Maps of p-gons with a ring of q-gons

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Abstract

We study 3-valent maps $M_n(p,q)$ consisting of a ring of n q-gons whose the inner and outer domains are filled by p-gons, for $p,q \geq 3$. We describe a domain in the space of parameters p, q, and n, for which such a map may exist. With four infinite sequences of maps - prisms $M_p(p \geq 3,4)$, $M_4(4,q \geq 4)$, $M_4(5,5t+2 \geq 7)$, $M_4(5,5t+3 \geq 8)$, we give 26 sporadic ones. The maps whose p-gons form two paths are first two infinite sequences and 5 maps: $M_{28}(7,5)$, $M_{12}(6,5)$, $M_{10}(5,6)$, $M_{20}(5,7)$, $M_2(3,6)$.

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Let $M_n(p,q)$ be a planar graph with all vertices of degree 3 having only p-gonal and q-gonal faces such that the q-gonal faces form a ring R_n of n q-gons. We want to know for which n, p and q such maps exist. Clearly, the dual of $M_n(p,q)$ is a triangulation, in which all vertices are p-valent, except n q-valent vertices forming an induced simple circuit.

We dismiss trivial maps $M_3(2,2+4t)$ (with t consecutive digons on each of 3 edges of dual triangle) and $M_3(p,p)$, $3 \leq p \leq 5$, of simple Platonic polyhedra. So, one can consider only the case $q \geq 4$ since the map $M_3(3,3)$ is unique $M_n(p,3)$. We represent the ring R_n as a ring of quadrangles as follows. Each quadrangle has two pairs of opposite edges. Edges of one pair belong to neighboring quadrangles. Edges of another pair belong to the outer and inner boundaries of the ring. In order to transform a quadrangle into a q-gon, we set $q-4$ vertices on the edges of the quadrangle belonging to the outer and inner boundaries. If (i,j) is the pair of numbers of the vertices on the outer and inner edges of quadrangles, then the following pairs are possible: $(0,q-4)$, $(1,q-5)$,...,$(q-4,0)$. So any of our maps will be a decoration of a n-gonal prism.

Denote by I_n and O_n the parts of $M_n(p,q)$ consisting of the inner and outer domains of R_n, with the common boundary of R_n.

If $q=4$, then the map $M_n(p,4)$ exists only for $n=p$. In this case I_n and O_n each is a p-gon, and $M_p(p,4)$ is the map of a p-gonal prism with quadrangle lateral faces. Hence, from now on we suppose that $q \geq 5$.

Consider the part I_n of $M_n(p,q)$ consisting of p-gons in the inner domain of R_n. Vertices of I_n have degrees 2 and 3. Let v_2 and v_3 be the numbers of vertices of degree 2 and 3 lying on the common boundary of I_n and R_n. The vertices of degree 2 are end-vertices of edges common to two adjacent q-gons. Obviously, there are $v_2 = n$ such vertices. Let x be the number of interior vertices of I_n.

We are interested especially in maps $M_n(p,q)$ such that I_n and O_n each is a path of p-gons. We say in this case that $M_n(p,q)$ *has two paths of p-gons*.

Now we apply results of [1] to the map I_n. The boundary of I_n generates a *p-gonal boundary sequence* studied in [1]. Each p-gonal sequence has the form $a = a_1a_2...a_k$, where k is the length of the sequence a. Here a_i is the number of vertices of degree 2 between the *tails* t_i and t_{i+1}. Namely, the tail t_i is an edge of I_n having exactly one end-vertex (of degree 3) on its boundary.Hence we have

$$k = v_3 \text{ and } \sum_{i=1}^{k} a_i = v_2. \tag{1}$$

Note that if $p = n$, then there is a degenerate case, when one of the domains I_n and O_n or both (p-prism) may consists of one p-gon. In this case $v_3 = 0$, what implies $k = 0$. The corresponding p-gonal sequence a is degenerate also.

Similarly, let v'_2, v'_3, x' be the numbers of the corresponding vertices of the outer domain O_n. The boundary of O_n generates a boundary sequence of length $k' = v'_3$. Of course, we have

$$v'_2 = v_2 = n, \quad v'_3 + v_3 = (q-4)n. \tag{2}$$

The equalities (1) and (2) imply

$$\sum_{i}^{k} a_i = n, \quad \sum_{i}^{k'} a'_i = n, \quad k + k' = (q-4)n. \tag{3}$$

We call the sequence a' the *q-complement* of a, and vice versa. A sequence a is called *self q-complemented* if a' is obtained from a by shifting or/and reversing.

