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Face-regular bifaced polyhedra☆

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Dedicated to S.S. Shrikhande

Abstract

Call *bifaced* any k-valent polyhedron, whose faces are p_a a-gons and p_b b-gons only, where $3 \leqslant a < b$, $0 < p_a$, $0 \leqslant p_b$. We consider the case $b \leqslant 2k/(k-2)$ covering applications; so either $k = 3 \leqslant a < b \leqslant 6$, or $(k; a, b, p_a) = (4; 3, 4, 8)$. Call such a polyhedron aR_i (resp., bR_j) if each of its a-gonal (b-gonal) faces is adjacent to exactly i a-gonal (resp., j b-gonal) faces. The preferable (i.e., with isolated pentagons) fullerenes are the case aR_0 for $(k; a, b) = (3; 5, 6)$. We classify all a- or b-face-regular bifaced polyhedra, except aR_0 for $(k; a, b) = (3; 4, 6), (3; 5, 6), (4; 3, 4)$ and aR_1 for $(k; a, b) = (3; 5, 6), (4; 3, 4)$. For example, we list all 13,6,4,10,26 polyhedra bR_j for all five possible cases: $k = 4$; $k = 3$, $b < 6$; $k = 3$, $b = 6$, $a = 3, 4, 5$.

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1. Introduction

Denote by $(k; a, b; p_a, p_b)$ and call *bifaced* any k-valent polyhedron whose faces are only p_a a-gons and p_b b-gons with $3 \leqslant a < b$ and $0 < p_a$, $0 \leqslant p_b$.

Any polyhedron $(k; a, b; p_a, p_b)$ with n vertices has $\frac{1}{2}kn = \frac{1}{2}(ap_a + bp_b)$ edges and satisfies the Euler relation $n - \frac{1}{2}kn + (p_a + p_b) = 2$, i.e.,

$$p_a(2k - a(k-2)) + p_b(2k - b(k-2)) = 4k. \tag{1}$$

Note, that if $a \geqslant 2k/(k-2)$, then $b > 2k/(k-2)$, and the left-hand side of the above equality is less than zero. Hence $a < 2k/(k-2) = 2 + 4/(k-2)$ and (3,3), (3,4), (3,5), (4,3), (5,3) are only possible (k, a). We consider only the case when, moreover, $b \leqslant 2k/(k-2)$; it covers bifaced polyhedra mentioned in chemical applications. It is

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easy to see, that all such possible bifaced polyhedra belong to one of the following three classes:

(A) If $k = 3 \leqslant a < b \leqslant 5$, then (1) takes the form $p_a(6-a) + p_b(6-b) = 12$ and, for $p_b > 0$, the class consists only of the following 6 simple polyhedra (all but no. 2 are duals of all 5 non-Platonic convex deltahedra):
1. Prism$_3$ for $(a, b) = (3, 4)$ with $p_3 = 2$, $p_4 = 3$, $n = 6$;
2. the Dürer octahedron (i.e., the cube truncated in 2 opposite vertices) for $(a, b) = (3, 5)$ with $p_3 = 2$, $p_5 = 6$, $n = 12$;
   and 4 polyhedra for $(a, b) = (4, 5)$:
3. dual of 2-capped APrism$_4$ with $p_4 = 2$, $p_5 = 8$, $n = 16$;
4. dual of 3-augmented Prism$_3$ with $p_4 = 3$, $p_5 = 6$, $n = 14$;
5. dual of bidisphenoid with $p_4 = 4$, $p_5 = 4$, $n = 12$;
6. Prism$_5$ with $p_4 = 5$, $p_5 = 2$, $n = 10$.

(B) If $k = 3 \leqslant a < b = 6$, then (1) takes the form $p_a(6-a) = 12$ and there are 3 infinite families a_n: 3_n, 4_n, 5_n with $(a, p_a) = (3,4)$, $(4,6)$, $(5,12)$, respectively, and with the unbounded number p_6 of hexagons. (Recall that n is the number of vertices of the corresponding polyhedron).

(C) If $k = 4$, then $3 \leqslant a < b \leqslant 2k/(k-2) = 4$ implies $a = 3$, $b = 4$ and (1) takes the form $p_a = 8$, i.e., there is only one infinite family $(3,4)_n$ with $(a, b, p_a) = (3, 4, 8)$ and unbounded p_4.

The minimal polyhedra of the families 3_n, 4_n, 5_n and $(3,4)_n$ are polyhedra 3_4, 4_8, 5_{20} and $(3,4)_6$ with $p_b = 0$, when $n = ap_a/k$. Clearly, they are Platonic polyhedra: tetrahedron, cube, dodecahedron and octahedron, respectively.

Grünbaum (1967) gives that other 3_n, 4_n, 5_n and $(3,4)_n$ exist if and only if $12 \leqslant n \equiv 0 \pmod 4$, $12 \leqslant n \equiv 0 \pmod 2$, $24 \leqslant n \equiv 0 \pmod 2$ and $8 \leqslant n$, respectively. The equality $kn = ap_a + bp_b$ implies that the number n of vertices is equal to

$$n(3_n) = 4 + 2p_6, \; n(4_n) = 8 + 2p_6, \; n(5_n) = 20 + 2p_6 \text{ and } n((3,4)_n) = 6 + p_4. \quad (2)$$

Polyhedra 5_n are *fullerenes* well known in Chemistry (see, for example, Goldberg, 1935; Fowler and Manolopoulos, 1995); polyhedra 4_n are mentioned in Gao and Herdon (1993). In Deza et al. (1998), Deza and Grishukhin (1997, 1999b), we consider isometric (up to scale 2) embedding of skeletons of some bifaced polyhedra into the vertex-set of hypercubes. It turns out that all known fullerenes such that it or its dual is embeddable are face-regular in the sense considered below. In fact, 5_{20}=20:1, 5_{26}=26:1, 5_{44} = 44:73, 5_{80} = 80:7 and duals of 20:1, 28:2, 36:15, $5_{60}(\mathrm{I_h})$ are embeddable into a half m-cube for $m = 10, 12, 16, 22$ and $6, 7, 8, 10$, respectively. (The notations n:k of a 5_n are taken from Fowler and Manolopoulos, 1995). Moreover, all known (see Deza and Grishukhin, 1999b) bifaced polyhedra such that it or its dual is embeddable turn out to be face-regular: ## 1, 2, 5, 6 and duals of ## 1, 2 in case (A); 4 polyhedra 4_n ($n = 12, 24, 32, 32$), 5 dual 3_n ($n = 12, 16, 16, 28, 36$); all t-elongated octahedra and their duals, in addition to 6 others $(3,4)_n$ ($n = 8, 9, 10, 16, 16, 24$) and to the dual cuboctahedron embeddable into H_4.

The graphs of all polyhedra of classes (A)–(C) and their duals (except undecided 5_n and dual $(3,4)_n$) have a Hamiltonian circuit; this follows from the results surveyed in Section 5.3 of Bayer and Lee (1993).

Here we want to identify aR_i and bR_j bifaced polyhedra. We denote as aR_i (resp. bR_j) the fact that each a-face (b-face) is edge-adjacent to exactly i a-faces (resp., j b-faces). Sometimes, it is convenient to distinguish aR_i and bR_j bifaced polyhedra by graphs G_a and G_b of the edge-adjacency of a- and b-faces, respectively. These graphs have p_a and p_b vertices, respectively. *Face-regular* bifaced polyhedra are those having regular graphs either G_a or G_b. (This combinatorial notion has nothing to do with the affine notion of *regular-faced* polyhedra.)

A motivation for this work comes from fullerenes studies; see, for example, Fowler and Manolopoulos (1995) (pentagonal and hexagonal indices and their connection to the steric strain). In fact, these indices give the number of vertices with degree $0,\dots,5$ and $0,\dots,6$ for graphs G_a and G_b of fullerenes; they where introduced in Raghvachari (1992) as an attempt to measure the steric strain of isomers of 5_{84}.

See Grünbaum (1967) for terms used here for polyhedra. We identify a polyhedron with the graph of its skeleton. According to the famous Steinitz Theorem, a graph is the skeleton of a (three-dimensional) polyhedron if and only if it is planar and 3-connected. For a simple polyhedron P, we denote by cham$P$ and call chamfered P the polyhedron obtained by putting prisms on all faces of P and deleting original edges (see Deza et al., 1998; Deza and Grishukhin, 1999b for more details).

For $t \geqslant 1$, denote by 2-Prism$_4^t$ the *t-elongated* octahedron, i.e., the column of t cubes, capped in 2 most opposite faces. It is $(3,4)_{4t+6}$, and besides it is $3R_2$ in our terms.

Similarly, for $t \geqslant 1$, denote by $(\text{APrism}_3^{t+1})^*$ the t-hex-elongated cube, i.e., the cube with t triples of hexagons inserted as belts between 2 triples of squares incident to 2 opposite vertices (in other words, the dual of the column of $t+1$ octahedra $\beta_3 = \text{APrism}_3$). It is tetrahedral 4_{6t+8}, which is $4R_2$.

