis of $P(G)$. If the scale of G is larger than 1, then $P(G)$ is a Cartesian product of polytopes of half-cubes and cocktail-party graphs (the latter polytopes are the well-known cross-polytopes). An affine basis of a Cartesian product is a union of affine bases of the components with one point in common. Any affine basis of a cross-polytope of dimension n contains an $(n-1)$-dimensional simplex. The skeleton of this simplex is the complete graph K_n. Hence, if G is planar, the corresponding direct product $P(G)$ can only contain cross-polytopes of dimension smaller than 5. The skeletons of such polytopes are isometric subgraphs of half-cubes.

An L_1-embeddable graph G is called *rigid* [12] if it has an essentially unique L_1-representation. Every isometric subgraph of a hypercube is L_1-rigid [12]. On the other hand, every L_1-rigid graph is scale 2 embeddable into a hypercube [28]. This follows from the following characterization of L_1-rigidity [28]: *an L_1-graph G is rigid if and only if in the isometric embedding of G into the Cartesian product of complete graphs, half-cubes and cocktail-party graphs all factors are L_1-rigid.* This is equivalent to the condition that no factor is isomorphic to a complete graph K_m or a cocktail party graph $K_{m,2}$ for $m > 3$. By Lemma 1 this happens when the L_1-graph G does not contain 4-cliques. Then its projections on factors are K_4-free, thus the factors (being irreducible) do not contain 4-cliques. Therefore, the following is true.

Corollary 2. *Every L_1-graph without K_4 is L_1-rigid. In particular, any 3-partite L_1-graph is L_1-rigid.*

3. Alternating cuts in planar graphs

Let G be a planar locally finite (all vertices have finite degree), and assume that a plane drawing of G is given. As usual, an *interior face* of G is a cycle of G which bounds a simply connected region. Denote by G^* the graph dual to G. Suppose that a plane drawing of G^* is given, such that the vertices and the edges of G^* belong to the corresponding faces of G. For a cut $\{A,B\}$ of G, let $E(A,B)$ be the set of edges with one end in A and another one in B. Evidently, removing $E(A,B)$ from G we obtain a graph with at least two connected components, i.e. $E(A,B)$ is a cutset of edges. Let $Z(A,B)$ be the family of interior faces of G which are crossed by $\{A,B\}$. We will say that $Z(A,B)$ is the *zone* of the cut $\{A,B\}$. Finally, let $C(A,B)$ be a partial subgraph of G^* defined in the following way: the vertices of $C(A,B)$ are the faces of $Z(A,B)$ and two such faces are adjacent in $C(A,B)$ if and only if they share a common edge from $E(A,B)$.

In [26] has been shown that any planar graph in which all interior faces have length larger than 4 and the degrees of interior vertices are larger that 3 is L_1-embeddable. In this section we investigate another set of local conditions under which a planar graph is L_1-embeddable. Namely, we consider planar graphs G in which all interior faces are isometric cycles. For them we define a special collection of cuts and show that often it defines a scale 2 embedding into a hypercube. First we establish a simple property of convex cuts of planar graphs.

Lemma 2. *If $\{A,B\}$ is a convex cut and F is a face of a planar graph G, then $|E(A,B)\cap E(F)| = 0$ or 2. In particular, $C(A,B)$ is either a path or a cycle.*

Proof. Suppose by way of contradiction that $\{A,B\}$ cuts at least three edges (a_1,b_1), (a_2,b_2) and (a_3,b_3) of F, where $a_1,a_2,a_3 \in A$ and $b_1,b_2,b_3 \in B$ (then $|E(A,B)\cap E(F)| \geqslant 4$, because it is an even integer). We can assume, without loss of generality, that one of two subpaths of F connecting the vertices b_1 and b_2 belongs to the set B. Pick an arbitrary shortest path P between the vertices a_1 and a_2. Then any path between b_3 and b_1 either contains one of the vertices a_1, a_2 and a_3 or separates a_1 from a_2. Therefore, every shortest path between b_3 and b_1 intersects P, yielding a contradiction with $A\cap B = \emptyset$. To prove the second assertion it suffices to establish that $C(A,B)$ is connected. But this is clear: otherwise, removing $E(A,B)$ we obtain a graph with at least three connected components, hence one of A and B is not convex. $\square$

Geometrically, Lemma 2 asserts that if we cut the plane along $C(A,B)$, then once entering a face of $Z(A,B)$ we must exit this face through some other edge and we will never visit this face again. In particular, the line along which we cut has no self-intersections. Furthermore, the sets $A \cap Z(A,B)$ and $B \cap Z(A,B)$ are either paths or cycles. We will denote them by $bd(A)$ and $bd(B)$, and call them the *border lines* of the cut $\{A,B\}$.