Theorem 1 *For $p, q \geq 4$, a map $M_n(p,q)$ may exist only for the following values of the parameters n, p and q.*

A *If $q = 4$, then $n = p$ and $M_p(p,4)$ is the map of a p-prism.*

B *If $q \geq 5$, then $p \leq 7$, and if $p \geq 6$, then $q = 5$. Besides*

1) if $p = 7$, then $n \geq 28$ and unique $M_{28}(7,5)$ has two paths of 7-gons;

2) if $p = 6$, then all existing maps are four $M_{12}(6,5)$ and one of them has two paths of hexagons (see Figure 1);

3) if $p = 5$, then

a) if $q = 5$, then there exist only two maps $M_6(5,5)$ and one $M_5(5,5)$, all three realizing the dodecahedron;

b) if $q = 6$, then $n \leq 10$, and all existing maps are: $M_5(5,6)$, two maps $M_6(5,6)$, $M_8(5,6)$ and (with two paths of pentagons) $M_{10}(5,6)$ (see Figure 1);

c) if $q = 7$, then $n \leq 20$ and all existing maps are (except of undecided case $17 \leq n \leq 19$*) following nine: $M_4(5,7)$, $M_{10}(5,7)$ and (with two paths of pentagons) $M_{20}(5,7)$ (see Figure 2), and (from [3], see Figure 3) 4 maps $M_{12}(5,7)$, 2 maps $M_{16}(5,7)$;*

if $q \geq 8$, then if a map $M_n(5,q)$ exists, it has not paths of pentagons, and the following maps exist: $M_2(5,10)$, $M_3(5,8)$ and $M_4(5,q)$, for any $q \equiv 2,3 (mod\, 5)$, but not for $q = 9$;

4) if $p = 4$, then $n \leq 4$ and all existing maps are: $M_2(4,8)$, $M_3(4,6)$ and (with two paths of $q-3$ quadrangles) $M_4(4,q)$ for any $q \geq 4$;

5) if $p = 3$, then there exist exactly two maps $M_2(3,6)$ and $M_3(3,4)$, where the last map is the special case $p = 3$ of **A**.

Proof. The case **A** was described above. Consider the case **B**.

We use the Euler relation $v - e + f = 1$ for numbers of vertices v, edges e and faces f of I_n. We have $v = v_2 + v_3 + x$, $2e = 2v_2 + 3(v_3 + x)$, $pf = v_2 + 2v_3 + 3x = v_2 - v_3 + 3(v_3 + x)$. Substituting these values in the Euler relation, one get

$$2v_2 - (p-4)v_3 + (6-p)x = 2p.$$

A similar equality is valid for O_n. Summing them and using (2), we get

$$((4-p)(q-4) + 4)n + (6-p)(x + x') = 4p. \tag{4}$$

Consider at first the case $p \geq 7$. Rewrite (4) as follows:

$$(p-6)(x+x') = (4-(p-4)(q-4))n - 4p.$$

Since $p-6>0$ and $x+x' \geq 0$, one has $4-(p-4)(q-4)>0$. For $p \geq 7$, this inequality holds only if $q=5$ and $p=7$. For these values p and q, the equation (4) takes the form

$$x + x' = n - 28.$$

So, no map $M_n(7,5)$ exists for $n<28$. For $n=28$, $x+x'=0$. Hence if a map $M_{28}(7,5)$ exists, then I_{28} and O_{28} consist each of a path of heptagons. In fact, such map exists and it is unique for $n=28$.

Now we consider the case $p=6$. In this case (4) takes the form

$$(12-2q)n = 24.$$

Hence $12-2q>0$, i.e. $q<6$. Since $q \geq 5$, we have $q=5$ and $n=12$. There exist four maps $M_{12}(6,5)$, and only one of them has two paths of hexagons. (See Proposition 9.1 of [2].)

The case $p=5$ is the most rich case. Now (4) takes the form

$$x + x' = 20 - (8-q)n. \tag{5}$$

For $q<8$, this equality restricts n, since $x+x' \geq 0$.

Now we will use Table 1, which is Table 3 of [1] extended by the sequences of length $k=9$. For given n and q, we inspect pentagonal sequences of length $k \leq \frac{1}{2}(q-4)n$ and such that $\sum_1^k a_i = n$. Call such a sequence *feasible*. We seek feasible sequences having pentagonal q-complements. At right of a feasible sequence we give in parentheses the map related to this sequence.