Finally, for $t \geqslant 1$, denote by $(2\text{-APrism}_6^{t+1})^*$ the t-hex-elongated 5_{24}, i.e., the dual of the column of $t+1$ APrism$_6$'s capped on 2 opposite 6-faces. It is 5_{12t+24} and is $5R_2$.

We denote by K_n, C_n and P_n the complete graph, the circuit (cycle) and the path all on n vertices.

2. Face-regular maps with digons

In the Introduction we supposed that we consider only polyhedra with non-degenerate a- and b-faces with $b > a \geqslant 3$. The 1-skeletons of these polyhedra are planar regular graphs without parallel edges. But if a plane graph has parallel edges, then some pairs of parallel edges form 2-faces or digons. In this section, we consider planar regular graphs with 2- and b-faces, that are 1-skeletons of degenerated bifaced polyhedra.

The equality (1) for $a = 2$ takes the form

$$4p_a + p_b(k-2)\left(\frac{2k}{k-2} - b\right) = 4k, \qquad (3)$$

where $3 \leqslant b \leqslant 2k/(k-2)$ and $k \geqslant 3$. Note that if $b = 2k/(k-2)$, then there is no restriction on the number p_b of b-faces.

Consider at first the cases when $b < 2k/(k-2)$, strictly. Then, for a fixed value of the pair (k, b), there is a finite number of feasible values of parameters p_a and p_b satisfying (3). In the table below, we give feasible parameters p_a, p_b and the number $v = (ap_a + bp_b)/k$ of vertices of the corresponding graphs. Each column of the table corresponds to a graph.

- The first row of the table contains the order numbers of the graphs.
- The graphs no. 1, 4, 8 and 10 having $p_a = 1$ digon cannot be constructed.
- The graphs no. 2, 6 and 9 are the graphs K_4, the octahedron and C_4, where the edges of a perfect matching (i.e. of a set of mutually non-adjacent edges covering all vertices) are each changed by a digon.
- The graph no. 3 is K_3, where *each* edge is changed by a digon.
- The graph no. 7 is K_4, where (two) edges of a perfect matching are each changed by a bundle of two digons.

The graph no. 11 is obtained from K_4 by setting digons on edges of a perfect matching. (The notion *setting* is explained below in this section.)

No.	1	2	3	4	5	6	7	8	9	10	11
k	4	4	4	5	5	5	5	3	3	3	3
b	3	3	3	3	3	3	3	4	4	5	5
p_a	1	2	3	1	2	3	4	1	2	1	2
p_b	6	4	2	16	12	8	4	4	2	8	4
v	5	4	3	10	8	6	4	6	4	14	8
aR_i		0	0		0	0	1		0		0
bR_j		2	0		—	—	2		1		3

If $b = 2k/(k-2)$, then there are 3 infinite families of graphs, that we denote as 2_n, $(2,3)_n$ and $(2,4)_n$, where n is the number of vertices of the graphs. In fact, for $k = 3, 4$ and 6, the ratio $2k/(k-2)$ takes integral values 6,4 and 3, respectively. Hence when the pair (k, b) is one of the three pairs (3,6), (4,4) and (6,3), we obtain the families of maps 2_n, $(2,4)_n$ and $(2,3)_n$, respectively. The number p_2 of digons in a graph of each family is equal to k, the valency of the graph. The number of vertices of a graph of each family is equal to $v = 2(1 + p_b/(k-2))$. The smallest graphs of these families (i.e., the graphs with minimal number 2 of vertices) are bundles of k parallel edges.

There is the following sequence of graphs of the family $(2,3)_n$ with $n \equiv 2 \pmod 4$. The six digons of such a graph of type $(2,3)_n$ are partitioned into two groups of bundles of 3 digons each. We set two new vertices on edges of the middle digons of each group and connect them by 3 parallel edges. We obtain a graph of type $(2,3)_{n+4}$.

Similarly, we obtain the following two sequences of the graphs of type $(2,4)_n$ as follows:

Let $m \geqslant 2$ be even, and $G(m)$ be the following graph on m^2 vertices (i,j), $1 \leqslant i,\ j \leqslant m$. If $2 \leqslant i,\ j \leqslant m-1$, then the vertex (i,j) is adjacent to the 4 vertices $(i \pm 1, j)$, $(i, j \pm 1)$. The vertex $(1,j)$ for $2 \leqslant j \leqslant m-1$ is adjacent to the vertices $(1, j \pm 1)$, $(2,j)$ and $(1, m+1-j)$. The vertex (m,j) for $2 \leqslant j \leqslant m-1$ is adjacent to the vertices $(m, j \pm 1)$, $(m-1, j)$ and $(m, m+1-j)$. Similar adjacencies have the vertices with $j=1$ or m and $2 \leqslant i \leqslant m-1$. The vertex $(1,1)$ is adjacent to the vertices $(2,1)$, $(1,2)$, $(1,m)$ and $(m,1)$. The vertex (m,m) is adjacent to the vertices $(m-1,m)$, $(m,m-1)$, $(1,m)$, $(m,1)$. It is easy to see that $G(m)$ is a graph of type $(2,4)_n$ for $n=m^2$.

The graphs $R(k)$, $k \geqslant 0$, of the second sequence are constructed iteratively from $R(k-1)$. $R(0)$ is the bundle of 4 digons (=4 parallel edges). The graph $R(k)$ contains a ring R_k containing the 4 digons and $8k$ squares. For $k>0$, the digons and squares lie between two circuits C_{8k} such that each pair of digons is separated by $2k$ squares. Call edges of R_k connected vertices of distinct circuits C_{8k} *spikes*. $R(k)$ is obtained from $R(k-1)$ as follows. We set two new vertices on each spike. Connect new opposite vertices of each square and digon by new edges. Then each square is partitioned on 3 squares, and each digon is partitioned into 2 triangles separated by a square. Now, in each digon, we set on each of the two edges separating triangles and square by a vertex and connect the new vertices by two parallel edges. Now each digon is partitioned into 4 squares and a new digon. It is easy to see that $R(k)$ is of type $(2,4)_n$ for $n=2+8k(k+1)$.

It is proved in Grünbaum and Zaks (1974) that the number of hexagons of a 2_n is $p_6 = x^2+xy+y^2-1$ for some nonnegative integers x, y. Recall that $n=2+2p_6$, i.e., $n=2(x^2+xy+y^2)$.

The graph 2_n can be obtained by setting digons on edges of some cubic graphs. We say a digon is *set* on an edge e of a graph if we set two new vertices on e and connect them by a new edge parallel to the part of e between these new vertices.

All 2_n are face-regular. The minimal graphs 2_n are given below with the pairs (i,j) denoting that they are $2R_i$ and $6R_j$:

$n=2$, $(x,y)=(1,0)$, $p_6=0$, $(i,j)=(2,-)$; $2_2=K_3^*$, the dual of the triangle K_3; it consists of 3 parallel edges. Strictly speaking, it is not $6R_j$ since it has no hexagons.

$n=6$, $(x,y)=(1,1)$, $p_6=2$, $(i,j)=(0,1)$; 2_6 is obtained by setting 3 digons on the edge of a loop.

$n=8$, $(x,y)=(2,0)$, $p_6=3$, $(i,j)=(0,2)$; 2_8 is obtained by setting digons on the 3 edges of 2_2.

$n=14$, $(x,y)=(2,1)$, $p_6=6$, $(i,j)=(0,4)$; 2_{14} is obtained by setting digons on 3 edges of a cube $\gamma_3=4_8$ such that no pair of digons belongs to the same square of the cube.

$n=18$, $(x,y)=(3,0)$, $p_6=8$, $(i,j)=(0,-)$; 2_{18} is obtained by setting digons on alternating lateral edges of $\mathrm{Prism}_6=4_{12}$.

For all other 2_n, we have $(i,j)=(0,-)$. They are obtained by setting digons on edges common to two adjacent quadrangles of 4_{n-6} having 3 disjoint pairs of adjacent quadrangles (i.e., with $4R_1$).

The expression $n = 2(x^2 + xy + y^2)$ is a special case of the following formula:

$$n = N_a(x^2 + xy + y^2), \quad 2 \leqslant a \leqslant 5 \tag{4}$$

for the number of vertices of the polyhedron a_n with a maximal symmetry in its class. In table below we give value of N_a, the minimal polyhedron P_a, its dual P_a^* and the groups of symmetries Gr, $\mathrm{Gr_h}$. (Two polyhedra are dual to each other if the vertices of one corresponds to the faces of the other and vice versa.)

a_n	P_a	P_a^*	N_a	$\mathrm{Gr_h}$,	Gr
2_n	$2_2 = K_3^*$	K_3	2	$\mathrm{D_h}$,	D
3_n	$2_4 = \alpha_3$	α_3	4	$\mathrm{T_h}$,	T
4_n	$4_8 = \gamma_3$	β_3	8	$\mathrm{O_h}$,	O
5_n	$5_{20} = \mathrm{Dod}$	Ico	20	$\mathrm{I_h}$,	I

Formula (4) and the coincidence of N_a with the number of vertices of P_a= number of faces of P_a^* is not an accident. This is the result of a construction of polyhedra a_n that goes back to Goldberg (Goldberg, 1937) (see also Fowler, 1993).