It does not seem possible to achieve much more in the full generality. Thus, we descend to a smaller, but quite natural class of planar graphs. Further we assume that G is a planar graph, embedded in the Euclidean plane with the property that

(a) *Any interior face is an isometric cycle of* G.

(Although natural, one can construct planar L_1-graphs which do not admit a planar embedding obeying this condition: for this take a book, i.e., a collection of at least three 4-cycles sharing a common edge.)

Two edges $e' = (u',v')$ and $e'' = (u'',v'')$ on a common interior face F are called *opposite* if $d_G(u',u'') = d_G(v',v'')$ and equal the diameter of the cycle F. If F is an even face, then any of its edges has an unique opposite, otherwise, if F has an odd length, then every edge $e \in F$ has two opposite edges e^+ and e^-. In the latter case, if F is clockwise oriented, for e we distinguish the *left opposite edge* e^+ and the *right opposite edge* e^-. If every face of $Z(A,B)$ is crossed by a cut $\{A,B\}$ in two opposite edges, then we say that $\{A,B\}$ is an *opposite cut* of G.

If a convex cut $\{A,B\}$ of G enters an interior face F through an edge e, then convexity of A and B and isometricity of F yield that $\{A,B\}$ exits F through an edge opposite to e. We will say that $\{A,B\}$ is *straight* on an even face F and this cut *makes a turn* on an odd face F. The turn is *left* or *right* depending which of the opposite edges e^+ or e^- we cross.

Lemma 3. *In a planar graph with isometric faces all convex cuts are opposite.*

If G is a planar graph with isometric faces of even length only (i.e., G is bipartite), then G is an L_1-graph if and only if every opposite cut is convex. This already presents an useful way to verify if G is L_1-embeddable or not. For example, using this we obtain that the first graph presented in Fig. 1 is not L_1-embeddable (this is the skeleton of the smallest convex polyhedron with an odd number of faces, all of which are quadrangles; see [19]).

An opposite cut $\{A,B\}$ of a planar graph G is *alternating* if the turns on it alternate. For an illustration of this concept in Figs. 2 and 3 we present a list of planar graphs, in particular of tilings (for other examples of L_1-graphs with isometric faces see [10, 15]). For many important planar L_1-graphs the convex cuts participating in the L_1-decomposition turn out to be alternating. This is so for planar graphs from [26]. Consequently, if G is bipartite then the alternating cuts are exactly the opposite cuts of G. By Lemma 3 every convex cut of G is alternating. Another class of planar graphs

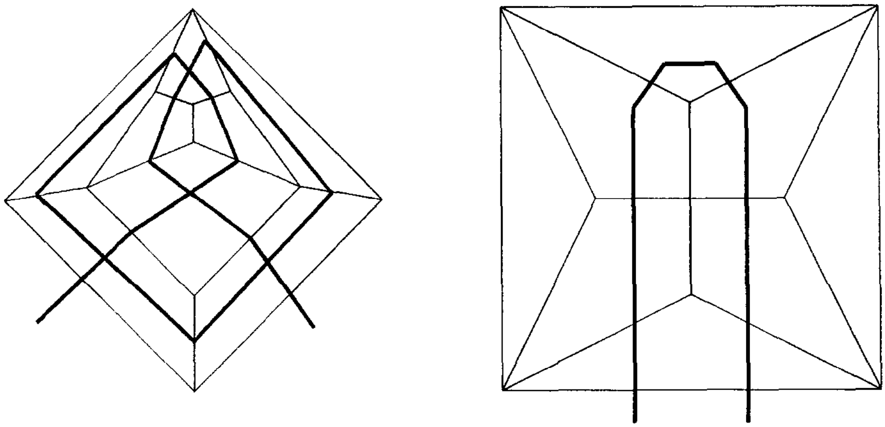

Fig. 1.