Consider at first the case $q=5$. The equality (5) gives $n \leq 6$. But now $M_n(5,5)$ is a partition of the plane into pentagons. There exists only one such partition and it is the dodecahedron. It can be realized as one map $M_5(5,5)$ with a degenerated sequence a, and two maps $M_6(5,5)$, one with the self 5-complemented sequence 222 and another with the sequence 2121 and its 5-complement 33.

Table 1. Pentagonal sequences of length k at most 9.

k	n	$n=\sum_1^k a_i$, pentagonal sequences					
2	6	33	$(M_6(5,5))$				
3	6	222	$(M_6(5,5))$				
4	6	2121	$(M_6(5,5))$				
	7	3130					
5	5	11111	$(M_5(5,6))$				
	6	21120					
	7	30220					
6	2	100100	$(M_2(5,10))$				
	3	101010	$(M_3(5,8))$				
	4	110110	$(M_4(5,7))$				
	5	201110					
	6	202020	$(M_6(5,6))$	210210	$(M_6(5,6))$		
	7	301210					
	8	311300		310310			
7	4	2001100					
	5	2010200					
	6	2101200					
	7	3011200					
	8	3102200		3020300			
8	4	20002000	$(M_4(5,8))$				
	6	30011100		21002100			
	7	30102100		30020200			
	8	31012100		30103010	$(M_8(5,6))$	22002200	
	9	31113000		31103100		31013010	
9	5	300011000					
	6	300102000		210012000			
	7	301012000					
	8	310112000		310021100		301103000	220012100
	9	311022000		310203000		310022010	300300300
	10	311002201					

Let $q = 6$. The equality (5) takes the form $x + x' = 20 - 2n$. Hence $n \leq 10$. One has to consider feasible pentagonal sequences of length $k \leq n$. For $2 \leq n \leq 10$, we find the following feasible pentagonal sequences:

$n = 5$, $k = 5$: 11111. It is a self 6-complemented sequence giving $M_5(5,6)$.

$n = 6$: for $k = 2,3,4,5$ we find the feasible sequences 33, 222, 2121, 21120, respectively. (Note that the 5-complement of 222 is pentagonal giving a map $M_6(5,5)$ of the dodecahedron.) For $k = 6$, we find two feasible sequences 202020 and 210210 (both are self-complemented), giving two distinct maps $M_6(5,6)$.

For $n = 7$ all feasible sequences have no pentagonal 6-complement.

For $n = 8$ there is unique feasible sequence 30103010 of length 8, which is self 6-complemented and gives the map $M_8(5,6)$.

For $n = 9$: the 6-complement of unique feasible sequence is not pentagonal.

If $n = 10$, then $x + x' = 0$. There exists a map $M_{10}(5,6)$ having two paths of pentagons. It is unique for $n = 10$.

Let $q = 7$. The equality (5) takes the form $x + x' = 20 - n$. Hence $n \leq 20$. We have to consider feasible sequences of length $k \leq \frac{3}{2}n$. For $n = 2$ and $n = 3$ there are no feasible sequences. Hence maps $M_n(5,7)$ do not exist for $n \leq 3$.

For $n = 4$, Table 1 shows that amongst of all sequences of length $k \leq 6$ there is unique feasible sequence, namely 110110, This sequence is self 7-complemented. Hence in this case both maps I_4 and O_4 are isomorphic and contain 8 pentagons each. We obtain the map $M_4(5,7)$, starting the sequence $M_4(5, 5t+2)$.

For $n = 5$, Table 1 shows that amongst all sequences of length $k \leq 7$ there are 3 feasible sequences 11111, 201110 and 2010200. But the 7-complements of all these sequences are not pentagonal. So, there is no map $M_5(5,7)$.

For $n = 6$, Table 1 gives 11 feasible sequences, but the 7-complements of all these sequences are not pentagonal.

There exist $M_{10}(5,7)$ with pentagonal sequence bb for $b = 3010010$; its 7-complement is pentagonal sequence bb for $b = 21001001$.

Harmuth ([3]) checked by computer all maps $M_n(5,7)$ with $n \leq 16$. He got six new maps (four with $n = 12$ and two with $n = 16$); see them on Figure 3. Amongst them, only 3rd and 4th have $I_n \neq O_n$. The 4th map $M_{12}(5,7)$ has non-periodic pentagonal sequence, other five have it of form bb.