This construction considers duals rather than the a_n themselves. Each three-valent vertex of a a_n corresponds to a triangular face of the dual a_n^*, and the face to an a- and 6-valent vertex of the deltahedral dual. The net of an arbitrary a_n is found by joining points of a sub-lattice L in the infinite equilateral triangular lattice $\mathrm{L_0}$. The points of the sub-lattice L correspond to a-valent vertices of the dual deltahedron a_n^*. The vector of L between a neighboring pair of a-valent points of any master polyhedron P_a^* must be a lattice vector of the triangular net $\mathrm{L_0}$. Let x, y be coordinates of the vector in a natural basis of $\mathrm{L_0}$. By folding the net $\mathrm{L_0}$ around the master polyhedron P_a^*, various vertices (and edges) collapse. There are two possible foldings using symmetry G and $\mathrm{G_h}$ of the master polyhedron P_a^*. Surface area of P_a^* is proportional to the number of small triangles cut from the net. It is easy to see that the number of triangles related to a triangular face of P_a^* is equal to $x^2 + xy + y^2$, and the total number of small triangles, i.e., the number n of vertices of a_n, is obtained by the multiplication of the above binomial on number N_a of faces of P_a^*.

All a_n, $2 \leqslant a \leqslant 5$, obtained by this construction have the highest possible (dihedral, tetrahedral, octahedral, icosahedral) symmetry. It turns out that for 2_n, and only for them, there are no others. In all four cases $a = 2, 3, 4, 5$, we have the full symmetry $\mathrm{Gr_h}$ if and only if either $x = y$ or $y = 0$ (equivalently, $x = 0$) and Gr, otherwise.

An analogue of the construction of a_n, $5 \geqslant a \geqslant 2$, of higher symmetry can be applied to dual $(3,4)_n$; namely, put infinite equilateral square lattice on the master polyhedron. For example, all octahedral $(3,4)_n$ come as duals of those obtained from the cube by this way.

3. Face-regular polyhedra 3_n, 4_n and 6 sporadic ones

One can verify by a direct check the following:

Fact. *All 6 bifaced polyhedra of the class (A) above are both aR_i and bR_j for $(i,j)=(0,2),(0,4),(0,4),(0,3),(1,2)$ and $(2,0)$, respectively. Their graphs G_b are, respectively: C_3, the skeletons of the octahedron, of* APrism$_4$, $K_2\times K_3$, C_4, C_5.

From now on we consider only classes (B) and (C). So $(k,b)=(3,6)$ or $(4,4)$, i.e., we consider only $3_n,4_n,5_n$ and $(3,4)_n$. Studying polyhedra with bR_j we, naturally, exclude the four Platonic polyhedra $3_4,4_8,5_{20}$ and $(3,4)_6$ having no b-faces.

First observations about bR_j give

Lemma 1. *For a bifaced polyhedron bR_j the following holds:*

(i) $p_b\leqslant ap_a/(b-j)$ *with an equality if and only if it is aR_0; if $k=3$ and $p_b>0$, then* $p_b\geqslant(p_a/b-j)$;
(ii) $p_b=(a-i)p_a/(b-j)$ *if the polyhedron is also aR_i;*
(iii) *if $(3,4)_n$ is $4R_j$ for $j=1,2$, then it is not $3R_0$;*
(iv) *if $k=3$, then bR_j for $j<(b/2)$ excludes aR_0 and bR_{b-1} implies aR_0; similarly, aR_i for $i<(a/2)$, excludes bR_0 and aR_{a-1} implies bR_0.*

Proof. At first, (i) comes from counting the number of a–a edges as $\frac{1}{2}(p_a a-p_b(b-j))\geqslant 0$. The lower bound comes from the upper bound $\frac{1}{2}p_a(a-1)$ on the number of a–a edges and from the fact that the only 3_n, 4_n and 5_n which contain an a-face surrounded by a-faces only, are Platonic polyhedra. Now, (ii) comes from double counting of the number of a–b edges. (iii) and (iv) can be easily checked case by case. □

The case (i) of Lemma 1 implies the finiteness of the number of bifaced polyhedra bR_j. The same upper bound holds even without our restriction $b\leqslant 2k/(k-2)$, but only this restriction limits p_a.

The double counting of the number of a–b edges also implies the equality $2e(G_a)-2e(G_b)=ap_a-bp_b$ for the numbers $e(G_a)$, $e(G_b)$ of edges of G_a, G_b. So, $e(G_a)=e(G_b)$ if and only if $kn=2ap_a$.

3.1. Face-regular polyhedra 3_n

For 3_n, Lemma 1 gives a full answer to our problem of classification of aR_i and bR_j.

Proposition 1. *Any 3_n (except the tetrahedron 3_4) is $3R_0$. There are exactly 4 polyhedra 3_n which are $6R_j$:*

- *the truncated tetrahedron 3_{12} for $j=3$,*
- *the chamfered tetrahedron 3_{16} and its twist 3_{16} (coming by the truncation of a cube on 4 vertices, pairwise at distance 2 for the first 3_{16} and on 4 vertices being end-vertices of two opposite edges for the second 3_{16}) for $j=4$,*

- *a* 3_{28} (*coming by the truncation of a dodecahedron on* 4 *vertices pair-wisely at distance* 3) *for* $j = 5$.

The graphs G_b *of these polyhedra are the skeletons of the tetrahedron, the octahedron and the icosahedron for* $j = 3, 4, 5$, *respectively.*

Proof. Recall that for 3_n, $p_3=4$ by (2), $n=4+2p_6$, and by Lemma 1(i) $p_6 \leqslant 12/(6-j)$. Hence, for $p_b > 0$, we have

$$12 \leqslant n \leqslant 4 + \frac{24}{6-j} \leqslant 28,$$

where, recall that the lower bound for $p_b > 0$ is given by Grünbaum (1967). If $j \leqslant 2$, the upper bound implies $n \leqslant 10$, i.e., we have a tetrahedron having no hexagons. Hence $j \geqslant 3$, and $n \geqslant 12$. We just checked all 3_n for $n \leqslant 28$ (there are 1, 2, 1, 2, 2 polyhedra 3_n for $n = 12$, 16, 20, 24, 28, respectively). □

3.2. Face-regular polyhedra 4_n

The cube is unique 4_n which is $4R_4$. There is no 4_n which is $4R_3$.

Proposition 2. *The only polyhedra* 4_n *which are* $4R_2$ *are either* $4_{12} = Prism_6$ *or the family* $(\mathrm{APrism}_3^{t+1})^* = 4_{8+6t}$, $t \geqslant 1$, *of t-hex-elongated cubes.*

Proof. Recall that $p_4 = 6$. Let q_0 be a quadrangle of 4_n with $4R_2$. Then q_0 is adjacent to two quadrangles q_1 and q_2. These quadrangles are adjacent to other quadrangles. There are two cases: either q_1 and q_2 are adjacent or not. In the first case, we obtain a configuration of 3 quadrangles surrounded by 3 hexagons. This configuration generates the family $(\mathrm{APrism}_3^t)^*$. In the second case, we obtain a ring of six quadrangles that uniquely gives $\mathrm{Prism}_6 = 4_{12}$. □

There is also an infinity of $4R_0$ and $4R_1$. If a polyhedron 4_n is $4R_1$, then the 6 quadrangles are partitioned into 3 pairs of adjacent quadrangles. The pairs are separated by hexagons. Each pair of quadrangles is surrounded by at least one ring of hexagons. There are 3 polyhedra 4_n for $n=18, 20, 26$, where each pair of quadrangles is surrounded exactly by one ring of hexagons. Remark that the deletion of the 3 edges, separating quadrangles in the 3 pairs, (and 6 their end-vertices) from above 4_{18} and 4_{26} produces bifaced polyhedra Prism_6 and dual of 3-augmented Prism_3, respectively.

Let P be the skeleton of a polyhedron 4_n with $4R_1$. Zaks (1982) noted a relation of P to graphs 2_{n+6}. In fact, by setting digons on each edge common to two adjacent quadrangles in P we get a cubic planar map (with $n + 6$ vertices) all faces of which are 3 digons and $p'_6 = n/2 + 2$ hexagons. Conversely, removing an edge of each of the 3 digons, we get either 4_n with $4R_1$ or 3-cube $= 4_8$ or $\mathrm{Prism}_6 = 4_{12}$.

The authors of Grünbaum and Zaks (1974) introduce the notion of an m-patch, $m > 0$, of a map consisting of a digon surrounded by m rings of hexagons. An m-patch contains $m(m+1)$ hexagons. Obviously, deleting an edge of the digon of an m-patch, we obtain a pair of adjacent quadrangles surrounded by $m-1$ rings of hexagons. Call this configuration also m-patch. So, our 1-patch consists of two adjacent quadrangles. Each 4_n with $4R_1$ contains three non-overlapping m_i-patches, $m_i \geqslant 1$, $i=1,2,3$.

