

# Voronoi's conjecture and space tiling zonotopes

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## Abstract

Voronoi conjectured that any parallelotope is affinely equivalent to a Voronoi polytope. A parallelotope is defined by a set of  $m$  facet vectors  $p_i$  and defines a set of  $m$  lattice vectors  $t_i$ ,  $1 \leq i \leq m$ . We show that Voronoi's conjecture is true for an  $n$ -dimensional parallelotope  $P$  if and only if there exist scalars  $\gamma_i$  and a positive definite  $n \times n$  matrix  $Q$  such that  $\gamma_i p_i = Q t_i$  for all  $i$ . In this case the quadratic form  $f(x) = x^T Q x$  is the metric form of  $P$ .

As an example, we consider in detail the case of a zonotopal parallelotope. We show that  $Q = (Z_\beta Z_\beta^T)^{-1}$  for a zonotopal parallelotope  $P(Z)$  which is the Minkowski sum of column vectors  $z_j$  of the  $n \times r$  matrix  $Z$ . Columns of the matrix  $Z_\beta$  are the vectors  $\sqrt{2\beta_j} z_j$ , where the scalars  $\beta_j$ ,  $1 \leq j \leq r$ , are such that the system of vectors  $\{\beta_j z_j : 1 \leq j \leq r\}$  is unimodular.  $P(Z)$  defines a dicing lattice which is the set of intersection points of the dicing family of hyperplanes  $H(j, k) = \{x : x^T(\beta_j Q z_j) = k\}$ , where  $k$  takes all integer values and  $1 \leq j \leq r$ .

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## 1 Introduction

A *parallelotope* is a convex polytope which fills the space facet to facet by its translation copies without intersecting by inner points. Such a filling by parallelotopes is a *tiling*. Any parallelotope  $P$  of dimension  $n$  has the following three properties:

(*symP*)  $P$  is centrally symmetric;

(*symF*) each facet of  $P$  is centrally symmetric;

(*proj*) the projection of  $P$  along any  $(n-2)$ -face is either a parallelogram or a centrally symmetric hexagon.

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Venkov [Ve54] (and, independently, McMullen [McM80]) proved that the above three properties are sufficient for a polytope to be a parallelotope. Aleksandrov [Al54], knowing the Venkov's result and using his main idea, simplified the proof of Venkov and generalized his result.

Coxeter [Cox62] noted that the condition (*proj*) is necessarily and sufficient for a zonotope to be a parallelotope. But he considered only dimensions  $n \leq 3$ . Shephard [Sh75] proved this assertion for  $n = 4$ . Coxeter remarked also in [Cox62] that the condition (*proj*) implies that, for  $n \leq 3$ , every parallelotope is a zonotope.

A parallelotope is called *k-primitive* if each its *k*-face (i.e. *k*-dimensional face) belongs to a minimal possible number  $n - k + 1$  of parallelotopes of its tiling. Besides, a *k*-primitivity implies the  $(k + 1)$ -primitivity. A 0-primitive parallelotope is simply called *primitive*. Obviously, any parallelotope is  $(n - 1)$ -primitive.

The definition of  $(n - 2)$ -primitivity shows that each  $(n - 2)$ -face of a parallelotope belongs to at least 3 parallelotopes of its tiling. The property (*proj*) above says that each  $(n - 2)$ -face belongs to at most 4 parallelotopes. Hence the property (*proj*) can be reformulated in the following form:

*(fhyp)* each  $(n - 2)$ -face of a parallelotope  $P$  is contained in at most 3 affine hyperplanes supporting  $(n - 1)$ -faces of its tiling.

A zonotope  $P(Z)$  is the Minkowski sum of a set  $Z \subset \mathbf{R}^n$  of  $n$ -vectors. Every zonotope and all its faces are centrally symmetric. Hence, every zonotope satisfies the properties (*symP*) and (*symF*) of a parallelotopes. This implies that, for every  $k$ ,  $1 \leq k \leq n - 1$ , the family of *k*-faces of  $P(Z)$  is partitioned into *zones* of mutually parallel *k*-faces. For a zonotope  $P(Z)$ , every its zone of *k*-faces is in a one-to-one correspondence with a *k*-flat of the matroid  $M(Z)$  represented by the set  $Z$ . Hence for a zonotope  $P(Z)$  the property (*fhyp*) is equivalent to the following property of the matroid  $M(Z)$ :

*(bin)* every  $(n - 2)$ -flat of  $M(Z)$  is contained in at most three  $(n - 1)$ -flats.