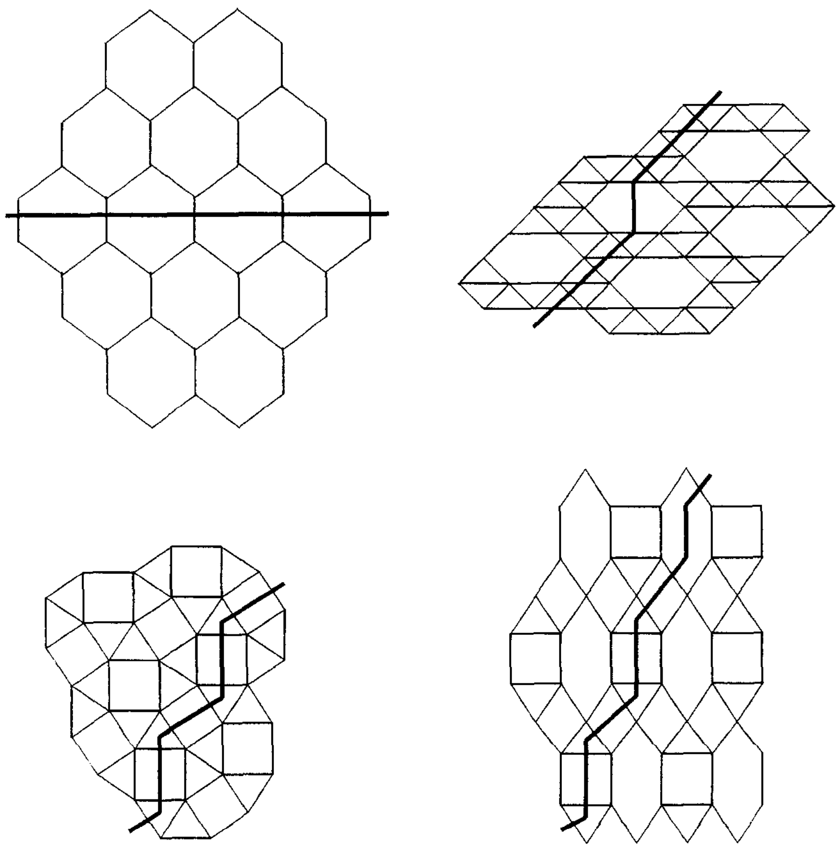

Fig. 2.

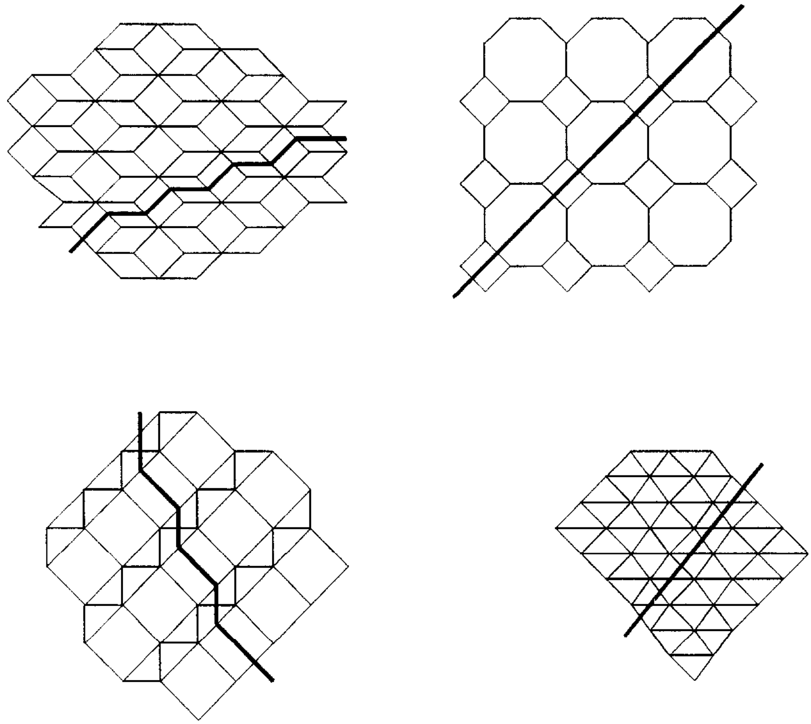

Fig. 3.

with this property is given in the next result. (Two interior faces are *incident* if they share an edge.)