There are the two following cases (see Grünbaum and Zaks, 1974).

Case A: The three patches form a ring R such that the inner D_1 and outer D_2 domains of R are partitioned into hexagons. In this case $m_1=m_2=m_3=m$, and the domains D_1 and D_2 have equal partitions and contain equal numbers of $k+1$ hexagons, where $6k$, $0 \leqslant k \leqslant m$, is the number of edges of each of the boundaries of D_1 and D_2. Hence, in case A,

$$p_6' = 3m(m+1) + k(k+1).$$

Case B: The three patches form a string of the outer domain which is partitioned into hexagons. The string consists of two end-m-patches and one central $(m+1)$-patch. The outer domain contains $2\binom{k+1}{2} + m - k$ hexagons, $0 \leqslant k \leqslant m+1$. The number p_6' of all hexagons is $2m(m+1) + (m+1)(m+2) + k + 1 + m - k$, i.e.,

$$p_6' = 3(m+1)^2 + k^2 - 1.$$

Recall that our m-patch contains $m(m+1)-2$ hexagons. Hence $p_6 = p_6' - 6$, and the number $n = 8 + 2p_6$ of vertices of 4_n is given by the following two-parameter formulae:

Case A:

$$n = 2(3m(m+1) + k(k+1) - 2), \quad 0 \leqslant k \leqslant m, \quad m \geqslant 1,$$

Case B:

$$n = 2(3m(m+2) + k^2), \quad 0 \leqslant k \leqslant m+1, \quad m \geqslant 1.$$

The 3-cube 4_8, Prism$_6 = 4_{12}$ (which are not $4R_1$, and the six $4R_1$ polyhedra 4_n with $n < 46$ (see Table 1 below) also as $4R_1$ 4_{92} have the following parameters:

n	8	12	18	20	26	32	36	44	92	92
Case	A	A	B	B	B	A	A	A	A	B
m	1	1	1	1	1	2	2	2	3	3
k	0	1	0	1	2	0	1	2	3	1

$n = 92$ is the smallest number n that is given by both cases A and B.

If we set $m = (x+y-1)/2$, $k = (x-y-1)/2$ in case A, and $m = (x+y)/2 - 1$, $k = (x-y)/2$ in case B, then both the cases provide the following unique formula for the number p_6 of hexagons in 4_n:

$$p_6 = x^2 + xy + y^2 - 7, \quad 0 \leqslant y \leqslant x, \; x \geqslant 2.$$

Recall that $p_6' = p_6 + 6$. Here cases A and B correspond to pairs (x,y) of different and equal parity, respectively. If $x = y$ or $y = 0$, there is a unique 4_n with these parameters which is $4R_1$. Otherwise, there are two mirror-symmetries.

Table 1

n	$\#4_n$	$\#4R_0$	$\#4R_1$	$\# 4R_2$	$2K_2+2K_1$	K_2+4K_1	P_3+P_3
12	1	—	—	1	—	—	—
14	1	—	—	1	—	—	—
16	1	—	—	—	—	—	1
18	1	—	1	—	—	—	—
20	3	—	1	1	—	—	1
22	1	—	—	—	1	—	—
24	3	1	—	—	1	—	1
26	3	—	1	1	1	—	—
28	3	—	—	—	1	1	1
30	2	1	—	—	1	—	—
32	8	2	1	1	3	—	1
34	3	1	—	—	1	1	—
36	7	3	1	—	1	1	1
38	7	1	—	1	3	2	—
40	7	2	—	—	2	2	1
42	5	2	—	—	2	1	—
44	14	3	1	1	8	1	—

Proposition 3. (i) *All triples* $(i,j;n)$ *such that there exists* 4_n, *both* $4R_i$ *and* $6R_j$ *are* $(2,0;12),(2,2;14),(1,3;20),(2,4;20),(0,3;24),(1,4;26),(0,4;32),(0,5;56)$.

(ii) *Each of above* 8 *cases is realized by unique* 4_n, *except the case* $(0,4;32)$ *realized by the chamfered cube and its twist.*

(iii) Prism$_6$=4_{12} (*the first case in* (i) *above*) *is only* $6R_0$. *The dual tetrakis snub cube* (*the last case in* (i)) *is unique* $6R_5$. *There are no* $6R_1$. *The unique* $4_{14}=(\text{APrism}_3^2)^*$ (*which is also* $4R_2$) *and unique* 4_{16} *are only* $6R_2$. *Only* $6R_3$ *are cases* $(1,3;20)$ *with* G_b *being* Prism$_3$, *and the truncated octahedron* $(0,3;24)$ *above. All* $6R_4$ *are the cases* $(2,4;20)=(\text{APrism}_3^3)^*$, $(1,4;26)$ *with* G_b *being the unique* $(3,4)_9$ *and* $(0,4;32)$ (*the chamfered cube and its twist having* G_b *as the cuboctahedron and its twist*). □

All examples of 4_n for $n\leqslant 44$ are given in Table 1. The last 3 columns of Table 1 give numbers of 4_n, $n\leqslant 44$, with the graph G_a of the edge-adjacency of 4-gons equal to $2K_2+2K_1$, K_2+4K_1, P_3+P_3, respectively. The polyhedra 4_n, $n\leqslant 44$, are taken from Dillencourt (1996).

So, all face-regular 4_n (except two infinite sets for $4R_0$, $4R_1$) are 9 polyhedra from Proposition 3(i), the unique 4_{16} and all t-hex-elongated cubes for $t>2$.

4. Face-regular fullerenes 5_n

The dodecahedron 5_{20} is the unique fullerene which is $5R_5$.

- The hexagonal barrel Barrel$_6=5_{24}$ is the unique 5_n which is $5R_4$. It is also $6R_0$.

- The only fullerenes with $5R_3$ are $5_{28}(\mathrm{T_d})$ and $5_{32}(\mathrm{D_{3h}})$, which are $6R_0$ and $6R_2$, respectively. All fullerenes $5R_i$ for $i=3,4$ are the first 3 cases in (i) of Proposition 6 below.

The fullerenes which are $5R_2$ are distinguished by graphs G_a which are cycles formed by pentagons. There are the following 5 cases:

(1) $G_a=4C_3$, (2) $G_a=2C_3+C_6$, (3) $G_a=C_3+C_9$, (4) $G_a=2C_6$, (5) $G_a=C_{12}$.

Proposition 4. *The fullerenes which are* $5R_2$ *are as follows*:

(i) *the fullerenes with* $G_b=4C_3$ *correspond one-to-one to the polyhedra* 3_{n+8}, $n\geqslant 0$; *so*, n *is divided by* 4;
(ii) *there is no fullerene with* $G_a=2C_3+C_6$;
(iii) *there is the unique fullerene with* $G_a=C_3+C_9$, *namely* $5_{38}(C_{3v})$ (*which is* $6R_2$);
(iv) *the fullerenes with* $G_a=2C_6$ *are t-elongated* 5_{24}, *i.e.*, $(2\text{-APrism}_6^{t+1})^*=5_{12t+24}$, $t\geqslant 1$;
(v) *the fullerenes with* $G_a=C_{12}$ *are exactly* 4 *fullerenes* $5_{36}(\mathrm{D_{2d}})$, $5_{44}(\mathrm{D_{3d}})$, $5_{44}(\mathrm{D_2})$ *and* $5_{48}(\mathrm{D_{6d}})$ (*which are not* $6R_j$ *for all* j) *exist if and only if* $n=4t$, $t\geqslant 10$;

Proof. (i) Clearly, any fullerene 5_n with $G_a=4C_3$ comes by collapsing into a point of all 4 triangles of a 3_{n+8}, since, by Proposition 1, any 3_n for $n>4$ is $3R_0$.

(ii) The 6-cycle of pentagons can be considered as a ring with 6 *tails*, i.e., edges connecting the vertices of the ring with other vertices. Similarly, the boundary of each 3-cycle of pentagons is a circuit of 9 vertices, six of which are endpoints of 6 tails. We have to connect the 6 tails of the 6-cycle with 12 tails of two 3-cycles to obtain a net of hexagons.

The 6-cycle C_6 has two domains: outer and inner. There are two cases: either the two 3-cycles lie in distinct domains or both lie in the same, say, outer, domain. In the first case, by symmetry, we can consider only outer domain. The Euler relation shows that the boundary circuit of the ring of pentagons should have 3 tails. It is easy to verify that it is not possible to form a net of hexagons using 3 tails of the 6-cycle and 6 tails of the 3-cycle.

