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Archimedean polycycles

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Any two-dimensional manifold pasted together out of r-gons so that the degrees of interior vertices are equal to q and the degrees of exterior vertices are $\leqslant q$, where $r \geqslant 3$, $q \geqslant 3$, will be called an (r,q)-*polycycle*. We shall say that an (r,q)-polycycle is *ordinary* if it is planar and simply connected. In [1]-[6] we studied ordinary (not just finite but also infinite) polycycles. Each of these is a cellular complex, see [6] (the cells are closed [7]). We call an (r,q)-polycycle P, respectively, i-*isogonal*, j-*isohedral*, and k-*isotoxal* (briefly, i-IG, j-IH, and k-IT) if the group $\operatorname{Aut} P$ has exactly i orbits on the vertices, j orbits on the faces, and k orbits on the edges. Clearly, 1-IG implies j-IH and k-IT with $j,k \leqslant q$, 1-IH implies k-IT and i-IG with $k,i \leqslant r$, and 1-IT implies i-IG and j-IH with $i,j \leqslant 2$. Along with an (r,q)-polycycle P, an unbranched covering $\widetilde{P}$ is also an (r,q)-polycycle. The covering map $\varphi\colon \widetilde{P} \to P$ and a non-trivial automorphism $h\colon P \to P$ generate the covering map $h\varphi$ (first φ, then h). If the covering is universal, that is, $\widetilde{P} = \widehat{P}$, then by its uniqueness it follows that there is an automorphism $\widehat{h}\colon \widehat{P} \to \widehat{P}$ such that $h\varphi = \varphi\widehat{h}$. We shall represent P as a factorisation of the universal covering $\widehat{P}$ with respect to its translation group $\widehat{N}$. We find that $\operatorname{Aut}P \simeq \widehat{H}/\widehat{N}$, where $\widehat{H}$ is the normaliser of $\widehat{N}$ in the group $\operatorname{Aut}\widehat{P}$. The polycycle P (with respect to the group $H = \operatorname{Aut} P$) is i-IG, j-IH, and k-IT with the same values i, j, k as the polycycle $\widehat{P}$ with respect to the group $\widehat{H} \subseteq \operatorname{Aut}\widehat{P}$. We call an (r,q)-polycycle P *proper* if it is simultaneously 1-IG, 1-IH and 1-IT. It is clear that an individual r-gon is an ordinary proper (r,q)-polycycle with special parameter $q=2$ (a monocycle). All partitions (r^q) are proper (r,q)-polycycles, which are ordinary in the case of the plane and non-ordinary for the sphere.[1] We shall call them *trivial* polycycles.

Theorem 1. *All vertices of a proper non-trivial polycycle are interior vertices.*

Only 9 proper polycycles, namely the 5 Platonic bodies and 4 partitions of the projective plane can be covered by the sphere. The universal covering of any other (r,q)-polycycle is an ordinary polycycle. In particular, if P is an ordinary polycycle, then $\widehat{P} = P$. But if an (r,q)-polycycle P is not ordinary nor one of the indicated 9 polycycles, then the universal covering $\widehat{P}$, its translation group $\widehat{N}$ and the group $\operatorname{Aut}\widehat{P}$ are infinite.

The present note is devoted to *Archimedean*, that is, 1-isogonal (r,q)-polycycles.

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[1]By (r^q) we denote the proper partitions of the sphere and the Euclidean or Lobachevskii plane.

Theorem 2. *The edge skeleton of an arbitrary ordinary non-trivial Archimedean (r,q)-polycycle is an infinite outerplanar graph.*

If all the vertices of a Archimedean (r,q)-polycycle P are interior vertices (this includes all regular polycycles), then it constitutes a finite or an infinite surface without boundary. Its universal covering $\widehat{P}$ has the form (r^q). The group $\mathrm{Aut}(r^q)$ is the Coxeter group $T^*(2,q,r)$. We shall now proceed to enumeration. Let $\widehat{H}$ be a subgroup of $T^*(2,q,r)$ transitive on the vertices of the partition (r^q), and let $\widehat{N}$ be a torsion-free normal subgroup of this group. Then the factorisation of the polyhedron $\widehat{P}=(r^q)$ with respect to $\widehat{N}$ is an (r,q)-polycycle P. In this way one can obtain all P's. The surfaces of 'regular toroids', given in [8], serve as examples of such (r,q)-polycycles: those in Figs. 2 (the Czaszar polyhedron), 8, 9a, 16 (the Heawood map) are all 1-IG, 1-IH, 1-IT, those in Figs. 1, 7, 9b, 10, 11, 12 are 1-IG, 1-IH, 2-IT, and those in Figs. 13 and 14 are 1-IG, 1-IH, 3-IT.