Note the extremal case $n = 20$. In this case $k + k' = 60$. There exists a feasible self 7-complemented sequence of length 30. This sequence has the form bb, where $b = 300100101011011$. The inner and outer maps I_{20} and O_{20} of the map $M_{20}(5,7)$ are isomorphic. In this case $x = x' = 0$ and $v_3 = v_3' = \frac{3}{2}n = 30$. Hence each of them consists of a path of $f = 1 + \frac{1}{2}v_3 = 1 + \frac{1}{2}v_3' = 16$ pentagons. This map is unique for $n = 20$; its symmetry is S_4.

If $q \geq 8$, the equality (5) gives no restriction on n. There exist maps $M_3(5,8)$, $M_2(5,10)$, corresponding to pentagonal self q-complemented sequences with parameters $(k,n) = (6,3), (6,2)$, respectively. Moreover, there exist maps $M_4(5,q)$ for any $q = 5t + 3 \geq 8$ and $q = 5t + 2 \geq 7$. They have self q-complementary sequences bb with $b = 20...0$ ($5t - 2$ zeros) and $b = 110...0$ ($5t - 4$ zeros), respectively. The first is a decorated (not on the 4-ring) map $M_4(4, 5t+3)$; the second is a decorated $M_4(4,5)$. See the smallest case $t = 1$ (i.e. $q = 7,8$) of those maps on Figure 2; the larger ones come by repetition of their I_n, O_n t times. One can check non-existence of $M_4(4,9)$, using that its sequence a should be of form bb (since it period divides 4) and has $k = 10$ (since Table 1 has no needed sequence with $n = 4$ and $k \leq 9$ or $k' \leq 9$).

Let $p = 4$. In this case (4) takes the form

$$n = 4 - \frac{1}{2}(x + x'),$$

that implies that $n \leq 4$. An inspection shows that there exist the following maps

$$M_2(4,8), \quad M_3(4,6), \quad M_4(4,q),\ q \geq 4.$$

Since $x + x' = 0$ for $n = 4$, the inner and outer domains of the map $M_4(4,q)$, each consists of a path of $q-3$ quadrangles.

The map $M_4(4,q)$ can be obtained from the prism $M_p(p,4)$ for $p = q$ as follows. Select a 4-gon Q in the ring R_q of $M_q(q,4)$ and put in Q new $q-4$ edges parallel to edges of Q common with the two q-gons. Then Q will be replaced by a path of $q-3$ quadrangles. The two 4-gons, which were neighboring to Q, became q-gons that form a ring of four q-gons together with the old two q-gons. (The dual of $M_4(4,5)$ is the map of *snub dishpenoid*; its faces are 12 regular triangles.)

Now, consider the last case $p = 3$. In this case (4) takes the form

$$x + x' = 4 - \frac{1}{3}qn.$$

It implies that $qn \leq 12$ and qn is a multiple of 3. Since $q \geq 4$, only two pairs $(n,p) = (2,6)$ and $(n,p) = (3,4)$ are possible. Both corresponding maps $M_2(3,6)$ and $M_3(3,4)$ exist with isomorphic I_n and O_n. □

Clearly, $M_n(p,q)$ has two paths of p-gons if and only if $x + x' = 0$ and such paths consist of $\frac{n-4}{p-4}$ p-gons. So, in this case (4) implies $n = \frac{4p}{4-(p-4)(q-4)}$ and $(p,q) = (5,6),(5,7),(6,5),(7,5),(3,6)$. Any such map comes from $M_4(4, \frac{n-4}{p-4}+3)$ by addition of $n-4$ edges on the 4-ring.

Corollary *Besides two infinite sequences of maps: $M_p(p,3)$ and $M_4(4,q)$, there are only 5 maps $M_n(p,q)$, having two paths of p-gons: $M_2(3,6)$, one of 4 maps $M_{12}(6,5)$, $M_{10}(5,6)$, $M_{28}(7,5)$, $M_{20}(5,7)$ (see Figures 1,2).*□

Remarks:

(i) Maps $M_n(5,6)$ and $M_n(6,5)$ are instances of *fullerenes*, known in Organic Chemistry; they are 9 maps (of cases **B**2) and **B**3)b) of Theorem 1) given on Figure 1. In notation $F_m(Aut)$, stressing their number of vertices (carbon atoms) and the maximal symmetry, those 9 maps are: $F_{36}(D_{2d})$, $F_{44}(D_{3d})$, $F_{48}(D_{6d})$, $F_{44}(D_2)$ and $F_{30}(D_{5h})$, $F_{32}(D_2)$, $F_{40}(D_2)$, $F_{32}(D_{3d})$, $F_{36}(D_{2d})$.