In the second case by the Euler relation we have the 6-cycle with 6 tails and two 3-cycles C_3^{A} and C_3^{B} each with 6 tails. Suppose there is a fullerene containing this configuration. Then, in this fullerene, there are chains of hexagons connecting a pentagon of the 6-cycle and a pentagon of a 3-cycle. Consider such a chain of minimal, say, q, length. In this case, the 6-cycle is surrounded by q rings each containing 6 hexagons. If we dissect the qth ring of hexagons into two 6-cycles each with 6 tails, we obtain the 6-cycle surrounded by $q-1$ rings. The boundary of the $(q-1)$th ring contains 6 tails.

Let the chain of q hexagons connect the 6-cycle with C_3^{A}. At least two tails of the $(q-1)$th ring correspond (are connected) to tails of C_3^{A}. Since the boundary of the $(q-1)$th ring with 6 tails is similar to the boundary of the 6-cycle with 6 tails, our problem is reduced to the case when two tails of the 6-cycle are connected to two tails of C_3^{A}. There are two cases: either endpoints of the two tails of C_3^{A} are separated on the boundary of C_3^{A} by a vertex or not. We obtain two configurations each consisting of a circuit with vertices having or not having tails. Both these configurations have unique enlarging by hexagons which cannot be glued with the cycle C_3^{B} having 6 tails.

(iii) This assertion is proved in Deza and Grishukhin (1999b).

(iv) Let C_6 and C_6' be the inner and the outer circuits of a ring of 6 adjacent quadrangles. We have to set 6 new vertices on edges of C_6 and C_6' such that the 6 quadrangles of the ring are transformed into 6 pentagons. It is not difficult to verify that all the new vertices should be set on one of the circuits C_6 and C_6'. Let C_6 have no new vertices. Then it is the boundary of a hexagon. A similar assertion is true for the other 6-cycles of pentagons. This configuration of two 6-cycles of pentagons and the condition $5R_2$ uniquely give the family $(2\text{-APrism}_6^{t+1})^*$ consisting of a 6-cycle of pentagons surrounded by t rings of hexagons (each containing 6 hexagons) and by the other 6-cycle of pentagons.

(v) This assertion is proved in Deza and Grishukhin (1999b). □

Remark. Note the similarity of the assertion (i) of Proposition 4 to the following assertion:

There is a one-to-one correspondence between polyhedra 4_n which are $4R_1$ and the polyhedra 2_{n+6}.

There are infinitely many fullerenes with $G_a=4C_3$: $5_{48}(\mathrm{D}_2)$ and at least one tetrahedral 5_n for $n=4(a^2+ab+b^2)-8$ starting with $5_{40}(\mathrm{T_d})$, $5_{44}(\mathrm{T})$, $5_{56}(\mathrm{T_d})$, $5_{68}(\mathrm{T})$, $5_{76}(\mathrm{T})$; see Fowler and Cremona (1997) for the case (1). The fullerenes $5_{44}(\mathrm{T})$ and $5_{56}(\mathrm{T_d})$ are also $6R_3$ and $6R_4$, respectively. Actually, all face-regular 5_n (besides $26:1$, $30:1$, $38:16$ and $(0,-)$, $(1,-)$) have n divisible by 4.

It looks too hard to describe all fullerenes $5R_0$, and even simpler fullerenes $5R_1$. (All 130 of such $5R_1$ fullerenes for $n\leqslant 72$ are listed in Fowler (1993); the two smallest are a $5_{50}(\mathrm{D}_3)$ and the $5_{52}(\mathrm{T})$.) Let X be a configuration consisting of a hexagon adjacent by its two opposite edges to two pentagons. Clearly, any fullerene 5_n which is $5R_1$ comes from a 5_{n-12} having the 6 configurations X by adding in each X an edge end-vertices of which are the midpoints of the edges of the hexagon of X by which it is adjacent to pentagons.

Now we list all 26 fullerenes which are $6R_j: 3,2,8,5,7,1$ for $j=0,\dots,5$.

It is not difficult to show that the barrel 5_{24}, the 5_{26} and a $5_{28}(\mathrm{T_d})$ are the only fullerenes which are $6R_0$. A not great enumeration shows that only fullerenes which are $6R_1$ are the $5_{28}(\mathrm{D}_2)$ and the $5_{32}(\mathrm{D}_3)$.

$6R_2$-configurations G_b of hexagons are unions of cycles C_m.

Proposition 5. *The only fullerenes which are* $6R_2$ *are the following fullerenes*:

$5_{32}(\mathrm{D_{3h}})$ *with* $G_b=2C_3$,
$5_{38}(\mathrm{C_{3v}})$ *with* $G_b=C_3+C_6$,
$5_{40}(\mathrm{D_{5d}})$ *with* $G_b=2C_5$,
$5_{30}(\mathrm{D_{5h}})$ *with* $G_b=C_5$,

$5_{32}(\mathrm{D_{3d}})$ *and* $5_{32}(D_2)$ *both with* $G_b = C_6$,
$5_{36}(\mathrm{D_{2d}})$ *with* $G_b = C_8$,
$5_{40}(\mathrm{D_2})$ *with* $G_b = C_{10}$.

Proof. It can be done using $n \leqslant 50$ and the inequality for $6R_2$-fullerenes from Lemma 1(i) and scanning the list of small fullerenes in Fowler and Manolopoulos (1995). But we cannot assert that the list contains all $6R_2$-fullerenes. Hence we give below a geometrical proof.

Let a fullerene contain a triple C_3 of mutually adjacent hexagons. Then the triple is surrounded by a ring of 9 pentagons. There are two cases: either the ring of pentagons is surrounded by a ring of hexagons, or not. In the first case we obtain uniquely the fullerene $5_{38}(\mathrm{C_{3v}})$ with $G_b = C_3 + C_6$, which is the case (3) of fullerenes that are $5R_2$. In the second case, we obtain uniquely the fullerene $5_{32}(\mathrm{D_{3h}})$ with $G_b = 2C_3$ such that the ring of 9 pentagons is surrounded by 3 pentagons and 3 mutually adjacent hexagons.

Now consider fullerenes containing only rings C_m of hexagons for $m > 3$. Of course, there is a ring R containing only pentagons inside. We distinguish cases by the number p of pentagons contained inside the ring R. If $p = 1$, then the ring R consists of 5 hexagons. Since the fullerene is $6R_2$, R is surrounded by a ring of 10 pentagons. We obtain a configuration which uniquely defines the fullerene $5_{40}(\mathrm{D_{5h}})$ with $G_b = 2C_5$, where the ring of 10 pentagons is surrounded by the second ring of 5 hexagons, and the outer boundary of the ring is a pentagon.

If $p = 2$, we obtain, as above, uniquely the fullerene $5_{40}(\mathrm{C_2})$, which is not $6R_2$.

For $p = 3$, there are 2 configurations of 3 pentagons: three mutually adjacent pentagons and a chain of pentagons. We obtain again the fullerene $5_{38}(\mathrm{C_{3v}})$ in the first case. The other case does not give fullerenes which are $6R_2$.

For $p = 4$ and 5, there are 4 and 7 configurations of pentagons, respectively. None of them gives a fullerene which is $6R_2$.

There are 18 connected configurations of 6 pentagons. Only 5 of them give fullerenes which are $6R_2$: $5_{30}(\mathrm{D_{5h}})$ with $G_b = C_5$, $5_{32}(\mathrm{D_{3d}})$ and $5_{32}(\mathrm{D_{2h}})$ both with $G_b = C_6$, $5_{36}(\mathrm{D_{2d}})$ with $G_b = C_8$, $5_{40}(\mathrm{D_2})$ with $G_b = C_{10}$.

For $7 \leqslant p \leqslant 12$, we do not obtain fullerenes which are $6R_2$. □

In Deza and Grishukhin (1999a), we give another proof of above proposition for the cases when G_b consists of one circuit C_k, $k = 5, 6, 8, 10$.

The following 5 fullerenes are $6R_3$: $5_{36}(\mathrm{D_2})$, $5_{44}(\mathrm{T})$ which is also $5R_2$, $5_{48}(\mathrm{D_3})$, $5_{52}(\mathrm{T})$ which is $5R_1$, and $5_{60}(\mathrm{I_h})$ which is $5R_0$. Remaining $6R_3$-fullerenes 5_n should have $52 \leqslant n \leqslant 58$, but they do not exist (computer check in Brinkmann, 1998).