This condition on a matroid is equivalent to its binarity. Since  $M(Z)$  is trivially represented over the real field  $\mathbf{R}$ ,  $M(Z)$  is regular. This implies that

*(mzr)* a zonotope  $P(Z)$  is a parallelotope if and only if the matroid  $M(Z)$  is regular.

The regularity of  $M(Z)$  implies that, for  $z \in Z$ , there is a positive scalar  $\beta_z$  such that the system of vectors  $\{\beta_z z : z \in Z\}$  is unimodular.

There is a special well known case of a parallelotope, namely, the Voronoi polytope  $P_V(t_0)$  related to a point  $t_0$  of a lattice  $L$ . It is the set of all points of the space that are at least as near to  $t_0$  as to any other lattice point. Here the distance between two points  $t$  and  $t_0$  is the Euclidean norm  $(t - t_0)^2$  of the vector  $t - t_0$ . Using an arbitrary positive definite quadratic form  $f$  as a new norm  $f(t - t_0)$  of the vector  $t - t_0$ , we obtain a parallelotope  $P_f$ , which we call the *Voronoi polytope with respect to the form f*. But any quadratic form  $f(x)$  is affinely equivalent to the Euclidean form  $x^2$ . The corresponding affine transformation maps  $P_f$  into  $P_V$ .

The famous conjecture of Voronoi is that

*(Voc)* any parallelotope  $P$  is affinely equivalent to a Voronoi polytope  $P_V$ .

A parallelotope is defined by a set of  $m$  facet vectors  $p_i$  and defines a set of  $m$  lattice vectors  $t_i$ ,  $1 \leq i \leq m$  (see (1) below). We show that Voronoi's conjecture for an  $n$ -

dimensional parallelotope  $P$  is equivalent to the following condition

( $pQt$ ) there exist scalars  $\gamma_i$  and a positive definite  $n \times n$  matrix  $Q$  such that  $\gamma_i p_i = Qt_i$  for all  $i$ .

In this case the quadratic form  $f(x) = x^T Qx$  is the metric form of  $P$ .

Giving an explicit matrix  $Q$ , Voronoi proved his conjecture for primitive parallelotopes. Since a primitive parallelotope  $P$  is  $(n-2)$ -primitive, the projection of  $P$  along any  $(n-2)$ -face gives a centrally symmetric hexagon. On the other hand, if any such projection gives a hexagon, then the parallelotope is  $(n-2)$ -primitive. Zhitomirskii [Zh29] remarked that the Voronoi's proof in [Vo908] uses only  $(n-2)$ -primitivity. This allowed him to extend the result of Voronoi for  $(n-2)$ -primitive parallelotopes.

At this time all parallelotopes of dimension  $n \leq 5$  are known, and, since each of these parallelotopes is affinely equivalent to a Voronoi polytope, the Voronoi's conjecture is true for  $n \leq 5$ .

The two parallelotopes in  $\mathbf{R}^2$ , namely a centrally symmetric hexagon (primitive) and a parallelogram (non-primitive), are known as pavings since the antique.

In 1885, Fedorov described all 5 parallelotopes of dimension 3 including one primitive.

Delaunay [De29] enumerated 51 four-dimensional parallelotopes (including known to Voronoi 3 primitive ones), and Shtogrin [Sht73] found the last missed by Delaunay 52th non-primitive parallelotope.

Ryshkov and Baranovskii [RB76] found 221 primitive 5-dimensional parallelotopes. The missed 222th one is identified in [EG02]. Engel in [En98] and [En00], using a computer, enumerated 179 372 parallelotopes of dimension 5 (including 222 primitive).

Engel's proof of Voronoi's conjecture is based on a program which verified whether a given polytope is a parallelotope or not, and if it is, this program find an affinely equivalent Voronoi polytope. Similarly to a zonotope, a parallelotope has zones of mutually parallel edges. Such edge zone is called *closed* if any two-dimensional face has two or none edges of the zone. Otherwise, the zone is called *open*. Each closed zone can be contracted to an open zone. Conversely, some open zones can be extended to closed zones. Hence parallelotopes are partially ordered by inclusion of sets of closed zones. Maximal members of this order are primitive parallelotopes. So, any parallelotope can be obtained from primitive one by contraction of some closed zones. Note that a contracted Voronoi polytope is a parallelotope but, in general, it is not a Voronoi polytope. By such a manner, Engel enumerated all 5-dimensional parallelotopes. Giving a Voronoi polytope affinely equivalent to each parallelotope, Engel proved Voronoi's conjecture for dimension 5.

