Izvestiya: Mathematics **70**:3 1–18

Izvestiya RAN: Ser. Mat. **70**:3 3–22 DOI 10.1070/IM2006v170n01ABEH002301

Non-extendible finite polycycles

M. Deza, S. V. Shpectorov, M. I. Shtogrin

Abstract. We give a fairly simple proof of a result announced in [2] that there are only seven non-extendible finite (r,q)-polycycles: the tetrahedron without a face, the cube without a face, the octahedron without a face, the dodecahedron without a face, the icosahedron without a face, the split-vertex octahedron and the split-vertex icosahedron.

Introduction

A non-separable [1] planar graph G along with its interior faces is called an (r,q)-*polycycle* and is denoted by $P(G)$ (see [2]–[4]) if the following independent conditions hold.

(i) All interior faces are r-gons, for fixed $r \geqslant 3$.

(ii) All interior vertices are of degree q, for fixed $q \geqslant 3$.

(iii) The degree of every boundary vertex is at most q (and at least 2).

All the interior faces of G are called *faces* of $P(G)$. Every sufficiently small neighbourhood of an interior vertex of $P(G)$ is homeomorphic to a disc. Every finite (r,q)-polycycle $P(G)$ is also homeomorphic to a disc.

As was shown in [4], the vertices, edges and faces of any (r,q)-polycycle $P(G)$ form a *cell complex* (see[2] [5], § 67) such that the intersection of any two cells is again a cell of the complex. This condition holds for *cubic complexes*, which were introduced by Novikov in 1986, and for more general complexes formed by arbitrary convex polyhedra (see [6], Russian p. 47). As to their intersections, the cells of all these complexes behave like simplices in *simplicial complexes* (see [5], Russian p. 60).

(r,q)-polycycles are a broad generalization of regular partitions (traditionally denoted by (r^q)) of the sphere, Euclidean plane or Lobachevsky plane. They are widely used in chemistry, crystallography and theoretical physics.

For the sphere $\mathbb{S}^2$, a regular partition (r^q) is a partition of $\mathbb{S}^2$ into congruent equilateral triangles, squares or regular pentagons. The corresponding parameters (r,q) are said to be *elliptic*. They are integer solutions $(r,q \in \mathbb{N})$ of the inequality

[1] For terminology and numerous facts from graph theory, see [1].

[2] In the definition in [5], Russian p. 298, what is meant is that a one-dimensional cell is an interval, and its boundary consists of two points.

This paper was written with the financial support of the Russian Foundation for Basic Research (grant no. 05-01-00170), the programme "Leading Scientific Schools" (grant no. NSh-2185.2003.1) and the programme "Modern Problems in Theoretical Mathematics" of the Mathematical Sciences Division of the RAS.

AMS 2000 *Mathematics Subject Classification.* 05C10, 05C38, 52B70.

$rq < 2(r+q)$ such that $r \geqslant 3$ and $q \geqslant 3$. The elliptic parameters (r,q) can take the values $(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, $(5,3)$.

For the Euclidean plane $\mathbb{R}^2$, a regular partition (r^q) is a partition of $\mathbb{R}^2$ into congruent equilateral triangles, squares or regular hexagons. The corresponding parameters (r,q) are said to be *parabolic*. They are integer solutions $(r,q \in \mathbb{N})$ of the equation $rq = 2(r+q)$ such that $r \geqslant 3$ and $q \geqslant 3$. The parabolic parameters (r,q) can take the values $(3,6)$, $(4,4)$, $(6,3)$.

For the Lobachevsky plane $\mathbb{H}^2$, a regular partition (r^q) is a partition of $\mathbb{H}^2$ into congruent regular r-gons with any fixed r, where $3 \leqslant r < \infty$. The corresponding parameters (r,q) are said to be *hyperbolic*. They are integer solutions $(r,q \in \mathbb{N})$ of the inequality $rq > 2(r+q)$ such that $r \geqslant 3$ and $q \geqslant 3$. The hyperbolic parameters (r,q) are given by the following inequalities: $q \geqslant 7$ if $r = 3$, $q \geqslant 5$ if $r = 4$, $q \geqslant 4$ if $r = 5$ or 6, and $q \geqslant 3$ if $r \geqslant 7$.

In the planar case, every regular partition (r^q) is an (r,q)-polycycle (see [2]), but is not finite. In the spherical case, every regular partition (r^q) is finite but is not an (r,q)-polycycle (see [2]): the partition (r^q) is homeomorphic to the sphere while every finite (r,q)-polycycle $P(G)$ is homeomorphic to a disc.

The simplest visual example of a finite (r,q)-polycycle is the surface of a Platonic solid with a face deleted. The number of ways of deleting a face is equal to the number of faces. Deleting a face from the surface of a Platonic solid (homeomorphic to the sphere) gives a surface with boundary (homeomorphic to a disc), which can be embedded in the plane. Hence we get a polycycle. Its combinatorial structure does not depend on the choice of the face to be deleted. The edge skeleton of this polycycle has a unique cycle of minimal length which is not spanned by a face, namely, the boundary of the deleted face. For any other (r,q)-polycycle $P(G)$, every edge cycle of minimal length bounds a face. Hence we have the following uniqueness theorem: if a planar graph G is the edge skeleton of an (r,q)-polycycle $P(G)$, then this polycycle is unique. More generally, *every planar graph G is the edge skeleton of at most one (r,q)-polycycle $P(G)$* (see [2]).

An (r,q)-polycycle $P(G)$ is said to be *non-extendible* if it is not a proper *subpolycycle* of a larger (r,q)-polycycle. In other words, an (r,q)-polycycle $P(G)$ is non-extendible if and only if every *superpolycycle* of $P(G)$ with the same parameters (r,q) coincides with $P(G)$.