The following 5 fullerenes are $6R_4$: a $5_{40}(\mathrm{D_{5d}})$, $5_{56}(\mathrm{T_d})$ which is $5R_2$, $5_{68}(\mathrm{T_d})$ which is $5R_1$, $5_{80}(\mathrm{I_h})$ which is $5R_0$, a $5_{80}(\mathrm{D_{5h}})$ which is $5R_0$. Remaining $6R_4$-fullerenes 5_n should have $52 \leqslant n \leqslant 78$. Computer check by Brinkmann (1998) produced exactly two $6R_4$ in this range: a $5_{68}(\mathrm{D_{3d}})$ coming from $5_{60}(\mathrm{I_h})$ by *triakon* capping (i.e., from an interior point of 6-face to the mid-points of its 3 alternated edges) of 2 opposite 6-faces, and a $5_{72}(D_{2d})$. Their G_a are $2K_3+6K_1$ and $4K_2+4K_1$; their G_b are skeletons of 4-valent polyhedra with $p=(p_3=18, p_5=6, p_6=2)$ and $p=(p_3=20, p_5=p_6=4)$. (So, in their duals, isolated 5-gons are caps of 5-faces of G_b while triples or pairs of 5-gons go on 6-faces of G_b.) $6R_4$-fullerenes $5_{68}(T_d)$ and a $5_{72}(D_{2d})$ come from $5_{56}(T_d)$ by the addition of *bridges*, i.e., new edges connecting the mid-points of opposite edges of a hexagon. Any two triples of pentagons in $5_{56}(T_d)$ are separated by a hexagon. Addition of a bridge on each of those 6 hexagons, produces $5_{68}(T_d)$. (By the same addition of 4 bridges on hexagons separating 4 triples of pentagons, $6R_3$, $5R_1$ fullerene $5_{52}(T)$ comes from $6R_3$, $5R_2$ fullerene $5_{44}(T)$.) Addition of 4 bridges, corresponding to 4 triples (of adjacent pentagons) seen as a 4-cycle, gives a 5_{64}, having $G_a=4K_2+4K_1$, with 4-cycle of pairs of adjacent pentagons separated by a hexagon. Addition of 4 bridges, one on each separating hexagon, gives $6R_4$-fullerenes $5_{72}(D_{2d})$.

The fullerenes 5_n which are both $5R_i$ and $6R_j$ have $n=20+24(5-i)/(6-j)$. Taking such fullerenes from the above list of fullerenes which are $5R_i$ and/or $6R_j$, we obtain

Proposition 6. (i) *All* $(i,j;n)$ *such that there exists* 5_n, *both* $5R_i$ *and* $6R_j$ *are*: $(4,0;24)$, $(3,0;28)$, $(3,2;32)$, $(2,2;38)$, $(2,3;44)$, $(1,3;52)$, $(2,4;56)$, $(0,3;60)$, $(1,4;68)$, $(0,4;80)$, $(0,5;140)$.

Their G_b *are*, *respectively*, $2K_1$, $4K_1$, $2C_3$, C_{10}, *truncated tetrahedron*, *chamfered tetrahedron*, *a* 4-*valent polyhedron with* $p=(p_3=14,\ p_6=6)$, *the dodecahedron*, *a* 4-*valent polyhedron with* $p=(p_3=20,\ p_6=6)$, *the icosidodecahedron*, *its twist*, *snub dodecahedron*.

(ii) *Each of the above* 11 *cases is realized by the following* (*unique with their symmetry*) *fullerenes*:

$5_{24}(D_{6d})$, $5_{28}(T_d)$, $5_{32}(D_{3h})$, $5_{38}(C_{3v})$, $5_{44}(T)$, $5_{52}(T)$, $5_{56}(T_d)$, $5_{60}(I_h)$, $5_{68}(T_d)$, $5_{80}(I_h)$, $5_{80}(D_{5h})$, $5_{140}(I)$.

(iii) *For* $(i,j;n)=(0,4;80)$ *there is exactly one other fullerene*: $5_{80}(D_{5h})=$ *twisted chamfered dodecahedron. The fullerenes of other* 10 *cases are unique*.

(iv) $5_{140}(I)$ *is unique* $6R_5$; 5_{24}, 5_{26}, $5_{28}(T_d)$ *are only* $6R_0$ (*their duals and the icosahedron are called Frank–Kasper polyhedra in chemical physics*; *they appear also as disclinations* (*rotational defects*) *of local icosahedral order*).

For $n<82$, there are 15, 4, 4 fullerenes from 3 respective infinite series: $(0,-)$, $(2,-)$ with $G_5=2C_6$, $(2,-)$ with $G_5=4C_3$; there are 75 fullerenes $(1,-)$ for $n<74$.

In Table 2 we give also the graphs G_a and G_b (of adjacencies of 5- and 6-faces, respectively). We use the following notation:

C_n, P_n, K_n are the cycle, the path and the complete graph, all on n vertices. Truncated K_4 means the skeleton of the truncated tetrahedron. $\hat{C}_n$ is a wheel, i.e., C_n plus an universal vertex. We suppose below that the set of n vertices of a graph is the set $\{1,2,\ldots,n\}$:

$G_1=C_6$ with additional edges $(1,5)$, $(5,2)$, $(2,4)$;

$G_2=C_6$ with additional edges $(1,3)$, $(3,5)$, $(5,1)$;

$G_3=P_5$ with additional edges $(2,6)$, $(3,6)$, $(4,6)$, where 6 is a new vertex;

Table 2

No. in (Fowler and Manolopoulos, 1995)	Symmetry * if unique		(i,j) in $5R_i$, $6R_j$	G_a	G_b
(i) *All face-regular fullerenes with at most 50 vertices*					
20 : 1	I_h	*	$(5,-)$	Icosahedron	—
24 : 1	D_{6d}	*	$(4,0)$	$APrism_6$	$2K_1$
26 : 1	D_{3h}	*	$(-,0)$	G_7	$3K_1$
28 : 1	D_2	*	$(-,1)$	G_8	$2K_2$
28 : 2	T_d	*	$(3,0)$	Truncated K_4	$4K_1$
30 : 1	D_{5h}	*	$(-,2)$	$2\hat{C}_5$	C_5
32 : 2	D_2	*	$(-,2)$	$2G_1$	C_6
32 : 3	D_{3d}	*	$(-,2)$	$2G_2$	C_6
32 : 5	D_{3h}	*	$(3,2)$	G_4	$2C_3$
32 : 6	D_3	*	$(-,1)$	G_9	$3K_2$
36 : 2	D_2		$(-,3)$	$2G_1$	G_5
36 : 6	D_{2d}		$(-,2)$	$2G_3$	C_8
36 : 14	D_{2d}		$(2,-)$	C_{12}	$2P_4$
36 : 15	D_{6h}	*	$(2,-)$	$2C_6$	C_6+2K_1
38 : 16	C_{3v}	*	$(2,2)$	C_3+C_9	C_3+C_6
40 : 1	D_{5d}		$(-,4)$	$2\hat{C}_5$	$APrism_5$
40 : 38	D_2		$(-,2)$	$2P_6$	C_{10}
40 : 39	D_{5d}		$(-,2)$	$C_{10}+2K_1$	$2C_5$
40 : 40	T_d	*	$(2,-)$	$4C_3$	G_6
44 : 73	T	*	$(2,3)$	$4C_3$	Truncated K_4
44 : 85	D_2		$(2,-)$	C_{12}	$2G_1$
44 : 86	D_{3d}		$(2,-)$	C_{12}	$2G_2$
48 : 144	D_2		$(2,-)$	$4C_3$	G_{11}
48 : 186	D_{6d}		$(2,-)$	C_{12}	$2\hat{C}_6$
48 : 188	D_3	*	$(-,3)$	$3P_4$	G_{10}
48 : 189	D_{6d}		$(2,-)$	$2C_6$	$APrism_6+2K_1$
50 : 270	D_3		$(1,-)$	$6K_2$	G_{12}
(ii) *All fullerenes* 5_n *both* $5R_i$ *and* $6R_j$ *with* $n>50$					
52	T	*	$(1,3)$	$6K_2$	Chamfered K_4
56	T_d	*	$(2,4)$	$4C_3$	4-valent, $p=(p_3=14, p_6=6)$
60	I_h	*	$(0,3)$	$12K_1$	Dodecahedron
68	T_d	*	$(1,4)$	$6K_2$	4-valent, $p=(p_3=20, p_6=6)$
80	I_h	*	$(0,4)$	$12K_1$	Icosidodecahedron
80	D_{5h}	*	$(0,4)$	$12K_1$	Its twist
140	I	*	$(0,5)$	$12K_1$	Snub dodecahedron

$G_4=C_9$ with additional edges $(1,10)$, $(2,10)$, $(3,10)$, $(4,11)$, $(5,11)$, $(6,11)$, $(7,12)$, $(8,12)$, $(9,12)$, where 10, 11 and 12 are new vertices;

$G_5=C_6$ with additional edges $(1,7)$, $(2,7)$, $(3,7)$, $(4,8)$, $(5,8)$, $(6,8)$, where 7 and 8 are new vertices;

$G_6=C_8$ with additional edges $(1,9)$, $(9,5)$, $(3,10)$, $(10,7)$, where 9 and 10 are new vertices;

$G_7=C_6$ and C_6' with additional edges (1,3), (3,5), (5,1), $(1',3')$, $(3',5')$, $(5',1')$, $(2,2')$, $(4,4')$, $(6,6')$;

G_8= two isomorphic graphs G_1 and G_1' connected by edges (x,x') and (y,y'), where x,y and x',y' are pairs of vertices of valency 2 in G_1 and G_1', respectively;

$G_9 = C_8$ with additional edges $(1,9)$, $(2,9)$, $(9,10)$, $(10,11)$, $(5,11)$, $(6,11)$, where 9, 10, 11 are new vertices;

G_{10}=3 isomorphic graphs G_3, G_3' and G_3'' with identified vertices $x=x'=x''$ and $y=y'=y''$, where x, y, x', y' and x'', y'' are pairs of vertices of valency 1 in each of the 3 graphs;

G_{11}= two isomorphic graphs G_1 and G_1' with additional edges $(3,3')$, $(6,6')$, $(2,7)$, $(2',7)$, $(5,8)$, $(5',8)$; without the new vertices 7 and 8 it is G_8;

G_{12}= Dürer octahedron plus 3 new vertices; the midpoints of 3 edges which are disjoint pairwise and with each triangle.