McMullen [McM75] proved Voronoi's conjecture for zonotopal parallelotopes if segments form a unimodular system. Erdahl [Er99] gave complete proof using a notion of a *lattice dicing* (which forms a lattice whose Voronoi polytope is a zonotope).

We show that a zonotopal parallelotope  $P(Z)$  is a Voronoi polytope with respect to the quadratic form  $f(x) = x^T Qx$ , for  $Q = (Z_\beta Z_\beta^T)^{-1}$ , where the columns of the matrix  $Z_\beta$  are the vectors  $\sqrt{2\beta_z}z$  for  $z \in Z$ . Hence the transformation  $x \rightarrow Ax$  maps  $P(Z)$  into a Voronoi polytope if the matrix  $A$  is a solution of the equation  $A^T A = (Z_\beta Z_\beta^T)^{-1}$ .

The lattice of  $P(Z)$  is a *dicing lattice* obtained as the set of intersection points of the family of hyperplanes  $\{x : d^T x = m\}$ , where  $m$  is an integer and the family of vectors

$d = \beta_z Q z$ ,  $z \in Z$ , being a linear transform of the unimodular system  $\{\beta_z z : z \in Z\}$ , forms also a unimodular system.

## 2 Parallelotopes

Being a centrally symmetric polytope, any parallelootope  $P$  with  $m$  pairs of opposite facets has the following description.

$$P = P(0) \equiv \{x \in \mathbf{R}^n : -\frac{1}{2}p_j^T t_j \leq p_j^T x \leq \frac{1}{2}p_j^T t_j, 1 \leq j \leq m\}, \quad (1)$$

where  $\frac{1}{2}t_j$  is the center of the facet

$$F_j = \{x \in P(0) : p_j^T x = \frac{1}{2}p_j^T t_j\}$$

and  $p_j$  is a *facet vector* of the facet  $F_j$ . Obviously,  $p_j$  is determined up to a scalar multiple.

The parallelootope  $P(t_j)$  is obtained by translation of  $P(0)$  on the *translation vector*  $t_j$ .  $P(t_j)$  is adjacent to  $P(0)$  by the facet  $F_j$ . The set  $\mathcal{T} = \{t_j : 1 \leq j \leq m\}$  of translation vectors generates a lattice  $L(\mathcal{T})$  whose points are the centers of parallelotopes of the tiling. We denote by  $\pm\mathcal{T}$  the set of all translation vectors and their opposite.

If, for all  $j$ ,  $\gamma_j p_j = t_j$  with scalar  $\gamma_j > 0$ , then the parallelootope  $P(0)$  is the Voronoi polytope  $P_V(0)$  related to the zero point of the lattice  $L(\mathcal{T})$ .

Note that the usual Euclidean norm  $x^2 = x^T x$  is used in the definition of the Voronoi polytope  $P_V(0)$ . But we can use an arbitrary positive quadratic form  $f_Q(x) = x^T Q x$  as a norm of  $x$ , where  $Q$  is a symmetric positive definite  $n \times n$  matrix. Hence we call the parallelootope

$$P_Q(0) \equiv \{x \in \mathbf{R}^n : x^T Q x \leq (x - t)^T Q (x - t), t \in \pm\mathcal{T}\}.$$

the *Voronoi polytope with respect to the form  $f_Q(x)$* . Such a parallelootope relates to the lattice  $L(\mathcal{T})$ .

Now, if  $Q = I_n$ , i.e.  $Q$  is the unit matrix, then  $P_{I_n}(0) = P_V(0)$  is the Voronoi polytope of the lattice  $L(\mathcal{T})$ .

