Types and boundary uniqueness of polypentagons

M. Deza and M. I. Shtogrin

A connected plane graph taken with its interior faces forms an (r,q)-*polycycle* if the interior faces are all r-gons (where $r = \text{const} \geqslant 3$), while all interior vertices are of the maximal possible degree, denoted by q (where $q = \text{const} \geqslant 3$); see [1]. Any finite (r,q)-polycycle is homeomorphic to a disc and its *boundary* is homeomorphic to a circle. The cyclic sequence $q_1q_2\dots q_m$ formed by the degrees q_i, $i=1,\dots,m$, of all m boundary vertices of an (r,q)-polycycle is called the *boundary code*.[1] We call an r-gon *near-boundary* if at least one of its edges belongs to the boundary. If the intersection of a near-boundary r-gon with the boundary is connected, then the r-gon is *removable*, that is, the graph without this r-gon remains an (r,q)-polycycle; see [2]. Two (r,q)-polycycles with the same boundary are called *equiboundary*. Equiboundary (r,q)-polycycles consist of the same number p_r of r-gons; see [3]. A pair of equiboundary (r,q)-polycycles is said to be *irreducible* if every boundary edge belonging to a removable r-gon of one of the (r,q)-polycycles belongs to an unremovable r-gon of the other (r,q)-polycycle; see [2]. An example of an equiboundary pair of $(r,3)$-polycycles with the number $p_r = 4r$ is given in [4] for any $r \geqslant 5$; it is conjectured there that $4r$ is the minimal value for p_r. The conjecture is proved in [2] for $r=6$. Below we prove the conjecture for $r=5$, that is, for $(5,3)$-polycycles; we call such polycycles *polypentagons*.[2]

Theorem. (i) *There exists no pair of equiboundary polypentagons with* $p_5 \leqslant 19$.

(ii) *There exists only one* (*irreducible*) *pair of equiboundary polypentagons with* $p_5 = 20$. *Its boundary code is as follows*: $3^3232323^6232323^3232323^623232$.

(iii) *There exist three* (*reducible*) *pairs with* $p_5 = 21$. *Their boundary codes are*

$$3^3232323^62323^42323^623232, \quad 3^3232323^623^22^23^4232323^623232,$$
$$\textit{and} \quad 3^3232323^72^23^223^3232323^623232.$$

(iv) *There exist seventeen pairs of equiboundary polypentagons with* $p_5 = 22$.

(v) *For any* $p_5 \geqslant 23$ *there exist at least seventeen pairs obtained as extensions.*[3]

An interior edge of an arbitrary (r,q)-polycycle with both ends on the boundary is called a *bridge*. Any (r,q)-polycycle without bridges is called *elementary*.

Lemma 1. *Any* (r,q)-*polycycle cut along all its bridges is split up uniquely into elementary polycycles.*

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[1] This is a *vertex* boundary code. The paper [2] utilizes an *edge* boundary code.

[2] An example of a boundary code admitting exactly $n+1$ equiboundary polypentagons is known ([4], Proposition 3). Any such polypentagon has $40n+18$ vertices, $60n+23$ edges, and $20n+6$ faces.

[3] If the boundary code of a finite polypentagon differs from 3^5 then it is extendable; see [1].

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We call these elementary polycycles *elementary components* of the (r,q)-polycycle. The form and notation for elementary components of polypentagons may be found in Figure 5 in [1].

The *adjacency graph* of the elementary components of an (r,q)-polycycle is formed by assigning a vertex to each elementary component, with two vertices connected by an edge if the corresponding elementary components are adjacent, that is, have a common edge.

Lemma 2. *The adjacency graph of the elementary components of a polycycle is a tree.*

Any non-trivial tree of splices of elementary components has at least two hanging elementary components. One boundary edge of a hanging elementary component is a bridge of the enveloping (r,q)-polycycle, while all the other boundary edges of the component remain boundary edges of the polycycle.

We can represent the boundary code of an (r,q)-polycycle not only in the expanded form, but also in a compact form with the number of identical consecutive entries in the cyclic sequence indicated as an exponent. In the case of a polypentagon (that is, a $(5,3)$-polycycle) the compact boundary code consists of only twos and threes with some exponents. The collection of exponents of two is called the *type* of the polypentagon and is written in braces. For instance, in case (iii) of the theorem, polypentagons with the first boundary code are of type $\{1\}$ and polypentagons with the remaining two boundary codes are of type $\{1;2\}$ or $\{2;1\}$.