The 4 simple polyhedra with only 4- and 6-gonal faces and at most 16 vertices, correspond to 4 maps $M_4(4,4), M_6(6,4)$ (4- and 6-prisms) and $M_3(4,6), M_4(4,6)$.

The 26 maps $M_n(p,q)$, given here (which are not covered by families $M_p(p \geq 3,4)$, $M_4(4, q \geq 4)$, $M_4(5, q \equiv 2,3 (mod\, 5))$ are: 3 maps of the dodecahedron, 5 small maps $M_n(p,q)$ (with $(n;p,q)$=$(2;3,6)$, $(3;4,6)$, $(2;4,8)$, $(3;5,8)$,$(2;5,10)$; see Figure 2), 9 fullerenes and 9 *azulenoids* (a chemical term for the case $\{p,q\} = \{5,7\}$). All undecided maps $M_n(p,q)$ are either other azulenoids ($M_n(7,5)$, $n \geq 29$, or $M_n(5,7)$, $17 \leq n \leq 19$), or, prior to [5], other *fulleroids* (a chemical term for the case $p = 5$) $M_n(5,q)$ (with $q \geq 8$, $n \geq 4$).

[5] gave new constructions of fulleroids $M_n(5,q)$ for $n = 8, q \equiv 1,4(mod\, 5), q \geq 9$ (by S.Madaras) and four cases (by R.Sotak): 1) $n = 6, q \equiv 0,1,4(mod\, 5), q \geq 10$, 2) $n = 10, q \equiv 0(mod\, 10), q \geq 20$, 3) $n = 14 + 4k, q = 10, k \geq 0$, 3) $n = 12 + 4k, q \equiv 2,3(mod\, 5), q \geq 17, k \geq 0$. So any even $n \geq 0$ and any $q \geq 4$ are realized (separately) by some map $M_n(5,q)$, but amongst impossible pairs (n,q) there are, for example, $(4,9)$, $(n,6)$ for $n > 10$, $(n,7)$ for $n > 20$ and, using [4], $(n, q \equiv 0(mod\, 10))$ for odd n. All known $M_n(p,q)$ with *odd* n are, besides of prisms $M_n(n,4)$ with odd n, the map $M_3(4,6)$ and 3 maps $M_5(5,q)$ with $q = 4,5,6$. There is a $M_n(5,10)$ for each $n \equiv 2(mod\, 4)$ (except of undecided case $n = 10$) and an infinity of maps $M_n(5,q)$ for any $n \equiv 0(mod\, 4)$ and for $n = 6, 10$. The number of $M_n(5,q)$ with fixed q is finite for $q \leq 6$, but it is infinite for $q = 10$ and $q \equiv 2,3(mod\, 5), q \geq 17$.

(ii) Examples of two maps having the same graph G of p-gons in I_n, are: $M_3(4,6)$, one of 2 $M_6(5,5)$ (for $G = K_3$); $M_2(4,8)$, the second $M_6(5,5)$ (for

$G = K_4 - e$); $M_5(5,6)$, $M_5(5,5)$ (for G being 5-wheel); one of two $M_6(5,6)$, one of four $M_{12}(6,5)$ (for $G = K_{3\times 2} - K_3$).

The maps, given here (except of $M_5(5,5)$, one of two $M_6(5,5)$, $M_{10}(5,7)$ and 3rd, 4th $M_{12}(5,7)$ from Figure 3) are *self-complementary* (i.e. they have isomorphic domains I_n and O_n) since the corresponding p-gonal sequences are self q-complemented. Suppose, without loss of generality, that I_n contain no more p-gons than O_n. Then exceptional maps $M_5(5,5)$, $M_6(5,5)$, $M_{10}(5,7)$, $M_{12}(5,7)$ have (the graph of p-gons in I_n) $G = K_1, K_2$, $P_{0,1,\dots,9}+$ $(1,3)$+$(2,4)$+$(5,7)$+$(6,8)$, $P_{0,1,\dots,9}+$ $(0,2)$+$(7,9)$, $P_{0,1,\dots,9}+$ $(1,3)$+$(2,4)$+$(7,9)$.