As proved in [2], there are no finite non-extendible (r,q)-polycycles $P(G)$ with parabolic or hyperbolic parameters (r,q). For elliptic parameters (r,q) (that is, $(r,q) = (3,3)$, $(3,4)$, $(4,3)$, $(3,5)$, $(5,3)$), there are precisely 7 finite non-extendible (r,q)-polycycles $P(G)$: 5 *proper* (r,q)-polycycles (tetrahedron without a face,

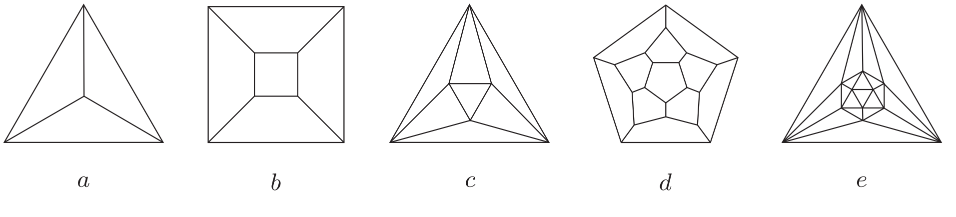
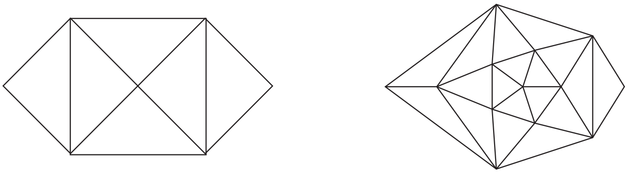


Figure 1

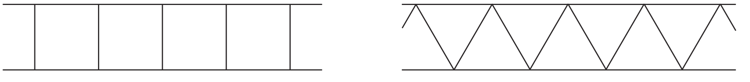
cube without a face, octahedron without a face, dodecahedron without a face, and icosahedron without a face; see Fig. 1), and 2 *improper* (r, q)-polycycles (split-vertex octahedron and split-vertex icosahedron; see Fig. 2).



a *b*

Figure 2

There are no other finite non-extendible (r, q)-polycycles. In [2] we omitted the proof of the absence of other finite non-extendible (r, q)-polycycles in the most difficult case $(r, q) = (3, 5)$. In this paper we give a relatively simple proof that the list above (in Figs. 1 and 2) is complete.



a *b*

Figure 3

There are no infinite $(3, 3)$-polycycles, precisely one infinite non-extendible $(4, 3)$-polycycle, and precisely one infinite non-extendible $(3, 4)$-polycycle (Fig. 3). The set of infinite non-extendible (r, q)-polycycles is uncountable whenever $(r, q) \neq (3, 3)$, $(3, 4)$ or $(4, 3)$ (see [2]).

Remark 1. Every proper (r, q)-polycycle $P(G)$ is a proper part of a regular partition (r^q), that is, $P(G) \subset (r^q)$. (An (r, q)-polycycle $P(G)$ is said to be proper if G is a partial subgraph of the edge skeleton of a regular partition (r^q).)

In what follows, we study (r, q)-polycycles $P(G)$ with $(r, q) = (3, 5)$ only. There are two finite non-extendible $(3, 5)$-polycycles: the icosahedron without a face (see Fig. 1, *e*) and the split-vertex icosahedron (see Fig. 2, *b*). Our purpose is to prove that there are no other finite non-extendible $(3, 5)$-polycycles.

§ 1. (3,5)-polycycles as two-dimensional triangulations

Consider a finite two-dimensional triangulation homeomorphic to a disc. It is a $(3, 5)$-polycycle if (see [2]) and only if (see [4]) every interior vertex belongs to 5 triangles and every boundary vertex belongs to 1, 2, 3 or 4 triangles. Two-dimensional cells of the triangulation are called *faces* of this $(3, 5)$-polycycle P. They are triangles: $r = 3$. The degree of every vertex in the edge skeleton of the triangulation does not exceed $q = 5$. If the triangulation has no interior vertices, then its edge skeleton is an *outerplanar* graph. A non-extendible $(3,5)$-polycycle is infinite whenever its edge skeleton is an outerplanar graph. In what follows we are interested only in finite non-extendible $(3,5)$-polycycles.

Suppose that a finite $(3,5)$-polycycle P contains at least two faces (that is, triangles). Then it has edges of two types: *interior* edges, which belong to two faces of P, and *boundary* edges, which belong to only one face of P. The boundary edges of the triangulation form the *boundary* of the $(3,5)$-polycycle P. The boundary of any finite $(3,5)$-polycycle P is homeomorphic to a circle, and the triangulation of P itself is homeomorphic to a disc.

Let us record the sequence of degrees of the boundary vertices of a $(3,5)$-polycycle P as we move along the boundary of the triangulation (these may take values 2, 3, 4, 5 only). Using this sequence, one can easily see whether or not the $(3,5)$-polycycle P is extendible (see below).

§ 2. Two ways to extend (3,5)-polycycles

There are two ways to extend $(3,5)$-polycycles, depending on the structure of the sequence of degrees of the boundary vertices.

The first way. Suppose that the sequence of boundary vertices of the $(3,5)$-polycycle P contains two neighbouring vertices whose degrees are smaller than 5. Then one can attach a new (hanging) triangle to P along the boundary edge connecting these two vertices. This is done by identifying the boundary edge with a side of the triangle. The other sides of the triangle are boundary edges of the extended $(3,5)$-polycycle. This extension of P results in replacing the sequence of boundary vertices of degrees $\dots, i, j, \dots$ (where $i < 5$ and $j < 5$) by a new sequence of boundary vertices of degrees $\dots, i+1, 2, j+1, \dots$ (where $i+1 \leqslant 5$ and $j+1 \leqslant 5$).

The second way. Suppose that the sequence of boundary vertices of the $(3,5)$-polycycle P contains a vertex of degree 5 such that the degrees of both neighbouring vertices are smaller than 5. Consider the boundary edges of P that are incident with the vertex of degree 5. We can identify them with two sides of a new triangle. The vertex of degree 5 becomes an interior vertex of the new $(3,5)$-polycycle, which is obtained from P by attaching the new triangle (the fifth at this vertex). This extension results in replacing the old sequence of boundary vertices of degrees $\dots, i, 5, j, \dots$ (where $i < 5$ and $j < 5$) by a new sequence of boundary vertices of degrees $\dots, i+1, j+1, \dots$ (where $i+1 \leqslant 5$ and $j+1 \leqslant 5$).

If neither of the previous conditions holds for the sequence of boundary vertices, then the finite $(3,5)$-polycycle P is non-extendible. In particular, if any closest boundary vertices of degree less than 5 are separated by at least two vertices of degree 5, then the $(3,5)$-polycycle P is non-extendible. This is why the split-vertex icosahedron (see Fig. 2, *b*) is non-extendible. Finally, if the degrees of all boundary vertices are equal to 5, then the $(3,5)$-polycycle P is again non-extendible. This is why the icosahedron without a face (see Fig. 1, *e*) is non-extendible.

Remark 2. One can consider (r,q)-polycycles on a closed surface of any genus instead of the plane. Then the surface of a Platonic solid without a face can be extended to the whole surface of the Platonic solid. However, this extension makes it homeomorphic to a sphere, not to a disc.

§ 3. Elementary (3,5)-polycycles

The *kernel* of a $(3,5)$-polycycle P is formed by all its vertices, edges and faces which are not incident with its boundary. A $(3,5)$-polycycle P is said to be *elementary* if, first, its kernel is connected and, second, P contains no proper subpolycycle with the same kernel. Any elementary $(3,5)$-polycycle with non-empty kernel contains at least 5 triangles. The unique elementary $(3,5)$-polycycle with empty kernel consists of one triangle. We denote it by d (see [2]). A list of all elementary $(3,5)$-polycycles is given in [2], Table 2.