Lemma 4. *Let G be a planar graph in which all interior faces are isometric cycles of odd length. If the union of each pair of incident faces is an isometric subgraph of G, then any convex cut of G is alternating.*

Proof. Suppose that a convex cut $\{A,B\}$ cuts two incident faces F_1 and F_2 of G. Since $F_1 \cup F_2$ is an isometric subgraph of G, every vertex of F_1 can be connected with every vertex of F_2 by a shortest path which contains at least one vertex of $F_1 \cap F_2$. Since A and B are convex, this implies that $\{A,B\}$ cuts an edge $e \in F_1 \cap F_2$. By Lemma 3, $\{A,B\}$ leaves both faces through edges $e' \in F_1$ and $e'' \in F_2$ opposite to e. Suppose that both $e' = (x', y')$ and $e'' = (y', y'')$ are left opposite edges of e. Denote by P and Q the intersections of A and B, respectively, with $F_1 \cup F_2$. Both P and Q are paths with one end in e' and another one in e'', and the longer one of them (say, P) has the length equal to the length of Q plus 2. We can connect the end vertices x' and x'' of P via Q. Since A is convex, the distance $d_G(x', x'')$ is smaller than the length of the path P, in contradiction with the assumption that $F_1 \cup F_2$ is an isometric subgraph of G. □

Perhaps the most known class of planar graphs verifying the conditions of Lemma 4 is that of *triangulations* (i.e., planar graphs in which all interior faces have length three) without 4-cliques. It has been established in [3] that any finite (planar) triangulation with the property that all vertices which do not belong to the exterior face have degree larger than 5 is L_1-embeddable. Moreover, all such graphs are L_1-rigid. From Lemma 4 we have the following property of planar triangulations.

Corollary 3. *If a triangulation G does not contain K_4 as an induced subgraph (i.e. all interior vertices have degrees $\geqslant 4$), then any convex cut of G is alternating.*

We continue with a property of cubic planar graphs (i.e., duals of triangulations). Replace every edge $e=(u,v)$ of a cubic planar graph G by two arcs $e'=(u,v)$ and $e''=(v,u)$. Denote the resulting oriented graph by Γ. A simple directed circuit C of Γ is said to be *alternating* if every face of G is either disjoint or shares with C exactly two consecutive arcs. The graph G^* dual to G is a planar triangulation. Every alternating circuit of Γ corresponds to an alternating cut of G^*, and, conversely, any convex alternating cut of G^* defines an alternating circuit of Γ. Therefore, we obtain the following property of G (for the definition of the cycle double covers of graphs see [22]).

Corollary 4. *If the dual of a finite cubic planar graph G is L_1-embeddable, then there is a family of alternating circuits of Γ such that any arc of Γ is covered by exactly one circuit. In other words, G has a double cover by alternating cycles.*

Now, we present an algorithmic procedure to find the alternating cuts (if they exist) crossing a given edge $e=(u,v)$ of a planar graph G. To do this we extend the cuts from the edge e, crossing face after face. We go away from e in two directions (or in only one direction if e belongs to the exterior face of G) until we arrive to odd faces. In this movement we go straight through even faces. Now, suppose that F' and F'' are the first odd faces which occur when moving in opposite directions. Then in one cut we make a left turn on F' and a right turn on F'', and in another cut we make a right turn on F' and a left turn on F''. After that we have only to alternate the directions when passing through odd faces of G. Namely, if say our last turn in one cut was to left, then comming to the next odd face this cut turns to right and conversely. Let $E'(e)$ and $E''(e)$ be two (not necessarily distinct) subsets of edges which we cross in this movement.

We assert that for any alternating cut $\{A,B\}$ which cuts the edge e either $E(A,B)=E'(e)$ or $E(A,B)=E''(e)$ holds. Indeed, $\{A,B\}$ cuts the edges from the common part of $E'(e)$ and $E''(e)$ until the faces F' and F''. In this moment, we have only two possibilities to continue the movement along $E(A,B)$, namely, $\{A,B\}$ cuts the faces F' and F'' in the same fashion as $E'(e)$ or $E''(e)$, say as $E'(e)$. In this case necessarily $E(A,B)$ and $E'(e)$ coincide everywhere. Concluding, we obtain the following result.