**Proposition 1** *Let  $P$  be a parallelootope, defined in (1),  $A$  be a non-singular  $n \times n$  matrix such that  $Q = A^T A$  is a symmetric positive definite matrix. Then the following assertions are equivalent:*

(i) *the map  $x \rightarrow Ax$  transforms  $P$  into a Voronoi polytope, i.e., Voronoi conjecture is true for  $P$ ;*

(ii)  *$P$  is the Voronoi polytope  $P_Q$  with respect to the positive definite quadratic form  $f(x) = x^T Q x$ ;*

(iii) *one can choose scalar  $\gamma_j$  such that the following equalities hold*

$$\gamma_j p_j = Q t_j \text{ for all } 1 \leq j \leq m. \quad (2)$$

**Proof.** (i) $\Leftrightarrow$ (ii) Let the map  $x \rightarrow x' = Ax$  map  $P$  into the Voronoi polytope

$$P_V = \{x' : x'^2 \leq (x' - t)^2, \text{ for } t \in \pm AT\}.$$

The converse transformation  $x' \rightarrow x = A^{-1}x'$  maps  $x'^2$  into

$$(Ax)^T Ax = x^T A^T Ax = x^T Qx.$$

Hence the transformation  $x' \rightarrow x$  maps  $P_V$  into the Voronoi polytope  $P_Q$  with respect to the form  $f(x) = x^T Qx$ , which should coincide with  $P$ .

(iii) $\Leftrightarrow$ (ii) Let  $t_j$  and  $\gamma_j p_j$  be connected by the relation (2). Then we can rewrite the inequality  $\gamma_j p_j^T x \leq \frac{1}{2} \gamma_j p_j^T t_j$ , determining the facet  $F_j$  of  $P$ , as  $0 \leq t_j^T Q t_j - 2t_j^T Q x$ , or, adding  $x^T Q x$  to both the sides of this inequality, as

$$x^T Q x \leq (x - t_j)^T Q (x - t_j).$$

In other words, we obtain that  $P = P_Q$ . This reasoning can be reversed. Hence we are done.

### 3 Matroids

Any set  $X \subset \mathbf{R}^n$  of vectors gives a representation (or a coordinatization) (over the field  $\mathbf{R}$ ) of a matroid  $M = M(X)$ . A matroid on a set  $X$  is uniquely defined by its *rank function*  $\text{rk}Y \leq |Y|$  for all  $Y \subseteq X$ . Rank of a subset  $Y \subseteq X$ , is dimension of the space spanned by  $Y$ . A non-zero vector  $x \in X$  has  $\text{rk}x = 1$  and is called a *point* of  $M(X)$ . Rank of the matroid  $M(X)$  is the rank of  $X$ . A maximal by inclusion subset of rank  $k$  is called a *k-flat*, or simply *flat*. So, a 1-flat is a point. 2-flats are called *lines*. Let  $\text{rk}M(X) = n$ . Then  $n - 2$ - and  $n - 1$ -flats are called *colines* and *copoints*, respectively. So, a copoint  $H$  spans a hyperplane  $h$  in  $\mathbf{R}^n$ . For a detailed information on matroids, see, for example, [Aig79] and [Wh87].

For our aims, binary and regular matroids are important. A matroid is called *binary* if it is represented over a field of characteristic two. Another equivalent characterization of a binary matroid is (*bin*) in terms of copoints and colines (see Theorem 7.22(iv) of [Aig79]):

(*bim*) *A matroid is binary if and only if each its coline is contained in at most 3 copoints.*

A binary matroid is called *regular* (or *unimodular*) if it is represented also over a field of characteristic distinct from two (see Theorem 7.35(iii) of [Aig79]). Any regular matroid  $M(X)$  can be represented by a unimodular system of vectors  $X$ . A set  $X$  of  $n$ -dimensional vectors is called *unimodular* if for *every* subset  $B \subseteq X$  of  $n$  independent vectors any vector of  $X$  is represented as an integer linear combination of vectors from  $B$ . If we suppose that  $X$  and  $B$  are columns of the corresponding matrices, then the matrix  $B^{-1}X$  is *totally unimodular*, i.e. all its minors are equals to 0 or  $\pm 1$ .

One of many equivalent characterizations of a unimodular system is as follows (cf. Theorem 7.37(v) of [Aig79], Theorem 3.11(7) of [Wh87] and the condition V of [McM75]) (we denote by  $y^T x$  the scalar products of column vectors  $x$  and  $y$ ).

Let  $H \subset X$  be a copoint of  $M(X)$ . Then  $H$  generates a hyperplane  $h \subset \mathbf{R}^n$ . Let  $p_H$  be a vector orthogonal to  $h$ . Obviously, the length of  $p_H$  can be arbitrary.