A regular finite $(3,5)$-polycycle P with at least two faces has interior edges, not only boundary ones. An interior edge is said to be *through* if its endpoints lie on the boundary of P. There are only finitely many through edges in any finite $(3,5)$-polycycle P. If we cut P along all through edges, it decomposes into subpolycycles without through edges.

Lemma 1. *Any finite* $(3,5)$*-polycycle* P *without through edges is elementary.*

Proof. Suppose that P has a vertex of degree 2. Then both edges at this vertex are boundary for P. The third (closing) edge cannot be through for P. Hence P consists of only one triangle: it is the elementary polycycle d. Its kernel is empty.

Now suppose that there is a vertex whose degree exceeds 2. Then the degrees of all other vertices also exceed 2. Then every boundary vertex is incident with at least one interior edge of P. The second endpoint of this edge belongs to the kernel since P has no through edges. Therefore the kernel of P is non-empty. If the degree of the vertex is 4, then it is incident with a triangle of P. Consider the edge of this triangle opposite to the boundary vertex of P. This edge belongs to the kernel along with its endpoints. If the degree of the boundary vertex is 5, then P contains two such triangles. The polycycle P is homeomorphic to a disc, and its boundary is homeomorphic to a circle. All the triangles of P that are incident with the boundary are also incident with the kernel. In each triangle, we join the midpoints of the sides to get intervals parallel to the sides that lie in the boundary or in the kernel. These intervals form a circle which is disjoint from the kernel and encircles it. The kernel is connected (see the note below). The polycycle P is minimal among all $(3,5)$-polycycles with a given non-empty connected kernel. Hence, P is elementary. The lemma is proved.

Note. Let P be a finite $(3,5)$-polycycle without through edges, and suppose that the degrees of all the vertices of P exceed 2. Then the kernel of P is a *deformation retract* of P.

§ 4. Elementary summands

In general, an arbitrary $(3,5)$-polycycle P has many different elementary subpolycycles. The most important are those called *elementary summands* (see [2]). A triangle in P is a *trivial elementary summand* d if it is disjoint from the kernel of P. (It does not matter whether the kernel is empty or not.) All triangles in P that are incident with the same connected component of the kernel of P form a *non-trivial elementary summand* of P. The kernel of a non-trivial elementary summand is non-empty and coincides with a connected component of the kernel of P. Each triangle of P belongs to one and only one elementary summand. The decomposition of a $(3,5)$-polycycle P into elementary summands is unique (see [2]).

Lemma 2. *If we cut a* $(3,5)$*-polycycle* P *along all its through edges, then the resulting elementary polycycles are elementary summands of* P*.*

Proof. Consider any such elementary polycycle. Each of its boundary edges is either boundary for P or through for P. Hence the boundary of the elementary polycycle is disjoint from the kernel of P. If the elementary polycycle contains only one triangle of P, then its intersection with kernel of P is also empty. Hence the elementary polycycle (which is a triangle d in this case) is an elementary summand. But if the elementary polycycle contains several triangles, then the intersection of this elementary summand and the kernel of P is non-empty: it coincides with a connected component of the kernel of P (see the proof of Lemma 1). Hence the elementary polycycle is a non-trivial elementary summand. The lemma is proved.

Thus, every $(3,5)$-polycycle P decomposes uniquely into elementary summands. To obtain these summands, it suffices to cut P along all its through edges.

Remark 3. All boundary vertices of an elementary summand of a $(3,5)$-polycycle P belong to the boundary of P.

We note that an extension of a $(3,5)$-polycycle P in the first way (see above) results in adding a new elementary summand d to P while every old elementary summand remains an elementary summand. When we extend a $(3,5)$-polycycle P in the second way, the additional elementary polycycle d is united with all adjacent elementary summands of P to form a larger elementary summand of the extended $(3,5)$-polycycle.

§ 5. The adjacency graph of elementary summands

Any through edge divides a $(3,5)$-polycycle P into two subpolycycles in the same way as a diameter divides a disc into two half-discs. Each through edge of a $(3,5)$-polycycle P belongs to two elementary summands, which are adjacent along this edge. We now construct the so-called *adjacency graph* of elementary summands of an arbitrary $(3,5)$-polycycle P. Each elementary summand of P corresponds to a vertex of the adjacency graph. Two vertices of the adjacency graph are joined by an edge if and only if the corresponding elementary summands have a common edge.

Lemma 3. *The adjacency graph of elementary summands of any non-elementary* $(3,5)$*-polycycle* P *is a tree.*

Proof. Two vertices of the adjacency graph are joined by an edge if and only if the corresponding elementary summands are adjacent, that is, have a common edge (a through edge of the ambient $(3,5)$-polycycle P). This through edge divides P into two subpolycycles, whose intersection is precisely the through edge. Only two elementary summands of P are adjacent along this edge. No other pair of elementary summands in these subpolycycles (one from each subpolycycle) consists of adjacent summands. Hence each edge of the adjacency graph is a *bridge* (see [1], Russian p. 41), and so the adjacency graph is a *forest* (see [1], Russian p. 59).

Since P is a triangulation of a disc, the adjacency graph is connected. Hence the forest consists of only one connected component. Each component of a forest is a *tree* (see [1], Russian p. 48). The lemma is proved.

Remark 4. Any two vertices of a tree are connected by a unique simple (edge) chain (see [1], Theorem 4.1, (2)).

Remark 5. Every non-trivial tree has at least two hanging (terminal) vertices (see [1], Corollary 4.1, (a)).

§ 6. The gluing tree of elementary summands

As remarked above, we can obtain the elementary summands of a $(3,5)$-polycycle by cutting along all through edges. The adjacency graph of elementary summands is a tree. It is convenient to regard this tree as the result of gluing the elementary summands of the $(3,5)$-polycycle P. Let us consider this in more detail.

Take any pair of elementary summands of a $(3,5)$-polycycle P. Then only one of the following 3 possibilities holds: the elements of the pair are disjoint, or they have a common vertex, or they have a common edge (and are adjacent).

Remark 6. If we delete the endpoints of all through edges in an arbitrary $(3,5)$-polycycle P, then all non-adjacent elementary summands become disjoint.

Given a non-elementary $(3,5)$-polycycle P, we can pass from the triangulation to an enlarged decomposition (see [5], Russian p. 175) by taking the elementary summands of P for the enlarged cells. The edge skeleton of the enlarged decomposition is a planar graph H. The edges of H are only the through and boundary edges of P. We construct the graph geometrically dual to H (see [1], Russian p. 138) and delete the vertex (see [1], Russian p. 25) corresponding to the exterior face of H (see [1], Russian p. 127). We get a new graph,[3] which represents the adjacency graph of elementary summands. Since this graph is a tree, the enlarged decomposition may naturally be called the *gluing tree of elementary summands*. The gluing tree has at least two hanging elementary summands. One boundary edge of a hanging elementary summand is a through edge of the $(3,5)$-polycycle P. All other boundary edges of the hanging elementary summand are boundary edges of P.