Lemma 5. *Every edge e of a planar graph G is crossed by at most two alternating cuts, each of them defined by $E'(e)$ or $E''(e)$.*

Denote by $\mathscr{A}(G)$ the collection of all alternating cuts of G, where every cut which never has to turn is counted twice. In general, we can construct planar graphs without alternating cuts. This is due to the fact that $E'(e)$ and $E''(e)$ do not necessarily define cutsets of G. In Fig. 1 we present two examples of alternating "pseudo–cuts" constructed by our procedure, which are not cuts. These graphs are not L_1-graphs. The second graph is taken from [4] and is a skeleton of a space-filler. The skeletons of many others space-fillers listed in this paper represent L_1-graphs. However, if all $E'(e)$ and $E''(e)$ $(e \in E(G))$ are cutsets, then Lemma 5 infers that the family of alternating cuts $\mathscr{A}(G)$ is rather complete: every edge of G is crossed by exactly two cuts from $\mathscr{A}(G)$. Unfortunately, only this property together with (a) do not imply L_1-embeddability of a planar graph G, because alternating cuts can be non-convex. Neither it can exclude L_1-embeddability: there exists planar L_1-graphs without convex alternating cuts. To ensure the L_1-embeddability of G we have to impose a metric condition on the borders of alternating cuts constructed by our procedure (fortunately, these natural requirements are easily verified in many important particular cases):

(b) *The border lines $bd(A)$ and $bd(B)$ of any alternating cut $\{A,B\}$ are isometric cycles or geodesics.*

(By a *geodesic* is meant a (possibly infinite in one or two directions) simple path P with the property that $d_P(x,y) = d_G(x,y)$ for any $x,y \in P$.) Evidently, (b) implies the condition (a).

Proposition 2. *If G is a planar graph satisfying condition* (b), *then a cut $\{A,B\}$ of G is alternating if and only if it is convex.*

Proof. Let $\{A,B\}$ be an alternating cut, and assume by way of contradiction that the set A is not convex: then we can find two vertices $x,y \in A$ and a shortest path R between x and y such that $R \cap B \neq \emptyset$. We can suppose, without loss of generality, that among the vertices of A violating the convexity condition the vertices x and y are choosen as close as possible. Let x' and y' be the neighbours in R of x and y, respectively. Then x', y' and all vertices of R between them belong to the set B'. In particular, the edges (x,x') and (y,y') are crossed by (A,B). This implies that $x,y \in bd(A)$ and $x',y' \in bd(B)$. Since $bd(A)$ and $bd(B)$ are isometric subgraphs of G, $d_{bd(A)}(x,y) = d_G(x,y)$ and $d_{bd(B)}(x',y') = d_G(x',y')$. Since (A,B) is an alternating cut of G, one can easily conclude that

$$|d_{bd(A)}(x,y) - d_{bd(B)}(x',y')| \leqslant 1.$$

This contradicts our supposition that x' and y' lie on the common shortest path R connecting x and y.

Conversely, let $\{A,B\}$ be a convex cut. By Lemma 3 it is an opposite cut. Suppose, however, that it has two consecutive turns on odd faces F' and F''. By our algorithmic procedure we conclude that F' and F'' belong to the zone of some alternating cut $\{A',B'\}$, which cuts F' along the same edges as $\{A,B\}$. Moreover, $\{A,B\}$ and $\{A',B'\}$

enter the face F'' through a common edge, while exit this face through incident edges. As in the proof of Lemma 4 one can obtain a contradiction with the fact that $bd(A')$ or $bd(B')$ are isometric subgraphs. □

Corollary 5. *If G is a planar graph satisfying condition* (b), *then G is a rigid L_1-graph.*

Proof. From Lemma 4 and condition (b) we obtain that every edge of G is crossed by exactly two alternating cuts. By Proposition 2 we conclude that G is scale 2 embeddable into a hypercube. Since every convex cut of G is alternating, we deduce that this L_1-embedding of G is unique. □

To apply this result we have to construct the alternating cuts of a graph G, and to verify if all border lines are isometric cycles or geodesics. For example, if we consider $K_{2,3}$ with the vertices x_1, x_2, y_1, y_2, y_3, then for the alternating cut $A = \{x_1, y_2\}$, $B = \{y_1, x_2, y_3\}$ we have $bd(A) = (y_1, x_2, y_3)$ and $bd(B) = (x_1, y_2, x_1)$. The second path is not a geodesic (it is not even simple), so we cannot apply Corollary 5. On the other hand, from Corollary 5 one can easily deduce the L_1-embeddability of many nice planar graphs, in particular tilings (some of them were already presented in Figs. 2 and 3); in all these cases the border lines of alternating cuts represent geodesics.