If all the vertices of an Archimedean (r,q)-polycycle P are exterior vertices (all such non-trivial polycycles are k-IT with $k\geqslant 2$), then the polycycle is a finite or infinite surface with boundary. A connected component of the boundary is a cycle C_n of length n, where $n=r$ or $r\neq n\leqslant\infty$. By filling the holes in the polycycle by n-gons, for $n=r$ we obtain the aforesaid Archimedean (r,q)-polycycle without boundary having the universal covering (r^q), while for $r\neq n\leqslant\infty$ we obtain an Archimedean polyhedron without boundary, the universal covering of which is an Archimedean partition of the sphere or the plane. Conversely, the removal of faces brings us back to the original polycycle. The removal of faces can be applied to construct polycycles of this kind. Thus, for example, one of the Archimedean $(4,4)$-polycycles can be obtained from the Archimedean polyhedron (3.4^3) by removing triangular faces. Two Archimedean $(4,4)$-polycycles can be obtained from the square lattice (4^4) by removing squares whose centres constitute a square or rhombic sublattice of index 4. Universal coverings give two more Archimedean $(4,4)$-polycycles. Among their factorisations are all proper $(4,4)$-polycycles with boundary.

Theorem 3. *Each ordinary non-trivial* 1-IH*, but not* 1-IT*, Archimedean (r,q)-polycycle is a universal covering of a proper partition pierced by cutting off vertices or a lateral surface of an antiprism.*

Among non-trivial Archimedean (r,q)-polycycles with $(r,q)=(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, $(5,3)$ only 4 are closed. These are the factorisations of the surface of the octahedron, cube, icosahedron, and dodecahedron with respect to central symmetry. All others have a boundary.

Theorem 4. *All non-trivial Archimedean $(4,3)$-polycycles with boundary are listed below*:

(i) *the lateral surfaces of n-gonal prisms, $1\leqslant n<\infty$, and their universal covering, $n=\infty$ (for $n=1$ and $n=2$ these are not partitions into squares, for $n=4$ the polycycle is extendable*);

(ii) *the factorisations of the polycycles listed under* (i) *for n even with respect to central symmetry (non-orientable, for $n=2$ and $n=4$ these are not partitions into squares, for $n=4$ the polycycle is extendable*).

Theorem 5. *All non-trivial Archimedean $(3,4)$-polycycles with boundary are listed below*:

(i) *the lateral surfaces of n-gonal antiprisms and their universal covering (for $n=1$ and $n=2$ these are not triangulations, for $n=3$ the polycycle is extendable*);

(ii) *the factorisations of the polycycles listed under* (i) *for $n\equiv 1 \pmod 2$ with respect to central symmetry (non-orientable, for $n=1$ and $n=3$ these are not triangulations, for $n=3$ the polycycle is extendable*).

Theorem 6. *All non-trivial Archimedean $(3,5)$-polycycles with boundary are listed below*:

(i) *the twisted*[2] *partitions (n^3) with n-gonal faces removed, $2\leqslant n<\infty$, and their universal covering, $n=\infty$ (for $n=3$ the polycycle is extendable*);

(ii) *the factorisations of the polycycles listed under* (i) *for $6\leqslant n\leqslant\infty$ (which are orientable*).

[2]The cases $n=3,4,5$ correspond to the known twisted tetrahedron, cube, and dodecahedron.

The factorisation of the dodecahedron is the unique non-trivial Archimedean $(5,3)$-polycycle. There are no non-trivial Archimedean $(3,3)$-polycycles. Any Archimedean (r,q)-polycycle with boundary admits each of the three metrics of constant curvature, namely the locally spherical, locally Euclidean, and locally hyperbolic metrics. A closed polycycle admits only one.

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