Any two elementary summands of a $(3,5)$-polycycle P can be joined by a simple chain[4] of elementary summands in which every two neighbouring elementary summands are adjacent to each other. The intersection of two adjacent elementary summands of a $(3,5)$-polycycle P is always a through edge of the ambient polycycle P. This edge is *open* (see [2]) for both elementary summands, that is, the degrees of its endpoints are smaller than 5. In an elementary polycycle d (with empty kernel), all the three boundary edges are open. Consider an elementary summand which is not a triangle d. Its open edges are of two types. A *strongly* open edge of an elementary summand has both endpoints of degree 3. (Any elementary summand may be adjacent to any other elementary summand along a strongly open edge.) A *weakly* open edge of an elementary summand has endpoints of degrees 3

[3] This graph is a deformation retract of the $(3,5)$-polycycle P (see the note above).
[4] This chain is unique in the gluing tree (see Remark 4).

and 4 or 4 and 4. (Along a weakly open edge, an elementary summand may be adjacent only to an elementary summand d.)

Remark 7. If two open edges of an elementary summand of an arbitrary $(3,5)$-polycycle P have a common vertex of degree 4, then they cannot both be through edges.

It is convenient to denote an edge of a $(3,5)$-polycycle P by a sequence of two digits that belong to the set $\{2,3,4,5\}$ and are equal to the degrees of the endpoints of the edge. For example, the split-vertex icosahedron determines a non-extendible $(3,5)$-polycycle denoted by $a_3 \cup d$ (see [2], Table 4, and Fig. 2, b). It has one through edge, which divides $a_3 \cup d$ into two elementary summands: a_3 and d. We denote this edge by 44 in the ambient polycycle $a_3 \cup d$, by 33 in the elementary summand a_3, and by 22 in the elementary summand d.

§ 7. Maximal chains

Consider a finite non-elementary $(3,5)$-polycycle P. Choose any elementary summand of P, denote it by x_1, and take it for the initial summand. Any other elementary summand of P can be joined to the initial summand by a simple chain $x_1x_2\dots x_{n-2}x_{n-1}x_n$, where the neighbouring elementary summands x_j and x_{j+1} ($j=1,\dots,n-1$) are adjacent to each other along a common edge $x_j \cap x_{j+1}$. Since P is finite, the gluing tree of its elementary summands contains a chain of elementary summands of maximal length. Such a chain is said to be *maximal*.

Lemma 4. *Let $x_1x_2\dots x_{n-2}x_{n-1}x_n$ be a maximal chain. Then the elementary summand x_n is hanging.*

Proof. One boundary edge of x_n is a through edge of P. All other boundary edges of x_n are boundary edges of P. The proof is by contradiction. Assume that x_n has a boundary edge (different from $x_{n-1} \cap x_n$) which is not a boundary edge of P. Then P contains an elementary summand x_{n+1} adjacent to x_n and different from x_{n-1}. Consider the chain $x_1x_2\dots x_{n-2}x_{n-1}x_nx_{n+1}$ joining x_1 and x_{n+1}. It is the only such chain and is longer than the maximal chain $x_1x_2\dots x_{n-2}x_{n-1}x_n$. This contradicts the maximality of $x_1x_2\dots x_{n-2}x_{n-1}x_n$ among all chains starting at x_1 in the polycycle P. Hence the elementary summand x_n is hanging.

Remark 8. If P contains elementary summands $x'_n, x''_n, \dots$ that are adjacent to the penultimate elementary summand x_{n-1} of a maximal chain and are different from x_{n-2}, then $x'_n, x''_n, \dots$ are also hanging.

Thus the last elementary summand x_n of a maximal chain $x_1x_2\dots x_{n-2}x_{n-1}x_n$ (with a fixed initial point x_1) is always hanging. One boundary edge of x_n is a through edge of P. All other boundary edges of x_n are boundary edges of the ambient $(3,5)$-polycycle P.

§ 8. Description of the non-extendible (3,5)-polycycles

Before passing to an investigation of the influence of various elementary summands on the extendibility or non-extendibility of a $(3,5)$-polycycle P, we compile a list of elementary summands.

All elementary $(3,5)$-polycycles are listed in [2], Table 2 (see also [3], Fig. 6). The list consists of a finite set a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d and one infinite sequence $e_1, e_2, e_3, e_4, \dots, e_n, \dots$. The elementary polycycle a_1 is given by Fig. 1, e. Many other elementary polycycles that are used as elementary summands below are depicted in Figs. 4–15. (The elementary polycycle d should be included in the infinite sequence as its first element by putting $e_0 = d$. The sequence has two limits: the one-sided limit $e_{|N|} = b_1$ and the two-sided limit $e_{|Z|} = a_6$. The elementary polycycle a_6 is an infinite non-extendible $(3,5)$-polycycle.)

The general list [2], Table 2, contains elementary summands of all (possibly infinite) $(3,5)$-polycycles. In particular, [2], Table 2, includes the infinite elementary $(3,5)$-polycycles a_6 and b_1. We shall not consider them since we are interested only in finite $(3,5)$-polycycles.

Consider the finite $(3,5)$-polycycles possessing at least one elementary summand a_i, where $i = 1,2,3,4,5$ (see [2], Table 2). There are exactly 8 such polycycles: a_1, a_2, a_3, $a_3 \cup d$, a_4, a_5, $a_5 \cup d$ and $d \cup a_5 \cup d$. This list contains 6 extendible $(3,5)$-polycycles and the 2 non-extendible $(3,5)$-polycycles mentioned above: a_1, the *icosahedron without a face* (see Fig. 1, e), and $a_3 \cup d$, the *split-vertex icosahedron* (see Fig. 2, b).

To study all other finite $(3,5)$-polycycles, we must exclude the elementary polycycles a_1, a_2, a_3, a_4, a_5, a_6, and b_1 from the general list of elementary summands. Therefore we shall study only those finite $(3,5)$-polycycles whose elementary summands belong to the following infinite series:

$$b_2, b_3, b_4, c_1, c_2, c_3, c_4, d, e_1, e_2, e_3, e_4, \dots, e_n, \dots . \tag{1}$$

The first 12 terms of this series are represented in Figs. 4–15 below.