We continue by establishing the L_1-embeddability of still another class of planar graphs. Recall that a (finite) planar graph G is *outerplanar* if there is an embedding of G in the Euclidean plane such that all vertices of G belong to the exterior face.

Proposition 3. *Any outerplanar graph G is a rigid L_1-graph.*

Proof. Indeed, G enjoys the conditions (a) and (b). In fact, every interior face of G is convex. In addition, for each edge e the sets $E'(e)$ and $E''(e)$ constructed by our algorithm define two alternating cuts $\{A', B'\}$ and $\{A'', B''\}$. Since the dual graph of G is a tree, the border lines of these cuts cannot be cycles. If one of them, say $bd(A')$ is not a geodesic, then we can find a shortest path L between two non-adjacent vertices $u, v \in bd(A')$, such that $L \cap bd(A') = \{u, v\}$. First suppose that L is disjoint from $bd(A'')$. But then at least one of the vertices of $bd(A')$ or $bd(A'')$ belongs to the interior of the region bounded by L and the second such path, contradicting that G is outerplanar. Otherwise, if L shares a vertex with $bd(A'')$, then one can deduce that L consists of two edges $(u,x), (v,y)$ with $x, y \in bd(A'')$ and the portion of $bd(A'')$ between x and y. By the algorithmic construction of alternating cuts we conclude that the length of L must be larger than that of $bd(A')$, contrary to our assumption. Therefore, we are in a position to apply Corollary 5. □

For a finite L_1-graph G, let size(G) denote $\min(n/\lambda)$ taken over all scale embeddings of G into a hypercube (here λ is the scale, while n is the dimension of the host hypercube).

Proposition 4. *Let H be a planar graph such that all border lines of alternating cuts are geodesics, and let G be a subgraph of H bounded by a simple (nondegenerated) cycle C of length p of H. Then*

(1) *G endowed with its own metric d_G is a rigid L_1-graph;*

(2) *$size(G) = p/2$.*

Proof. We show how to derive $\mathscr{A}(G)$ from $\mathscr{A}(H)$. Pick an alternating cut $\{A, B\}$ of H. From Lemma 5 and Proposition 2 we know that $\{A, B\}$ is defined by a cutset $E'(e)$, $e \in E(G)$. Let $Z_1, \ldots, Z_p$ be the connected components of the zone $Z(A, B)$. Each of them is the zone of a cut of G. Denote the resulting cuts by $(A_1, B_1), \ldots, (A_p, B_p)$, so that $Z_1 = Z(A_1, B_1), \ldots, Z_p = Z(A_p, B_p)$. By the definition, each of these cuts is an alternating cut of G. Suppose, without loss of generality, that (A_i, B_i) is defined by the cutset $E'(e_i)$ of G, where e_i is an arbitrary edge cut by (A_i, B_i). Since $bd(A)$ and $bd(B)$ are geodesics of H and $bd(A_i)$ and $bd(B_i)$ are subpaths of $bd(A)$ and $bd(B)$, respectively, we conclude that both $bd(A_i)$ and $bd(B_i)$ are geodesics of G (note that $d_G(u, v) \geqslant d_H(u, v)$ for any vertices u, v of G.) Thus, we are in a position to apply Corollary 5. This shows that G is an L_1-graph.

To prove (2) first note that every alternating cut of G starts and ends with edges which lie on C. If G is bipartite, then the alternating and opposite cuts coincide and G is isometrically embeddable into a hypercube of dimension $p/2$. Therefore, in this case size$(G) = p/2$. Otherwise, if G has an odd face, then G is scale 2 embeddable into a hypercube. Then every edge of C takes part in two alternating cuts (not necessarily distinct). This means that G has a scale 2 embedding into a hypercube of dimension p. This implies that again size$(G) = p/2$. □

In Fig. 4 we present a few examples of graphs verifying the conditions of Proposition 4. In fact, any part of a tiling from Fig. 4, bounded by a cycle, is a rigid L_1-graph.