*(uni)* A system of vectors  $X$  is unimodular if and only if, for any copoint  $H \subset X$  of the matroid  $M(X)$ , one can choose a scalar multiple  $\gamma_H$  of the vector  $p_H$  such that  $\gamma_H p_H^T x \in \{0, \pm 1\}$  for all  $x \in X$ .

A proof of *(uni)* is easily obtained from the fact that any vector of  $X$  has  $(0, \pm 1)$ -coordinates in any base of  $X$ .

Unfortunately, not every representation over  $\mathbf{R}$  of a regular matroid is unimodular. But *every* representation  $X$  over  $\mathbf{R}$  can be transformed into a unimodular representation by multiplying each vector  $x \in X$  on an appropriate positive scalar multiple  $\beta_x$ . Hence *(uni)* can be reformulated as follows.

*(reg)* A matroid  $M(X)$  is regular if and only if one can choose scalar multiples  $\gamma_H$  and  $\beta_x$  of the vectors  $p_H$  and  $x$ , respectively, such that  $\gamma_H p_H^T \beta_x x \in \{0, \pm 1\}$  for any copoint  $H \subset X$  of  $M(X)$  and all  $x \in X$ .

## 4 Zonotopes

A zonotope is the Minkowski sum of segments. If the zonotope is  $n$ -dimensional, then each segment  $S_i$  is defined by an  $n$ -dimensional vector  $z_i$  such that

$$S_i = \{x \in \mathbf{R}^n : -z_i \leq x \leq z_i\}.$$

Let a zonotope  $P(Z)$  be the Minkowski sum of  $r$  segments  $S_i$  and be defined by a system of column vectors  $\{z_i : 1 \leq i \leq r\}$  of an  $n \times r$  matrix  $Z$ . Then

$$P(Z) = \{x \in \mathbf{R}^n : x = Zy, \quad -1 \leq y_i \leq 1, \quad 1 \leq i \leq r\}.$$

Obviously,  $P(Z)$  is centrally symmetric. Each family of all mutually parallel  $k$ -faces of  $P(Z)$  is called a *zone*.

We can consider the matrix  $Z$  as a set of column vectors and denote the set by the same letter  $Z$ . Let  $M(Z)$  be the matroid represented by  $Z$ . Let  $X \subseteq Z$  be a  $k$ -flat. Obviously,  $X$  generates a sub-matrix of  $Z$  of rank  $k$ . The set  $X$  defines a zonotope  $P(X)$ . In particular,  $P(\{z_i\}) = S_i$ .

An important property connecting  $P(Z)$  and  $M(Z)$  is as follows

*(face)* Every  $k$ -face  $F$  of  $P(Z)$  has the form  $F = F(X)$  with

$$F(X) = \sum_{i: z_i \notin X} \varepsilon_i z_i + P(X),$$

where  $X$  is a  $k$ -flat of  $M(Z)$  and  $\varepsilon_i \in \{\pm 1\}$ . Conversely, every  $k$ -flat  $X$  of  $M(Z)$  defines a  $k$ -face  $F(X)$  of  $P(Z)$  for some  $\varepsilon_i \in \{\pm 1\}$ . If  $F(X)$  is a facet and  $p$  is a vector normal to the hyperplane supporting  $P(Z)$  in  $F(X)$ , then

$$\varepsilon_i = \begin{cases} 1, & \text{if } p^T z_i > 0, \\ -1, & \text{if } p^T z_i < 0, \\ 0, & \text{if } p^T z_i = 0. \end{cases} \quad (3)$$

This classical form of a face can be found in almost all papers on zonotopes. The fact that  $X$  is a flat is explicitly given, for example, in Proposition 2.2.2 of [B-Z92]. In [Jae83] this fact is given in terms of a representation of  $M(Z)$  by a chain group.

We see that any face of a zonotope  $P(Z)$  and  $P(Z)$  itself are centrally symmetric. The center of a facet  $F$  is given by

$$\frac{1}{2}t = \sum_{i=1}^r \varepsilon_i z_i. \quad (4)$$

Theorem 1 below is well known. This is Proposition 3.3.4 of [Wh87] and Theorem 2.2.10 of [B-Z92], both given there without proofs. We give a very short proof.