Lemma 5. *All the elementary* $(3,5)$*-polycycles belonging to the infinite series* (1) *are extendible.*

Proof. Every elementary $(3,5)$-polycycle in (1) has an open edge and is therefore extendible in the first way. (Open edges appear in all the elementary $(3,5)$-polycycles in [2], Table 2, except for the non-extendible polycycles a_1 and a_6.) The lemma is proved.

Furthermore, d is the only polycycle with empty kernel in (1). All three edges of d are open and pairwise adjacent. The following series consists of all the elementary polycycles of (1) except d:

$$b_2, b_3, b_4, c_1, c_2, c_3, c_4, e_1, e_2, e_3, e_4, \dots, e_n, \dots . \tag{2}$$

The elementary summands of the series (2) have non-empty kernels and possess non-adjacent open edges. Any two non-adjacent open edges of such an elementary summand may *simultaneously* be through edges of the ambient $(3,5)$-polycycle.

Lemma 6. *If at least one hanging elementary summand of a* $(3,5)$*-polycycle* P *belongs to the series* (2)*, then* P *is extendible.*

Proof. Along with any open edge, every elementary polycycle in (2) contains a non-adjacent open edge. (This can easily be verified by examining all the elementary polycycles in (2); see [2], Table 2. However, we do not need to prove this fact separately since it will also be checked in the course of the proof of Lemma 7.) If one open edge coincides with a through edge, then the second open edge is necessarily a boundary edge of P. We can attach a new triangle to P along this edge. Hence P is extendible in the first way. The lemma is proved.

The results of all further investigations of finite (non-)extendible $(3,5)$-polycycles can be compressed into the following key lemma.

Lemma 7. *Every finite* $(3,5)$*-polycycle* P *whose elementary summands belong to the series* (1) *is extendible.*

Proof. Consider any finite $(3,5)$-polycycle P consisting of elementary summands belonging to the series (1). If P is elementary, then it is extendible by Lemma 5. If P is not elementary and every hanging elementary summand of P belongs to the sequence (2), then P is extendible by Lemma 6. It remains to consider the case when P is not elementary and all hanging elementary summands are triangles d. We take one of the triangles d as the initial element: $x_1 = d$. Then P contains an elementary summand whose distance to x_1 is maximal. If there are several elementary summands with this property, we take any of them. We denote it by x_n and consider a maximal chain $x_1x_2\dots x_{n-2}x_{n-1}x_n$. The last term x_n of a maximal chain is always hanging. Hence $x_n = d$. Since the chain $x_1x_2\dots x_{n-2}x_{n-1}x_n$ is not closed, we have $x_n \neq x_1$ (when $n \neq 1$).

We consider several cases, depending on the element of (1) which appears as the penultimate term x_{n-1} of the maximal chain

$$x_1x_2\dots x_{n-2}x_{n-1}x_n.$$

In particular either x_{n-1} belongs to (2) or $x_{n-1} = d$.

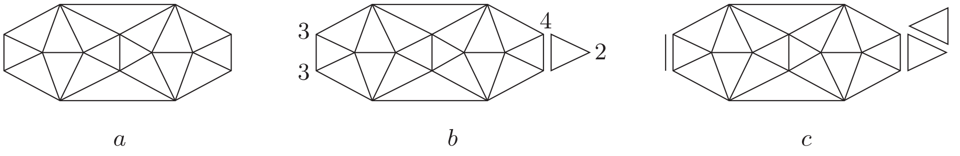


Figure 4

Case $x_{n-1} = b_2$ (Fig. 4). The elementary summand $x_{n-1} = b_2$ has two open edges 33. They are non-adjacent and are through edges of P, namely, $x_{n-2} \cap x_{n-1}$ and $x_{n-1} \cap x_n$. All other boundary edges of the elementary summand $x_{n-1} = b_2$ are boundary edges of P. The hanging elementary summand $x_n = d$ has a boundary edge 42 in P. Therefore P is extendible in the first way.

The elementary summand $x_{n-1} = b_2$ is shown separately in Fig. 4, a. The subpolycycle $x_{n-1} \cup x_n = b_2 \cup d$ of P with boundary edge 42 is shown in Fig. 4, b. It is cut along the through edge $x_{n-1} \cap x_n$ into elementary summands $x_{n-1} = b_2$ and $x_n = d$. The subpolycycle $x_{n-1} \cup x_n \cup x_{n+1} = b_2 \cup d \cup d$ of the extended $(3,5)$-polycycle $P \cup x_{n+1}$ is shown in Fig. 4, c. The polycycle $x_{n-1} \cup x_n \cup x_{n+1}$ is obtained from $P \cup x_{n+1}$ by cutting along the through edge $x_{n-2} \cap x_{n-1}$.

The polycycle $x_{n-1} \cup x_n \cup x_{n+1}$ is cut into 3 elementary summands: $x_{n-1} = b_2$, $x_n = d$, $x_{n+1} = d$.

All this is completely explained by Fig. 4, *b*. Therefore we shall need only this figure to illustrate everything said above.

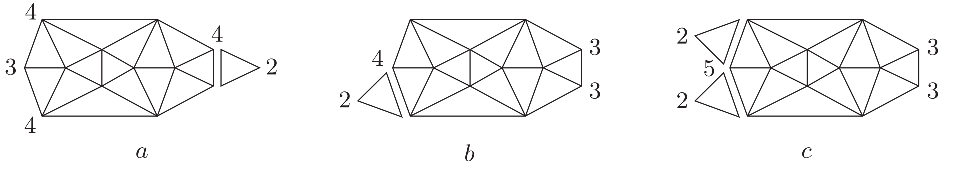


Figure 5

Case $x_{n-1} = b_3$ (Fig. 5). The elementary summand $x_{n-1} = b_3$ has 3 open edges: 43, 34 and 33. At least two of them are through edges of P. The remaining boundary edges of $x_{n-1} = b_3$ are boundary edges of P. If the open edge 33 of $x_{n-1} = b_3$ is a boundary edge of P, then P is extendible in the first way. If the open edge 33 of $x_{n-1} = b_3$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has boundary edge 42 in P and, therefore, P is extendible in the first way (see Fig. 5, *a*). Now suppose that the open edge 33 of $x_{n-1} = b_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Then we have one of the following two cases. If the open edge 43 is a boundary edge and the open edge 34 coincides with the through edge $x_{n-1} \cap x_n$ of P, then P is extendible in the first way since the hanging elementary summand $x_n = d$ has boundary edge 42 in P (see Fig. 5, *b*). If the open edges 43 and 34 coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$ of P, then P is extendible in the second way since the boundary vertex $x_n \cap x'_n$ of P is of degree 5 while the degrees of the neighbouring boundary vertices are equal to 2 (see Fig. 5, *c*).