A *corona* $Cor(p, q)$ (p and q are positive integers, $p \geqslant 4$) is the graph defined in the following way. $Cor(p, 1)$ is the cycle of length p. Then $Cor(p, q)$ is obtained by surrounding $Cor(p, q-1)$ with a ring of p-cycles. $Cor(p, q)$ can be viewed as the finite part of a locally finite tiling of the plane with convex p-gons; see Fig. 3.1.6 of [20] for $Cor(7, 3)$. Computing the alternating cuts, from Corollary 5 we obtain that all coronas are L_1-graphs. Using the same statement we can establish the result of [1, 15] that the graphs of regular tilings of the hyperbolic plane are L_1-embeddable.

The assertion (1) of Proposition 4 does not hold for all planar L_1-graphs H. For example, let H be the prism $C_6 \times K_2$ embedded in the Euclidean plane, and suppose that G is obtained from H by deleting a boundary vertex. Then G is not an L_1-graph, however it is obtained from H using the operation from Proposition 4. It would be interesting to investigate the planar L_1-graphs which verify the hereditary property described in Proposition 4(1).

An operation in some sense inverse to the previous one is that of gluing planar L_1-graphs along common (isometric) faces. Again, it does not, in general, preserve

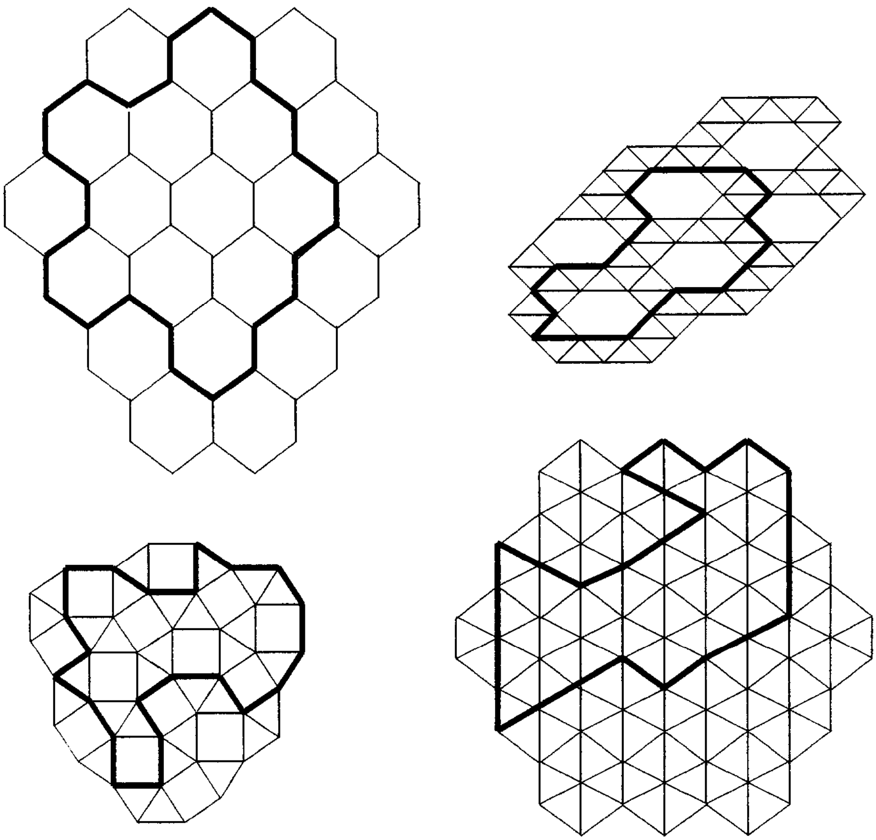

Fig. 4.

L_1-embeddability, so, the question is to find under which conditions the resulting planar graph is L_1-embeddable, too.