**Theorem 1** *Let  $Z$  be a set of vectors. The following assertions are equivalent:*

- (i) *the zonotope  $P(Z)$  is a parallelotope;*
- (ii) *the matroid  $M(Z)$  is regular.*

**Proof.** (i) $\Rightarrow$ (ii) If  $P(Z)$  is a parallelotope of dimension  $n$ , then its  $(n-2)$ -faces have the property (*fhyp*). Since by (*face*) each  $k$ -face of  $P(Z)$  uniquely determines a  $k$ -flat of  $M(Z)$ , the matroid  $M(Z)$  satisfies the condition (*bin*). Hence  $M(Z)$  is binary. Since  $M(Z)$  is obviously represented over  $\mathbf{R}$ , by definition of a regular matroid,  $M(Z)$  is regular.

(ii) $\Rightarrow$ (i) Since any zonotope is centrally symmetric and has centrally symmetric faces,  $P(Z)$  has the first two properties (*symP*) and (*symF*) of a parallelotope. Hence we have to prove that  $P(Z)$  has the property (*fhyp*). Since  $M(Z)$  is regular, it is binary and satisfies the condition (*bin*). This condition and (*face*) imply that  $P(Z)$  has the property (*fhyp*), too.

Note that in [McM75] and [Sh75], the condition (*fhyp*) of a parallelotope is formulated as the condition (*bin*) (see II of [McM75] and (11) of [Sh75]) but without mention of a notion of matroid.

The above proof of the implication (i) $\Rightarrow$ (ii) can be found in Section 4.2 of [Jae83].

Now we show that a zonotopal parallelotope  $P(Z)$  is a Voronoi polytope with respect to a quadratic form  $f_D(x)$  for some matrix  $D$ .

Let  $\beta_i$  be the positive multiple of  $z_i$  mentioned in the condition (*reg*). So, the system  $\{\beta_i z_i : 1 \leq i \leq r\}$  is unimodular. Let  $Z_\beta$  be an  $n \times r$  matrix whose columns are the vectors  $\sqrt{2\beta_i} z_i$ .

**Lemma 1** *Let  $P(Z)$  be a full-dimensional zonotopal parallelotope determined by a system  $Z$ . Let the facet vectors  $p_j$ ,  $1 \leq j \leq m$ , satisfy the condition (*reg*) (with  $\gamma_H = \gamma_j$ ). Then the following equalities hold*

$$\gamma_j p_j = Q t_j, \quad 1 \leq j \leq m, \quad \text{where } Q = (Z_\beta Z_\beta^T)^{-1}.$$

**Proof.** By (4), the center of a facet  $F_j$  is

$$\frac{1}{2}t_j = \sum_{i=1}^r \varepsilon_{ij} z_i,$$

where  $\varepsilon_{ij}$  is  $\varepsilon_i$  of (3) for  $p = p_j$ . If the facet vectors  $p_j$ ,  $1 \leq j \leq m$ , satisfy the condition (*reg*), then  $\varepsilon_{ij} = \gamma_j \beta_i p_j^T z_i$ . Hence we have

$$t_j = 2 \sum_{i=1}^r \beta_i \gamma_j (p_j^T z_i) z_i = \gamma_j \sum_{i=1}^r \sqrt{2\beta_i} z_i (\sqrt{2\beta_i} z_i^T p_j) = Z_\beta Z_\beta^T \gamma_j p_j.$$

Since  $P(Z)$  is full-dimensional, the matrix  $Z_\beta Z_\beta^T$  is a positive definite symmetric matrix. So, it is non-singular, and we have  $\gamma_j p_j = Q t_j$ . The result follows.

Hence we have the following

**Theorem 2** *The map  $x \rightarrow Ax$  transforms a zonotopal parallelotope  $P(Z)$  into a Voronoi polytope, where the matrix  $A$  is a solution of the equation  $A^T A = (Z_\beta Z_\beta^T)^{-1}$ .*