All p-gonal sequences of the maps in the paper (except of non-periodical one for 4th map on Figure 3) are: either 222222 (for a $M_{12}(6,5)$), or 11111 (for $M_5(5,5)$ and $M_5(5,6)$), or of form *bbb* (with b = 1, 10, 2, 20, 31 for 5 maps $M_3(4,6)$, a $M_6(5,5)$, $M_3(5,8)$, $M_6(5,6)$, a $M_{12}(6,5)$), or of form *bb* (including b = 1, 10, 100, 110...0, 20...0 (with any number of zeros), 21, 210, 321, 411, 3010, 30101, 3010010). Clearly, the period divides n and indicates the maximal symmetry of corresponding polyhedron.

(iii) Denote by $M_n^k(p,q)$ *k-valent* analogs of maps $M_n(p,q)$, i.e. k-valent (p,q)-map, such that q-gons form a simple circuit. Unique map $M_n^4(p,3)$ is the map $M_{2p}^4(p,3)$ of p-gonal antiprism; all $M_n^4(3,4)$ are 3 maps with $n = 4,6,8$, which are decorations of cube, octahedron and rhombic dodecahedron, respectively.

The only k-valent planar graphs, admitting more than one map $M_n^k(p,p)$, are the dodecahedron and the icosahedron. The dodecahedron has three maps: with $(n,G) = (5,K_1),(6,K_2)$ and (self-complementary one) with $(6,K_3)$. The icosahedron has six maps $M_n^5(3,3)$: with $(n,G) = (9,P_1)$, $(10,P_2)$, $(11,P_3)$ and (self-complementary ones) with $(12,P_4)$, $(12,K_{1,3})$, $(10,C_5)$.

(iv) Denote by $M_{n_1,\dots,n_t}(p,q)$ any 3-valent planar map, consisting of p- and q-gons only, where all q-gons are organized into $t, t \geq 2$, rings of length $n_1,\dots,n_t \geq 3$. We found 3 infinite families (of $M_{3,3}(6,4)$, $M_{6,6}(6,5)$, $M_{3,3,3,3}(6,5)$) and 30 sporadic maps: 3 fullerenes ($M_{3,3}(5,6)$, $M_{5,5}(5,6)$, $M_{3,6}(5,6)$= $M_{3,9}(6,5)$), 15 azulenoids (12 $M_{n_1,n_2}(5,7)$, $M_{3,3,3,3}(5,7)$, $M_{3,8,3}(5,7)$, $M_{6,9,3}(5,7)$=$M_{3,15,12}(7,5)$) and 12 maps $M_{3,\dots,3}(p,4)$ with (t,p)=

(4,7),(4,8),(4,8),(4,9),(6,10),(8,9),(8,9),(8,12),(12,11),(12,11),(16,13),(20,15).

(v)For $n = 2,3$ only the maps $M_2(p,2p)$ and $M_3(p,2p-2)$ for $p = 3,4,5$ exist. The maps with $n = 2$ are not polyhedral: two $2p$-gons have two edges in common.

The maps $M_2(p,2p)$ with $p = 3,4,5$ have symmetry D_{2h}; two maps from Figure 1 have symmetry D_{3d} (namely, fullerenes $M_{12}(6,5)$ and $M_6(5,6)$, which are the first and the last in the right column). Those five maps are only centrally-symmetric ones amongst the maps of Figures 1 and 2. They can be folded on real projective plane: the ring of n q-gons became Möbius band from $\frac{n}{2}$ q-gons, which is glued, by the boundary circle, with a disc of (the half of original) p-gons.