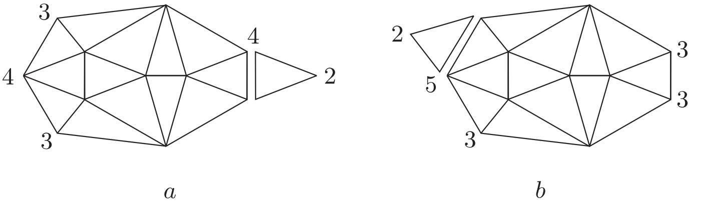


Figure 6

Case $x_{n-1} = b_4$ (Fig. 6). The elementary summand $x_{n-1} = b_4$ contains 3 open edges. Only two of them are through edges of P. If the open edge 33 of this elementary summand coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has boundary edge 42 in P, whence P is extendible in the first way (see Fig. 6, *a*). If the open edge 43 of $x_{n-1} = b_4$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has a boundary edge 25 in P. It is adjacent to the other boundary edge 53, whence P is extendible in the second way (see Fig. 6, *b*).

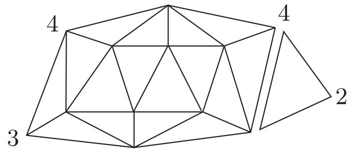


Figure 7

Case $x_{n-1} = c_1$ (Fig. 7). The elementary summand $x_{n-1} = c_1$ has two open edges. Both are through edges of P. One of them coincides with the through edge $x_{n-1} \cap x_n$. The hanging elementary summand $x_n = d$ has a boundary edge 42 in P, whence P is extendible in the first way (see Fig. 7).

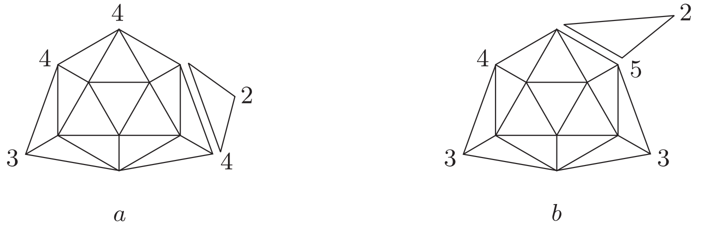


Figure 8

Case $x_{n-1} = c_2$ (Fig. 8). The elementary summand $x_{n-1} = c_2$ has 4 open edges. However, only two of them may coincide with through edges of P. If the open edge 34 of $x_{n-1} = c_2$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has a boundary edge 42 in P, and P is extendible in the first way (see Fig. 8, a). If the open edge 44 of $x_{n-1} = c_2$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has a boundary edge 25 in P. It is adjacent to a boundary edge 53, whence P is extendible in the second way (see Fig. 8, b).

Case $x_{n-1} = c_3$ (Fig. 9). The elementary summand $x_{n-1} = c_3$ has 6 open edges. At most 4 of them may coincide with through edges of P. This is because the adjacent edges 44 and 43 cannot both be through.

Suppose that the open edge 44 of $x_{n-1} = c_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Then we have one of the following two cases.

If the opposite open edge 44 of $x_{n-1} = c_3$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has boundary edge 25 in P. It is adjacent to another boundary edge 53, whence P is extendible in the second way (see Fig. 9, a).

Suppose that the open edge 43 of $x_{n-1} = c_3$ is not adjacent to $x_{n-2} \cap x_{n-1}$ and coincides with the through edge $x_{n-1} \cap x_n$ of P. Then the hanging elementary summand $x_n = d$ has a boundary edge 24 in P and, therefore, P is extendible in the first way (see Fig. 9, b).

Now suppose that the open edge 43 of $x_{n-1} = c_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Then we have one of the following 3 cases.

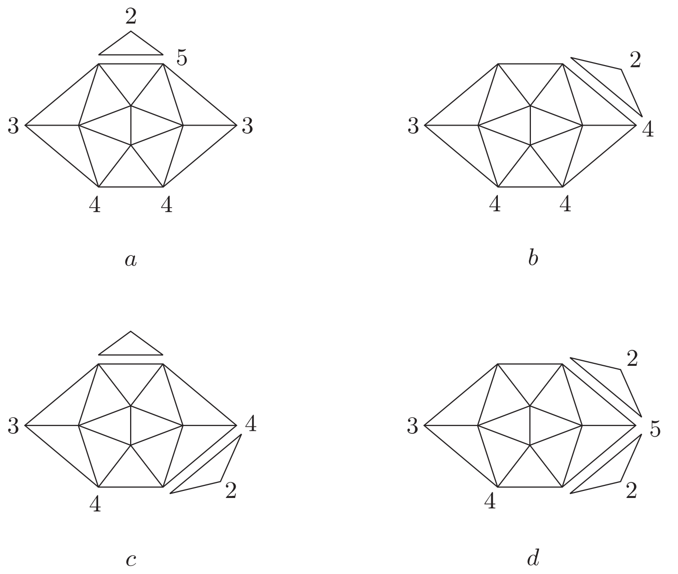


Figure 9

If the open edges 43 and 34 of $x_{n-1} = c_3$ are not adjacent to $x_{n-2} \cap x_{n-1}$ and are boundary edges of P, then at least one of them (closest to $x_{n-2} \cap x_{n-1}$) has vertices of degrees 4 and 3 in P. Hence P is extendible in the first way.

If only one open edge 43 of $x_{n-1} = c_3$ is not adjacent to $x_{n-2} \cap x_{n-1}$ and coincides with the through edge $x_{n-1} \cap x_n$ of P, then the hanging elementary summand $x_n = d$ has a boundary edge 42 in P. Hence P is extendible in the first way (see Fig. 9, b and c).

If open edges 43 and 34 of $x_{n-1} = c_3$ are not adjacent to $x_{n-2} \cap x_{n-1}$ and coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$ of P, then the intersection of the hanging elementary summands $x_n = d$ and $x'_n = d$ of P is a boundary vertex of degree 5 which is adjacent to two boundary vertices of degree 2. Hence P is extendible in the second way (see Fig. 9, d).

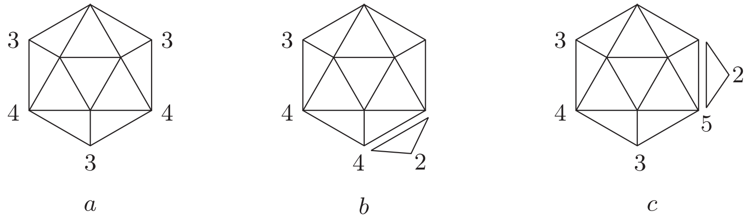


Figure 10

Case $x_{n-1} = c_4$ (Fig. 10). The elementary summand $x_{n-1} = c_4$ has 6 open edges. At most 3 of them may coincide with through edges of P. This is because boundary edges cannot be simultaneously through if they are adjacent to a vertex of degree 4.