## 5 Dicings

Let  $D$  be a set of vectors spanning  $\mathbf{R}^n$ . Then the set  $D$  defines a family  $\mathcal{H}(D)$  of parallel hyperplanes  $H(d, m) = \{x \in \mathbf{R}^n : x^T d = m\}$ , where  $d \in D$  and  $m$  is an integer. Let  $B \subseteq D$  be a basis of  $D$ . Then the set of intersection points of the hyperplanes  $H(d, m)$  for  $d \in B$  and all  $m$  is a lattice  $L(B)$ . Erdahl and Ryshkov [ErRy94] proved that the set of intersection points of the hyperplanes of the whole family  $\mathcal{H}(D)$  is a lattice  $L(D)$  (which then coincides with  $L(B)$ ) if and only if  $D$  is a unimodular system. (In [ErRy94] the notion of a unimodular system is not mentioned). In this case, the family  $\mathcal{H}(D)$  is called a *lattice dicing*. We call the lattice  $L(D)$  a *dicing lattice*. The family  $\mathcal{H}(D)$  dices  $\mathbf{R}^n$  into polytopes which are Delaunay polytopes of  $L(D)$ .

Now we show that the lattice  $L(\mathcal{T})$  formed by the centers of zonotopal parallelotopes is a dicing lattice.

Take the unimodular system  $\{\beta_i z_i : 1 \leq i \leq r\}$  considered in the preceding sections. Obviously, the system  $D = \{d_i = \beta_i Q z_i : 1 \leq i \leq r\}$  is also unimodular. Hence it defines a dicing lattice  $L(D)$ . We show that  $L(D) = L(\mathcal{T})$ , i.e. that  $L(D)$  is the lattice of the zonotopal parallelotope  $P(Z)$ .

Let  $I_j = \{i : p_j^T z_i = 0\}$ . So,  $p_j$  is the intersection of the hyperplanes

$$H_i = \{x \in \mathbf{R}^n : x^T z_i = 0\},$$

for  $i \in I_j$ . Similarly,  $t_j$  lies in the intersection of the hyperplanes  $G_i$ ,  $i \in I_j$ , of the family  $\mathcal{H}(D)$ , where

$$G_i = \{x \in \mathbf{R}^n : x^T \beta_i Q z_i = 0\}.$$

In fact, since  $Qt_j = \gamma_j p_j$ , we have, for  $i \in I_j$ ,

$$t_j^T Q z_i = (Qt_j)^T z_i = \gamma_j p_j^T z_i = 0.$$

Since  $Q$  is non-degenerate, dimension of the intersection of the hyperplanes  $G_i$ ,  $i \in I_j$ , coincides with dimension of the intersection of the hyperplanes  $H_i$ ,  $i \in I_j$  which is equal 1. Hence  $t_j$  spans the above intersection. Besides, if  $t_k$  does not lie in  $G_i$ , we have  $t_k^T \beta_i Q z_i = \beta_i \gamma_k p_k^T z_i = \varepsilon_{ik} \in \{\pm 1\}$ . In other words, the endpoints of these  $t_k$ 's lie in the hyperplanes of the family  $D$  parallel to  $G_i$  and neighbouring to it. This implies that  $L(\mathcal{T}) = L(D)$ .

Similarly to the facet vectors  $p_j$  determining facets of the parallelotope  $P(Z)$ , the dicing vectors  $d_i$  determines facets of Delaunay polytopes. Edges of these Delaunay polytopes are just the lattice vectors  $t_j$ .

Using the dicing vectors  $d_i = \beta_i Q z_i$ ,  $1 \leq i \leq r$ , and the equality  $Q^{-1} = Z_\beta Z_\beta^T = \sum_{i=1}^r 2\beta_i z_i z_i^T$ , we can represent the quadratic form  $f(x) = x^T Q x$  as a sum of quadratic forms of rank 1:

$$f(x) = x^T Q x = x^T Q Q^{-1} Q x = x^T Q \left( \sum_{i=1}^r 2\beta_i z_i z_i^T \right) Q x = \sum_{i=1}^r \frac{2}{\beta_i} (x^T d_i d_i^T x) = \sum_{i=1}^r \lambda_i (d_i^T x)^2,$$

where  $\lambda_i = \frac{2}{\beta_i}$ .

Erdahl proves in [Er99] that a parallelotope related to a dicing lattice defined by dicing vectors  $d \in D$  is a Voronoi polytope with respect to the quadratic form  $f(x) = \sum_{d \in D} \lambda_d (d^T x)^2$ ,  $\lambda_d > 0$ , which is a zonotope  $P(Z)$ . He gives linear expressions connecting the vectors  $z \in Z$  generating the zonotope  $P(Z)$  and the vectors  $d \in D$ .

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