The asterisk '*' in the symmetry column of Table 2 means that the fullerene is unique with this symmetry and corresponding number of vertices.

Call a fullerene *quasi*-$6R_j$ if the number of 5–6 edges divided by the number of hexagons is $6-j$. If it is, moreover, not $6R_j$, then call it *proper quasi*-$6R_j$. There are in all 65 proper quasi-$6R_j$ with at most 50 vertices: 3, 2, 4, 17, 1, 38 for $(n,j)=(36,2)$, $(38,2)$, $(40,3)$, $(44,3)$, $(48,4)$, $(48,3)$, respectively. The unique case of $j=4$ above is the fullerene 48 : 1. Among above 65 fullerenes only 38 : 10 and 38 : 14 are also *quasi*-$5R_i$, i.e., the number of 5–6 edges is divided by the number 12 of pentagons. In fact, they are both proper quasi-$5R_2$ and quasi-$6R_2$. The second one looks as a double spiral: its 5-graph G_a is the path P_{12} with an additional edge $(1,3)$ and its 6-graph G_b is the path P_9 with an edge $(1,3)$. The fullerenes 44 : 85, 86 in Table 2 are quasi-$6R_3$ while 32 : 2, 3 and 36 : 2 are quasi-$5R_3$.

Some similarities between the graphs G_a, G_b and between symmetries in Table 2 indicate examples of operations on fullerenes (Fig. 1):

(i) 30 : 1 and 40 : 1 have the same graph $G_a = 2\hat{C}_5$ and they belong (for $t=1,2$) to the sequence of t-hex-elongated dodecahedron 5_{20}, i.e., the dual of column of $t+1$ APrism$_5$ capped on 2 opposite 5-faces;

(ii) by deleting 4 points of $K_{1,3}$ in any triple of adjacent pentagons, we obtain 28 : 2, 40 : 40 from 44 : 73 and the $5_{56}(\mathrm{T_d})$, respectively;

(iii) by deleting all six 5–5 edges from the unique $5_{56}(\mathrm{D_{3d}})$ which is $5R_1$, we obtain 44 : 86;

(iv) all 5 (resp. 4) fullerenes having a cycle as G_b (resp. G_a) are given in Table 2. A cutting of each hexagon in the cycle into two pentagons produces 44 : 85, 44 : 86, 40 : 39 from 32 : 2, 32 : 3, 30 : 1, respectively. The same cutting of hexagons of the 6-cycle (of hexagons) produces 48 : 186 from 36 : 15. The same cutting of alternating hexagons of the 8-cycle (of hexagons) produces 5_{44} from 36 : 6. The 5-graph G_a of 40 : 38 is $2P_6$ while the 6-graph G_b of 36 : 14 is $2P_4$.

5. Face-regular polyhedra $(3,4)_n$

Proposition 7. *All polyhedra* $(3,4)_n$ *which are* $4R_j$ *are as follows*:

(i) 3 *polyhedra* $4R_0$: *APrism*$_4 = (3,4)_8$ *which is* $3R_2$ *with* $G_b = 2K_1$, *the* $(3,4)_9$ *with* $G_b = 3K_1$ *and the cuboctahedron* $(3,4)_{12}$ *which is* $3R_0$ *with* $G_b = 6K_1$;

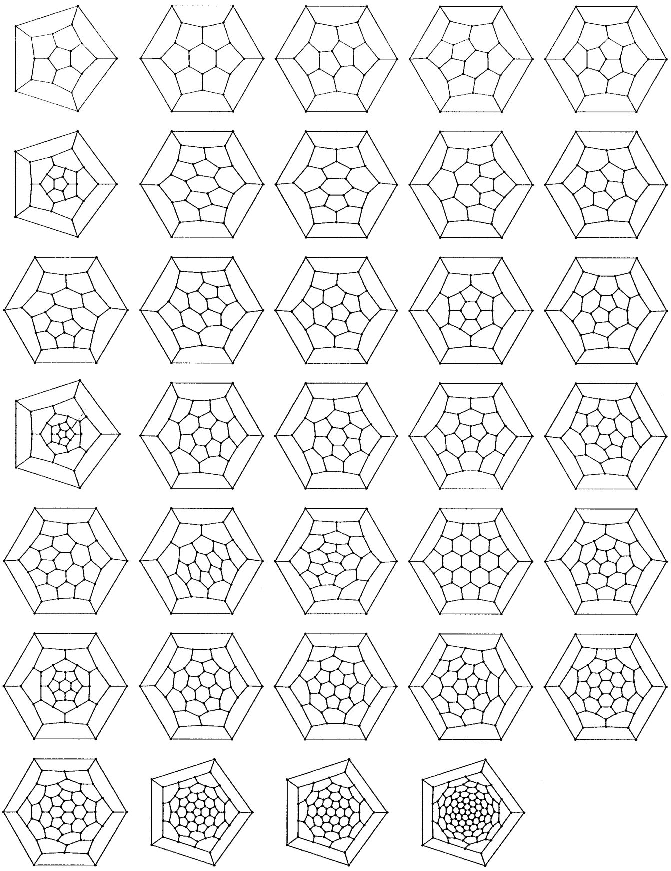

Fig. 1. All 34 face-regular fullerenes (i), (ii) of Table 2 in the order are given here.

(ii) 2 *polyhedra* $4R_1$: *a* $(3,4)_{10}$ *with* $G_b=2K_2$, $G_a=2P_4$ *and a* $(3,4)_{12}$ *which is the twisted cuboctahedron with* $G_b=3K_2$;

(iii) 5 *polyhedra* $4R_2$: 2-*Prism*$_4=(3,4)_{10}$ *which is* $3R_2$ *with* $G_b=C_4$, *two* $(3,4)_{12}$ *with* $G_b=C_6$ *and two non-isomorphic* $(3,4)_{14}$, *both are* $3R_1$, *with* $G_b=C_8$ *and* $G_b=2C_4$;

(iv) 3 *polyhedra* $4R_3$: 2-*Prism*$_4^2=(3,4)_{14}$ *which is* $3R_2$, *a* $(3,4)_{22}$ *which is* $3R_1$ *and the cross-capped truncated cube* $(3,4)_{30}$ *which is* $3R_0$. G_b *graphs of these polyhedra are planar cubic graphs consisting of quadrangles and hexagons. They are skeletons of the cube* 4_6, *the unique polyhedron* 4_{16}, *and the truncated octahedron* 4_{24}, *respectively.*

Proof. The properties $4R_i$ and $3R_j$ give rather restrictive conditions on adjacencies of triangles and quadrangles. Not very complicated enumeration of possible configurations provide the configurations described in items (i)–(iv) of the proposition. □

Taking from polyhedra of Proposition 7, ones that are $3R_i$, we obtain

Corollary 1. (i) *All* $(i,j;n)$ *such that there exists* $(3,4)_n$ *both* $3R_i$ *and* $4R_j$ *are as follows*: $(2,0;8)$, $(2,2;10)$, $(0,0;12)$, $(1,2;14)$, $(2,3;14)$, $(1,3;22)$, $(0,3;30)$.

(ii) *Each, except* $(1,2;14)$, *of the above 7 cases is realized by unique* $(3,4)_n$: *APrism*$_4 = (3,4)_8$, 2-*Prism*$_4 = (3,4)_{10}$, *the cuboctahedron*$=(3,4)_{12}$, 2-*Prism*$_4^2 = (3,4)_{14}$, *a* $(3,4)_{22}$ *and the* cross-capped *truncated cube*$=(3,4)_{30}$. □

It looks too hard to describe all $(3,4)_n$ which are $3R_0$; for example, the *ambo* (i.e., the convex hull of the mid-points of all edges) of $(3,4)_n$ is $(3,4)_{2n}$ which is $3R_0$.

There are other operations to obtain one $(3,4)_n$ from another. In Manca (1979), four operations are given that transform a 4-valent planar graph into another 4-valent planar graph with more vertices.

The first 3 of these operations can be applied to $(3,4)_n$ such that the obtained polyhedra belong again to the class $(3,4)_n$. But, unfortunately, there are polyhedra $(3,4)_n$ that cannot be obtained by these operations from a $(3,4)_{n'}$ with $n' < n$. In other words, these polyhedra can be obtained from another $(3,4)_{n'}$ by a sequence of these operations such that intermediate polyhedra do not belong to the class $(3,4)_n$.