Suppose that the (left vertical) edge 34 coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Consider the opposite (right vertical) edge and the (lower right) edge adjacent to it at a vertex of degree 4. One of the following 3 possibilities holds. If both open edges are boundary edges of P, then P is extendible in the first way since one can attach a new elementary summand d to the old $(3, 5)$-polycycle P along the lower right edge 34 (Fig. 10, a) as shown at Fig. 10, b. If the lower right open edge of the elementary summand $x_{n-1} = c_4$ coincides with the through edge $x_{n-1} \cap x_n$ of P, then the elementary summand $x_n = d$ has a boundary edge 42 in P and, therefore, P is extendible in the first way (see Fig. 10, b). If the vertical right open edge coincides with the through edge $x_{n-1} \cap x_n$ of P, then the elementary summand $x_n = d$ has a boundary edge 52 in P. This edge is adjacent to another boundary edge 53 and, therefore, P is extendible in the second way (see Fig. 10, c).

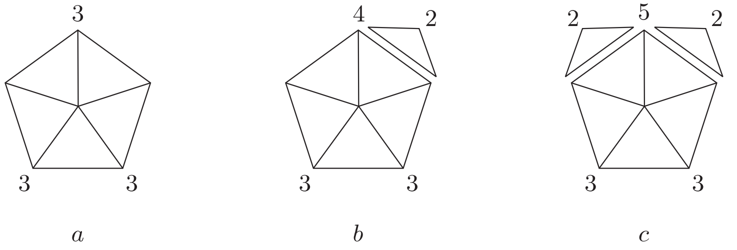


Figure 11

Case $x_{n-1} = e_1$ (Fig. 11). The elementary summand $x_{n-1} = e_1$ has 5 open edges. All of them may be through edges of P. One of the open edges 33 of $x_{n-1} = e_1$ coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. (This is the lower edge in Fig. 11.) Consider the vertex opposite to this edge in $x_{n-1} = e_1$. Its degree is 3 (see Fig. 11, a), and there are two open edges of $x_{n-1} = e_1$ at this vertex. For these edges, we have one of the following 3 cases. If both of them are boundary edges of P, then at least one of them (which is adjacent to the through edge $x_{n-1} \cap x_n$) has endpoints of degrees 3 and 4, whence the P is extendible in the first way. If one of them is boundary while the other coincides with the through edge $x_{n-1} \cap x_n$, then the hanging elementary summand $x_n = d$ has a boundary edge 42 in P, whence P is extendible in the first way (see Fig. 11, b). Finally, if the edges coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$, then P is extendible in the second way (see Fig. 11, c).

Case $x_{n-1} = e_2$ (Fig. 12). The elementary summand $x_{n-1} = e_2$ has 6 open edges: two strongly open edges 33 and four weakly open edges 34. At most 4 of them are through. Suppose that one of two adjacent open edges 33 or 34 coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Consider the other pair of adjacent edges 33 and 34 of the elementary summand $x_{n-1} = e_2$. (They are not adjacent to the previous two edges; see Fig. 12, a on the right.) If both of them are boundary edges of P, then one of them (the edge 33 of $x_{n-1} = e_2$) is a boundary edge 33 or 34 in P (if the upper horizontal open edge coincides with the through edge $x_{n-1} \cap x_n$) and, therefore, P is extendible in the first way. If one of them is a boundary edge and the second coincides with the through edge $x_{n-1} \cap x_n$, then the hanging elementary summand $x_n = d$ has a boundary edge 24 in P and, therefore, P is extendible in the first way (see Fig. 12, b and c). If they coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$, then P is extendible in the second way (see Fig. 12, d).

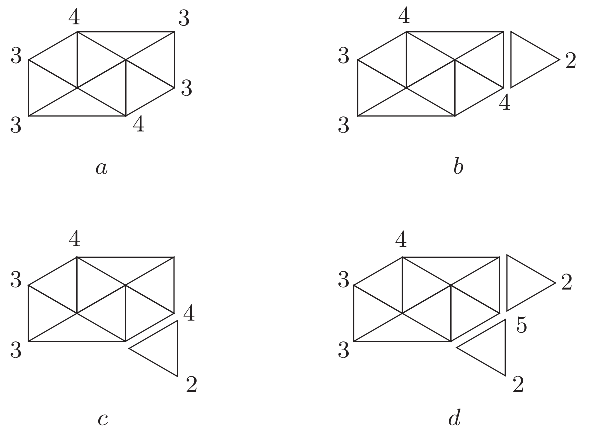


Figure 12

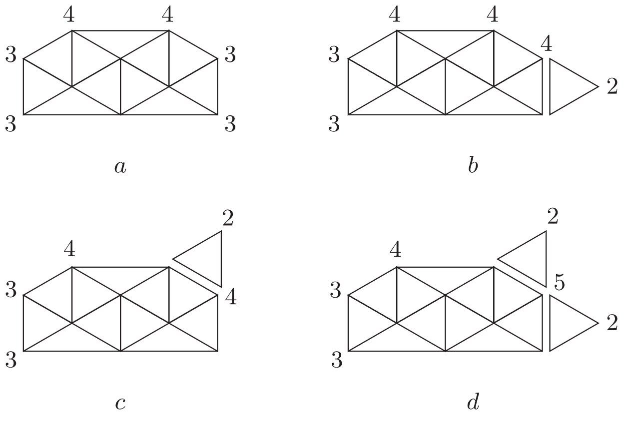


Figure 13

Case $x_{n-1} = e_3$ (Fig. 13). The elementary summand $x_{n-1} = e_3$ has 5 open edges. At most 4 of them are through edges. Suppose that the vertical left open edge 33 or the adjacent upper edge 34 of the elementary summand $x_{n-1} = e_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$. Then we consider the edges 33 and 34 opposite to them. If they are boundary edges of P, then the vertical right edge (see Fig. 13, a) has endpoints of degrees 3 and 3 in P and, therefore, P is extendible in the first way. If one of them is a boundary edge while the second coincides with the through edge $x_{n-1} \cap x_n$, then the hanging elementary summand $x_n = d$ has a boundary edge 42 in P and, therefore, P is extendible in the first way (see Figs. 13, b and c). If they coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$, then P is extendible in the second way (see Fig. 13, d).

Now suppose that the open edge 44 of the elementary summand $x_{n-1} = e_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$ of P. Then at least one open edge 33 of $x_{n-1} = e_3$ coincides with the through edge $x_{n-1} \cap x_n$. The hanging elementary summand $x_n = d$ has a boundary edge 42 in P, whence P is extendible in the first way (Fig. 13, b).