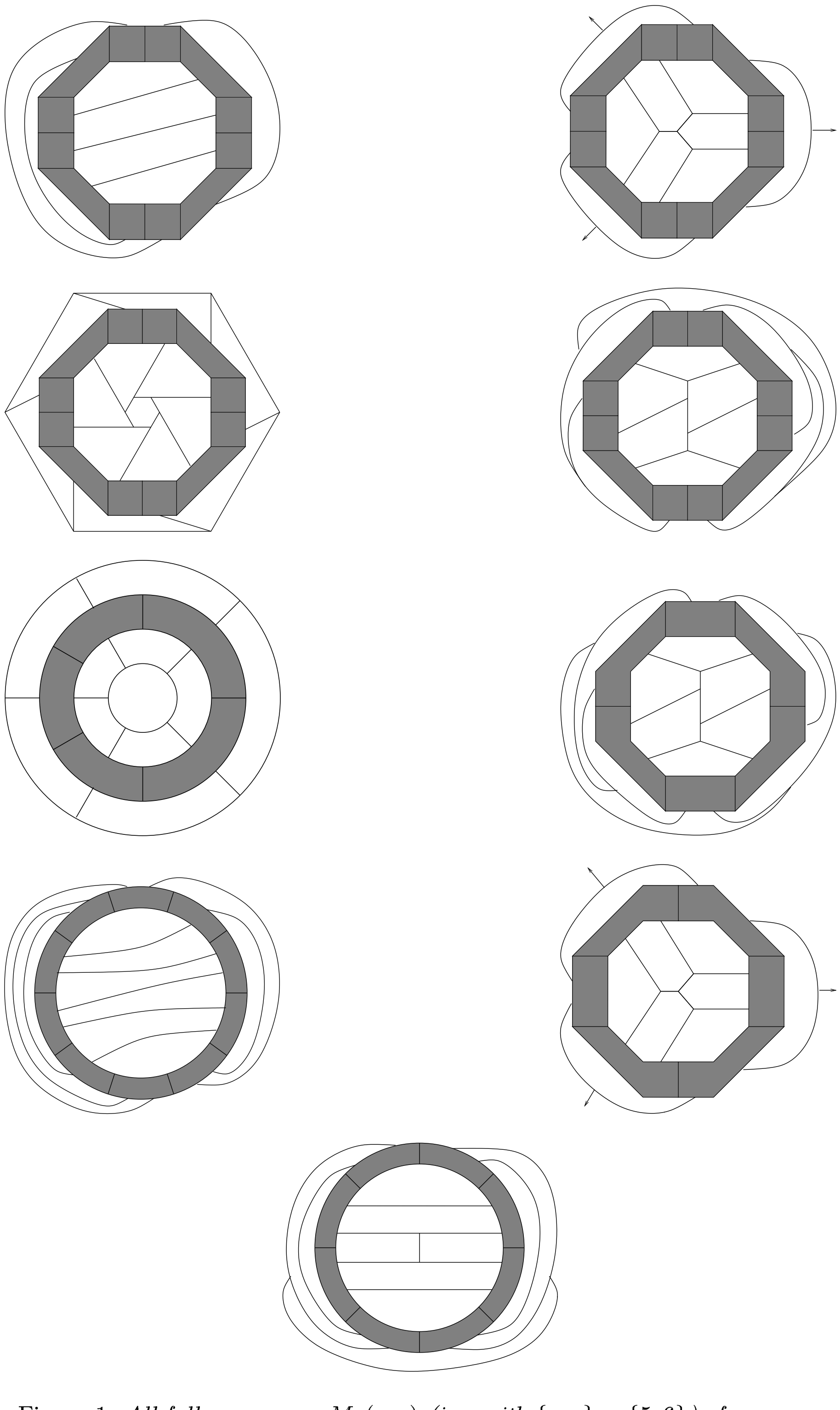

Figure 1: *All fullerene maps* $M_n(p,q)$ *(i.e. with* $\{p,q\}=\{5,6\}$*): four maps* $M_{12}(6,5)$ *and five maps* $M_n(5,6)$ *with* $n=5,6,10,6,8$.