There are operations that transform polyhedra of the class $(3,4)_n$ with $3R_1$ again into polyhedra of this class with $3R_1$ and more vertices. This shows that there are infinitely many polyhedra with $3R_1$.

There is the following bijection between polyhedra $(3,4)_n$ with $3R_1$ and polyhedra $(3,4)_n$ with $3R_0$ such that the 8 triangles of last polyhedra are partitioned into 4 disjoint pairs of triangles having exactly one common point. Delete, in any $(3,4)_n$ which is $3R_1$, 4 edges separating triangles in the 4 pairs. The dual of obtained polyhedron is a $(3,4)_{n-2}$ which is $3R_0$ with above adjacencies of triangles.

But the case of $3R_2$ is much simpler; we have the following:

Proposition 8. *The only polyhedra* $(3,4)_n$ *which are* $3R_2$ *are either* $(3,4)_8 =$ *APrism*$_4$ *or the family* 2-*Prism*$_4^t$, $t \geqslant 1$, *of t-elongated octahedra.*

Proof. Let t_0 be a triangle of $(3,4)_n$ with $3R_2$. Then t_0 is adjacent to two triangles t_1 and t_2. These triangles are adjacent to other triangles. There are two cases: either t_1 and t_2 have or have not a common second adjacent triangle. In the first case, we obtain a configuration of 4 triangles surrounded by 4 quadrangles. This configuration generates the family 2-Prism$_4^t$. In the second case, we obtain, uniquely, APrism$_4 = (3,4)_8$. □

All face-regular $(3,4)_n$ (except two infinite sets for $3R_0$, $3R_1$) are 8 polyhedra from Corollary 1, the unique $(3,4)_9$, the twisted cuboctahedron $(3,4)_{12}$, a $(3,4)_{10}$, a $(3,4)_{12}$ and all t-elongated octahedra for $t > 2$. The rhombicuboctahedron and its twist are $(3,4)_{24}$ which are $3R_0$. Their graphs G_b are the octahedron plus a new vertex on each edge and, respectively, $C_8 + K_{9;1,3,5,7} + K_{10;2,4,6,8}$. The snub cube is a 5-valent polyhedron with $p = (p_3 = 32,\ p_4 = 6)$ obtained from the rhombicuboctahedron by cutting 12 of its squares into two triangles; its graph G_b is the cube plus two new vertices on each edge.

Remark that dual $(3,4)_n$ are exactly the case $d=3$ of *almost simple cubical* d-polytopes in terms of Blind and Blind (1998). Actually, Blind and Blind (1998) characterizes those d-polytopes for $d \geqslant 4$. It is not difficult to enumerate them for $d=2$; they are $P_2 \times P_n$, for any finite n and $n = Z_+, Z$ and the cube with deleted vertex or edge. But this variety becomes too rich for $d=3$.

Heidemeier (1998) computed the number of polyhedra $(3,4)_n$ for all $n = 6, 7, \ldots, 50$. It is 1, 0, 1, 1, 2, 1, 5, 2, 8, 5, 12, 8, 25, 13, 30, 23, 51, 33, 76, 51, 109, 78, 144, 106, 218 for $n = 6, \ldots, 30$.

6. Some remarks

Among face-regular bifaced polyhedra, which are both aR_i and bR_j, the most interesting are those that satisfy the following conditions:

(a) $(i,j) = (0, b-1)$: dual 2-capped APrism$_4$, Dürer's octahedron, the dual 4-triakis snub tetrahedron 3_{28}(T), the dual tetrakis snub cube 4_{56}(O), the dual pentakis snub dodecahedron 5_{140}(I) and the cross-capped truncated cube $(3,4)_{30}$(O).

(b) $p_a = p_b$: dual bidishpenoid, the truncated tetrahedron 3_{12}, two 4_{20}, 5_{44}(T) and three $(3,4)_{14}$ with $(i,j;p_a) = (1,2;4)$, $(0,3;4)$, $(2,4;6)$, $(1,3;6)$, $(2,3;12)$, $(1,2;8)$, $(1,2;8)$, $(2,3;8)$, respectively.

(c) $i=j$: $4_{14} = (\text{APrism}_3^2)^*$, $5_{38}(C_{3v})$ and $(3,4)_{10} = $ 2-Prism$_4$, each has $i=j=2$ and consists of concentric belts of a- or b-gons (belts sizes are 3, 3, 3 for above 4_{14}, 3, 6, 9, 3 for $5_{38}(C_{3v})$ and 4, 4, 4 for 2-Prism$_4$).

All bifaced polyhedra with all a-faces forming a ring are: Prism$_5$, Prism$_6$, APrism$_4$ and 4 fullerenes with 36, 44, 44, 48 vertices. All bifaced polyhedra with all b-faces forming a ring are: Prism$_3$, dual of bidisphenoid, the 4_{14}, the 4_{16}, 2-Prism$_4$, a $(3,4)_{12}$, a $(3,4)_{14}$ and 5 fullerenes with 30, 32, 32, 36, 40 vertices.

It turns out that all bR_j bifaced polyhedra with $j > 2$ have as G_b, the skeleton of a $bR_{j'}$ bifaced (or regular) polyhedra. In particular, all three bR_4 polyhedra 4_n ($n = 26, 32, 32$) have as G_b, polyhedra $(3,4)_n$, and all three bR_3 polyhedra $(3,4)_n$ ($n = 14, 22, 30$) have as G_b, polyhedra 4_n.

The face-regularity aR_i or bR_j can be compared with other, relevant for applications of topological indices of a bifaced polyhedron. For example, it can be compared with description of vertices by the vertex type or with the pair (q_a, q_b), where q_a (resp. q_b) is the number of maximal connected sets of a-gons (resp. b-gons). Here we call a set of faces *connected* if its graph G_a (resp. G_b) is connected. For example, $q_a + q_b \leqslant p_a + p_b$ with equality for the cuboctahedron; $q_a = p_a$ or $\frac{1}{2} p_a$, if a bifaced polyhedron is aR_0 or aR_1, respectively.

It will be interesting to identify face-regular polyhedra among the following simple polyhedra, generalizing those considered in this paper (see Grünbaum, 1967 for the existence):

(1) with $p = (p_3 = 2, p_4 = 3, p_6)$; it exists unless $p_6 = 1, 3, 7$;

(2) with $p = (p_4, p_5 = 12 - 2p_4, p_6)$; it exists unless $(p_4, p_6) = (1, 0)$, $(i, 1)$ for

$i=0,1,5,6$; 3) with $p=(p_3, p_5=12-3p_3, p_6)$; it exists unless $(p_3, p_6)=(0,1)$, $(1,i)$ for $i=0,1,2,4$ or (2,1), $(3,i)$ for $i=0,2,4$ or (4,2), (4, any odd).

One can look for face-regular bifaced infinite polyhedra, i.e., partitions of Euclidean plane. For example, all 6 bifaced Archimedean plane partitions are face-regular. In fact, (3.6.3.6), (4.8.8), (3.12.12), (3.3.3.3.6), (3.3.3.4.4), (3.3.4.3.4) are (0,0), (0,4), (0,6), $(-,0)$, $(2,2)$, $(1,0)$, respectively.

There are face-regular simple polyhedra with $p=(p_5, p_b)$, $b>6$. Icosahedral polyhedra with $p=(p_5=72, p_{6+i}=60/i)$ for $i=1,2,3$ (so, $n=140+120/i$) with G_b being snub dodecahedron, icosidodecahedron, dodecahedron, respectively, are b-face-regular $(6+i)R_{(6-i)}$. On the other hand, following 3 simple bifaced polyhedra are (3,0) face-regular. They have $p=(p_5=12, p_6=4)$, $(p_5=24, p_8=6)$, $(p_5=60, p_{10}=12)$ and $n=28, 56, 140$, respectively. Their G_5 is truncated tetrahedron, truncated cube, truncated dodecahedron and they have tetrahedral, octahedral, icosahedral symmetry, respectively. They come as a triakon decoration of all $4, 8, 20$ hexagons of truncated tetrahedron, truncated octahedron, truncated icosahedron, respectively.

Finally, the quotient (by the antipodal map) of any centrally symmetric polyhedron can be realized (and its f- and p-vectors will be halved) on the real projective plane P^2. The centrally symmetric polyhedra have symmetry C_i, T_h, O_h, I_h or C_{nh}, D_{nh} for even n and D_{nd} for odd n; so the fullerenes among them have symmetry C_i, C_{2h}, D_{2h}, D_{6h}, D_{3d}, D_{5d}, T_h, I_h. The results of this paper imply similar classifications of face-regular bifaced maps on P^2; for example, the quotient of the cuboctahedron $(3,4)_{12}$ is the heptahedron, which is also $3R_0$ and $4R_0$. But face-adjacency should be counted, sometimes, with multiplicity: for example, any two 5-gonal faces of the quotient of Dürer octahedron (it is $5R_4$ and it is unique centrally symmetric one among 6 sporadic polyhedra, given in (A) of Section 1) intersect in *two* edges.

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