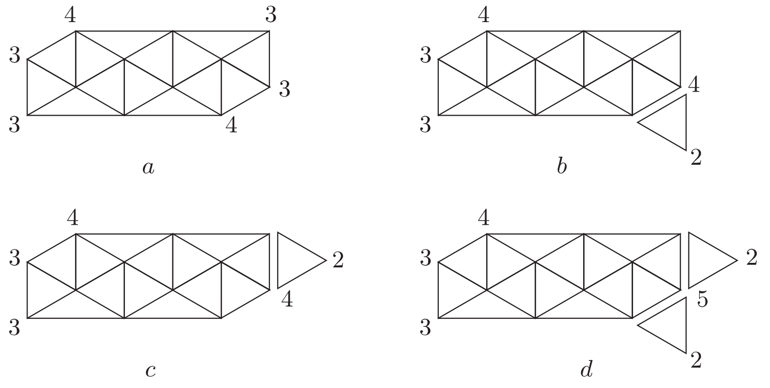


Figure 14

Case $x_{n-1} = e_4$ (Fig. 14). The elementary summand $x_{n-1} = e_4$ has 4 open edges. All of them may be through edges. Suppose that the vertical left open edge 33 or the adjacent edge 34 of $x_{n-1} = e_3$ coincides with the through edge $x_{n-2} \cap x_{n-1}$. Then we consider their opposite edges 33 and 34. If both of them are boundary edges in P, then the degrees of their endpoints remain the same in P (see Fig. 14, a) and, therefore, P is extendible in the first way. If one of them is a boundary edge while the second coincides with the through edge $x_{n-1} \cap x_n$, then the hanging elementary summand $x_n = d$ has a boundary edge 42 in P and, therefore, P is extendible in the first way (see Figs. 14, b and c). If the two edges coincide with the through edges $x_{n-1} \cap x_n$ and $x_{n-1} \cap x'_n$, then P is extendible in the second way (see Fig. 14, d).

All the arguments used in the case $x_{n-1} = e_4$ can be repeated verbatim in each of the cases $x_{n-1} = e_{k+4}$ for any $k \in \mathbb{N}$.

We have studied all cases when the elementary summand x_{n-1} coincides with a term of the series (2). It remains to consider the case when x_{n-1} coincides with the term d of the series (1).

Case $x_{n-1} = d$ (Fig. 15). The elementary summand $x_{n-1} = d$ has 3 open edges 22 (see Fig. 15, a). For every $n \geqslant 3$ one edge of the triangle d coincides with the through edge $x_{n-2} \cap x_{n-1}$ and another coincides with the through edge $x_{n-1} \cap x_n$. If the third edge of d is a boundary edge of P, then the hanging elementary summand $x_n = d$ has a boundary edge 23 in P and, therefore, P is extendible in the first way (see Fig. 15, b). If the third edge of d coincides with a through edge $x_{n-1} \cap x'_n$ of P, then the hanging elementary summand $x_n = d$ has a boundary edge 24 in P.

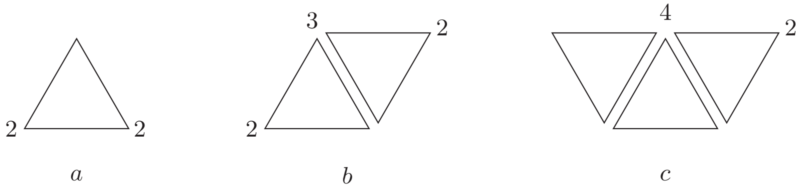


Figure 15

(The hanging elementary summands $x_n = d$ and $x'_n = d$ have a common vertex $x_n \cap x'_n$ of degree 4.) Hence P is extendible in the first way (see Fig. 15, c).

There are two more cases that occur when $x_{n-1} = d$ but are absent when x_{n-1} coincides with another (different from d) term of (1). Namely, if $n = 2$, then P is a subpolycycle of a polycycle consisting of 4 triangles, 3 of which are adjacent to the fourth triangle along its 3 sides. Hence the degrees of all vertices of P do not exceed 4 and, therefore, P is extendible in the first way. If $n = 1$ (this is the only case when we have $x_n = x_1$), then P is extendible by Lemma 5. The lemma is proved.

As a result, we get the following main theorem.

Theorem. *There are only two finite non-extendible* $(3,5)$*-polycycles: the icosahedron without a face, and the split-vertex icosahedron.*

Corollary. *There are only* 7 *finite non-extendible* (r,q)*-polycycles. Five of them are proper* (*the tetrahedron without a face, the cube without a face, the octahedron without a face, the dodecahedron without a face and the tetrahedron without a face; see Fig.* 1), *and two of them are improper* (*the split-vertex octahedron and the split-vertex icosahedron; see Fig.* 2).

This follows from the theorem and results in [2].

Remark 9. Only two finite non-extendible (r,q)-polycycles have hanging elementary summands. These summands are d and a_3 for the split-vertex icosahedron and two triangles for the split-vertex octahedron.

Remark 10. Only d can be a hanging elementary summand in a finite or infinite outerplanar $(3,5)$-polycycle P (here all elementary summands are d).

Remark 11. Every non-extendible outerplanar $(3,5)$-polycycle P is infinite.

Remark 12. Only d and b_1 can be hanging elementary summands in an [5] infinite non-extendible $(3,5)$-polycycle P.

[5] For any given t_1, t_2 with $0 \leqslant t_i \leqslant \infty$ $(i = 1,2)$ there is an uncountable set of infinite non-extendible $(3,5)$-polycycles $P(G)$ with t_1 hanging elementary summands d and t_2 hanging elementary summands b_1.

Bibliography

[1] F. Harary, *Graph theory*, Addison–Wesley, Reading, MA 1969; Russian transl., Mir, Moscow 1973.

[2] M. Deza and M. I. Shtogrin, "Extremal and non-extendible polycycles", *Trudy Mat. Inst. Steklov.* **239** (2002), 127–145; English transl., *Proc. Steklov Inst. Math.* **239** (2002), 117–135.

[3] M. Deza and M. Shtogrin, "Clusters of cycles", *J. Geom. Phys.* **40**:3–4 (2002), 302–319.

[4] M. Deza and M. I. Shtogrin, "A metric of constant curvature on polycycles", *Mat. Zametki* **78** (2005), 223–233; English transl., *Math. Notes* **78** (2005), 204–212.

[5] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig–Berlin 1934; Russian transl., Scientific Research Centre "Regular and Chaotic Dynamics", Izhevsk 2001.

[6] S. P. Novikov, *Topology*, Institute of Computer Studies, Moscow–Izhevsk 2002. (Russian)

M. Deza
E-mail: Michel.Deza@ens.fr

Received 2/AUG/05
Translated by A. V. DOMRIN

S. V. Shpectorov
E-mail: sergey@bgnet.bgsu.edu

M. I. Shtogrin
E-mail: stogrin@mi.ras.ru