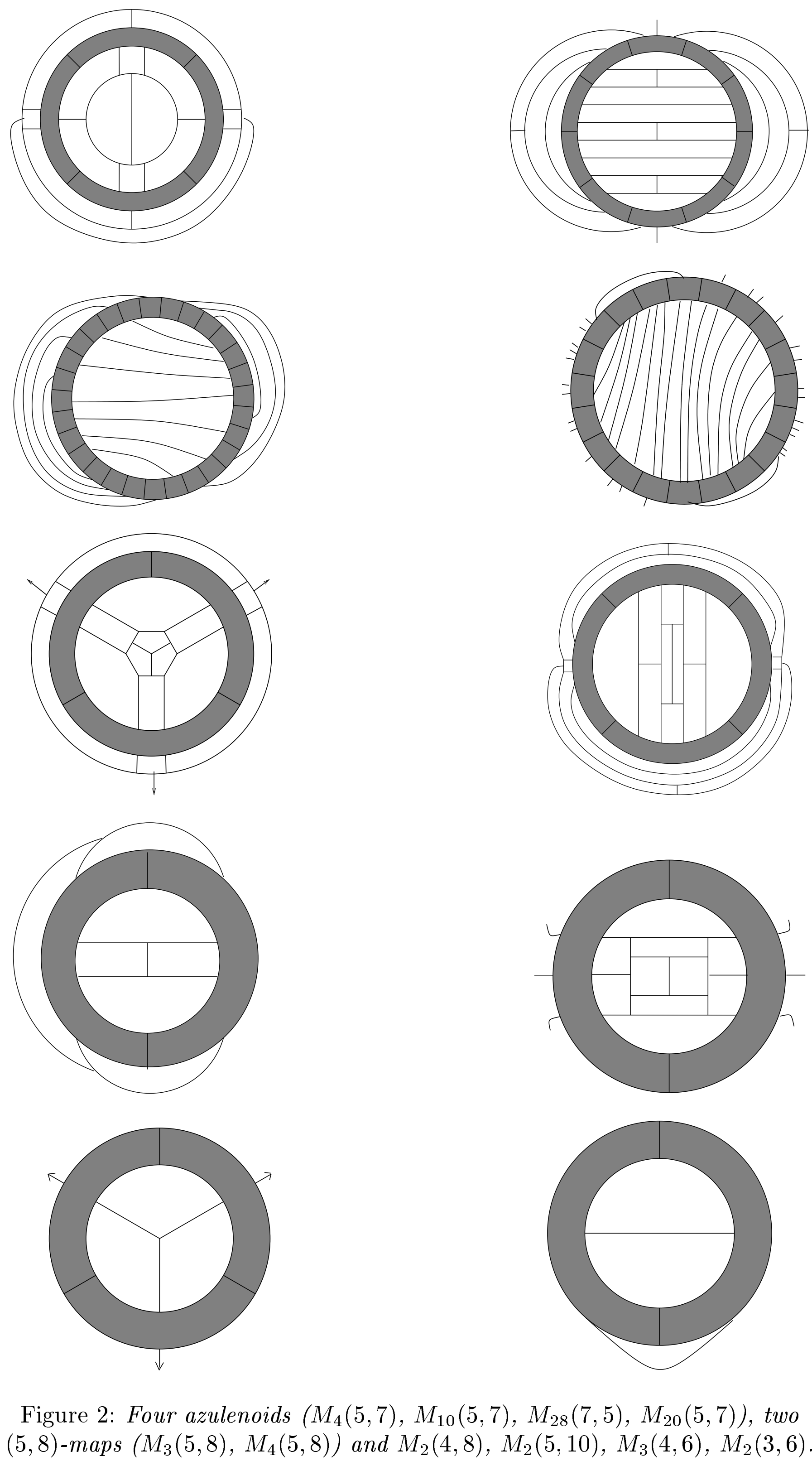

Figure 2: *Four azulenoids ($M_4(5,7)$, $M_{10}(5,7)$, $M_{28}(7,5)$, $M_{20}(5,7)$), two $(5,8)$-maps ($M_3(5,8)$, $M_4(5,8)$) and* $M_2(4,8)$, $M_2(5,10)$, $M_3(4,6)$, $M_2(3,6)$.

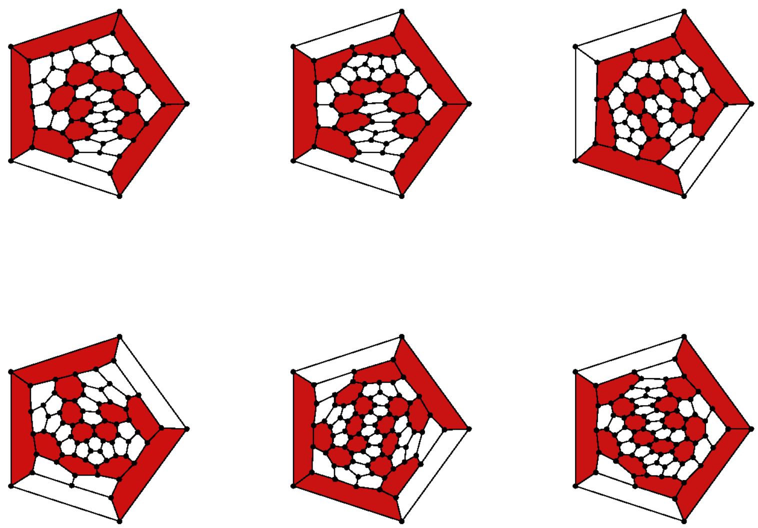

Figure 3: azulenoids $M_n(5,7)$ from [3]: 4 with $n = 12$ and 2 with $n = 16$.

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