All (finite and infinite) polypentagons are uniquely divided into nine types.

Type $\{5\}$. The unique polypentagon of type $\{5\}$ is D; see Figure 5 in [1].

Type $\{3\}$. Any polypentagon of type $\{3\}$ is not elementary and consists of elementary components C_1 (each C_1 is adjacent to two neighbouring elementary components), C_3, E_1 (each of these is adjacent to three neighbouring elementary components), and D (each D is hanging: it is adjacent to one neighbour). (In)finite polypentagons of type $\{1\}$ consist of at least (one) two elementary components D.

Type $\{2\}$. Polypentagons of type $\{2\}$ consist of elementary components C_1, C_3, and E_1.

Type $\{1\}$. Elementary polypentagons of type $\{1\}$ are A_2, A_3, A_4, A_5. Non-elementary polypentagons of type $\{1\}$ consist of elementary components C_3, E_1 (adjacent to three neighbours), C_1, C_2, D, E_j with $j \geqslant 2$ (adjacent to two neighbours), and B_1, B_2, B_3 (hanging).

Type $\{0\}$. Elementary polypentagons of type $\{0\}$ are the finite polypentagon A_1 and the infinite polypentagon A_6. Non-elementary polypentagons of type $\{0\}$ are infinite and consist of C_1 (adjacent to two neighbours) and C_3, E_1 (adjacent to three neighbours).

Type $\{3;2;1\}$. Polypentagons of type $\{3;2;1\}$ are not elementary and consist of elementary components B_1, B_2, B_3, C_1, C_2, C_3, D, and E_j with $j \geqslant 1$. At least one D is hanging.

Type $\{3;2\}$. Polypentagons of type $\{3;2\}$ are not elementary and consist of elementary components C_1, C_3, D, E_1. Each elementary component D is hanging.

Type $\{3;1\}$. Polypentagons of type $\{3;1\}$ are not elementary and consist of elementary components B_1, B_2, B_3, C_1, C_2, C_3, D, and E_j with $j \geqslant 1$. At least one D is hanging.

Type $\{2;1\}$. Elementary polypentagons of type $\{2;1\}$ are B_1, B_2, B_3, C_2, D, and E_j with $j \geqslant 2$. Non-elementary polypentagons of type $\{2;1\}$ consist of elementary components B_1, B_2, B_3, C_1, C_2, C_3, D, and E_j with $j \geqslant 1$. Each D is adjacent to two neighbours.

Any pair of equiboundary polypentagons of type $\{2;1\}$ or $\{2\}$ is reducible by means of the reduction $3\,2^2\,3 \to 2\,3\,2$ of a fragment of the boundary code.

Any pair of equiboundary polypentagons of type $\{3;2;1\}$, $\{3;2\}$, $\{3;1\}$, or $\{3\}$ is reducible by means of the reduction $3\,2^3\,3 \to 2^2$ of a fragment of the boundary code.

Lemma 3. *Polypentagons of an irreducible equiboundary pair are of type* $\{1\}$.

Corollary. *An equiboundary pair with minimal* p_5 *is of type* $\{1\}$.

Among elementary components of finite non-elementary polypentagons of type $\{1\}$ there are hanging B_2 and (or) B_3, and there can be non-hanging C_1 and (or) C_2, C_3, D, E_j with $j \geqslant 1$.

Bibliography

[1] M. Deza and M. Shtogrin, *J. Geom. Phys.* **40**:3–4 (2002), 302–319.

[2] X. Guo, P. Hansen, and M. Zheng, *Discrete Appl. Math.* **118**:3 (2002), 209–222.

[3] G. Brinkmann, O. Delgado-Friedrichs, and U. von Nathusius, *Graphs and discovery*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 69, Amer. Math. Soc., Providence, RI 2005, pp. 27–38.

[4] M. Dutour, M. Deza, and M. Shtogrin, "Fillings of a given boundary by p-gons and related problems", *Proceedings of the International Conference "General Theory of Information Transfer and Combinatorics"* (Bielefeld, 2004) (to appear).

M. Deza
École Normale Supérieure, Paris
E-mail: Michel.Deza@ens.fr

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M. I. Shtogrin
Steklov Mathematical Institute,
Russian Academy of Sciences
E-mail: stogrin@mi.ras.ru