A particular instance of this gluing operation is that of *capping* of a planar graph G (it corresponds to gluing a planar graph and a wheel): add a new vertex inside a given face and connect this vertex to all vertices of this face. An *omnicapping* of G is capping of all faces of G. When capping preserves L_1-embeddability? We know only that all partial cappings of skeletons of regular polyhedra are L_1-graphs, except the cube. Capping one, two, or three pairwise nonopposite faces of H_3 results in L_1-graphs; all other cappings give non-L_1-graphs.

4. Wiener index of L_1-graphs

A *benzenoid system* (alias *hexagonal system*) is a planar graph in which every (interior) face is bounded by a regular hexagon. Equivalently, a benzenoid system is a subgraph of the hexagonal grid which is bounded by a simple circuit of this grid. That the benzenoid systems are L_1-graphs (namely, they are isometrically embeddable into hypercubes) was established in [24]. Moreover, it has been shown how to apply this embedding to compute the Wiener index of a benzenoid system G. Recall that the *Wiener index* $W(G)$ (often used in mathematical chemistry) of G is the sum of distances $d_G(u,v)$ taken over all pairs of vertices u,v of G. In [7] is shown that actually the graphs of benzenoid systems are isometrically embeddable into the Cartesian product of three trees. Using this fact and the result from [24], a linear time algorithm for computing the Wiener number of a benzenoid is presented in [8]. Our final purpose is to extend the result from [24] to the whole class of L_1-graphs.

Proposition 5. *Let G be a finite graph scale λ embeddable into a hypercube and let $\mathscr{C}(G)$ be the family of (not necessarily distinct) convex cuts defining this embedding. Then*

$$W(G)=\frac{1}{\lambda}\sum_{\{A,B\}\in\mathscr{C}(G)}|A|\cdot|B|.$$

Proof. We will rewrite the expression for $W(G)$, taking into account that $\lambda\cdot d_G(u,v)=\sum_{\{A,B\}\in\mathscr{C}(G)}\delta_{\{A,B\}}(u,v)$ for any two vertices u,v of G:

$$\begin{aligned}W(G)&=\frac{1}{2}\sum_{u\in V}\sum_{v\in V}d_G(u,v)\\&=\frac{1}{2\lambda}\sum_{u\in V}\sum_{v\in V}\sum_{\{A,B\}\in\mathscr{C}(G)}\delta_{\{A,B\}}(u,v)\\&=\frac{1}{\lambda}\sum_{u\in A}\sum_{v\in B}|\{A,B\}\in\mathscr{C}(G)|\\&=\frac{1}{\lambda}\sum_{\{A,B\}\in\mathscr{C}(G)}|A||B|.\qquad\square\end{aligned}$$

Since the number of cuts of $\mathscr{C}(G)$ is normally much smaller than the number of pairs $\{u,v\}$, this formula significantly simplifies finding the Wiener index of an L_1-graph. In many cases it immediately produces the formula for computing $W(G)$. For example, $W(\frac{1}{2}H_n)=n\cdot 2^{2n-5}$ for $n\geqslant 2$. The *Johnson graph* $J(n,m)$ has all subsets of $\{1,\dots,n\}$ of cardinality m as vertices and two vertices A,B are adjacent if $|A\Delta B|=2$. Our formula gives $W(J(n,m))=\frac{1}{2}\binom{n-1}{m}\binom{n-1}{m}$. Proposition 5 provides an easy verification of the result of [5] that $W(Cor(6,q))=\frac{1}{5}(164q^5-30q^3+q)$. Using a formula from [24] combinatorial expressions for the Wiener index of compact pericondenced benzenoid hydrocarbons were given in [21].

Nowadays the chemical graph theory presents the richest source of planar graphs. Using our approach one can establish L_1-embeddability of many chemical graphs.

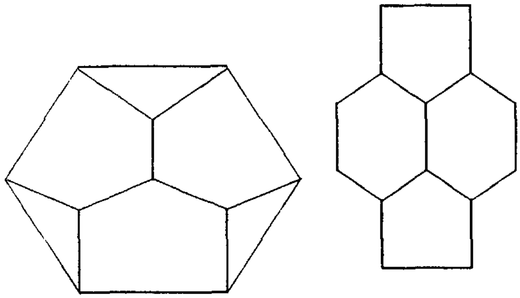

Fig. 5.

We wish to conclude our note with two examples of chemical L_1-graphs of size 5 and 10 taken from [17, 23].

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