# Hypermetric two-distance spaces

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**Abstract**

Any two-distance space is uniquely up to a multiple represented by a distance $d_{G,t}$ for a graph $G$ such that $d_{G,t}(ij)$ is equal to 1 or $t$ depending on $(ij)$ is an edge or non-edge of $G$. For a cone $C_n^A$ of $n$-point distance spaces, we set $t^A(G) := \max\{t : d_{G,t} \in C_n^A\}$. We consider the cut cone $Cut_n = C_n^C$, the hypermetric cone $Hyp_n = C_n^H$, and the cone of negative type $Neg_n = C_n^N$. The values of $t^N(G)$ (in other terms) are considered by many authors, and are determined by roots of some polynomials. We give bounds on $t^H(G)$, and consider some classes of graphs $G$ with a given value of $t^H(G)$, especially for $t^H(G) = 2$ and $t^H(G) = \frac{3}{2}$. The graphs $G$ with $t^H(G) = 2$ are exactly graphs having the hypermetric truncated distance $d_G^*$.

## 1 Introduction

A point set in an $m$-dimensional *Euclidean space* $\mathbf{R}^m$ is called a two-distance *set* if the pairwise distances between the points take only two values. We distinguish a two-distance *space* as an abstract distance space with two distances.

Two-distance sets have an intrinsic interest. Upper bounds on their cardinality may depend on the specific metric space where they are embedded, on its dimension, and on actual distances. Two-distance sets attract by their simplicity and their relation to some combinatorial objects, for example, to spherical designs.

Besides, a use of metric considerations simplifies combinatorial problems. For example, it is very interesting to compare the proofs in [17] and [11] of the non-existence of a 7-point two-distance set in $\mathbf{R}^3$. The metric proof of [17] is much simpler than the combinatorial proof of [11].

There are some optimization problems related to sets $V$ endowed by a distance function $d$. For example, the Traveling Salesman Problem is the problem to find an order $V \to \{1, 2, ..., |V|\}$ such that the sum $\sum_{i=1}^n d(i, i+1)$ (with $n+1 \equiv 1$) is minimal. This problem is NP-complete even for two-distance sets, since it is reduced to finding a Hamiltonian cycle in a graph. We think that many other problems are equally hard for two-distance spaces as for general distance spaces.

Any two-distance is, up to a multiple, the following distance $d_{G,t}$ for a graph $G$ and some nonnegative $t$:

$$d_{G,t}(ij) = \begin{cases} 1 & \text{if } (ij) \text{ is an edge of } G \\ t & \text{if } (ij) \text{ is a non-edge of } G \end{cases}$$

Let $C_n^A$ be a cone of distance spaces $(V, d)$ on $n = |V|$ points. As $C_n^A$, we consider three combinatorially significant cones, namely, the cut cone $C_n^C = Cut_n$, the cone of $l_1$-embeddable semi-metrics; the hypermetric cone $C_n^H = Hyp_n$; and the cone $C_n^N = Neg_n$, the cone of distances of negative type (i.e. squared Euclidean distances). For a given graph $G$, we set

$$t_0^A(G) = \min\{t : d_{G,t} \in C_n^A\},\ t^A(G) = \max\{t : d_{G,t} \in C_n^A\}.$$

Since $d_{\overline{G},t} = t d_{G,\frac{1}{t}}$, we obtain

$$t_0^A(G) t^A(\overline{G}) = t_0^A(\overline{G}) t^A(G) = 1. \qquad (1)$$

So if we know values of $t^A(G)$ for all $G$, we know $t_0^A(G)$, too. Hence we are restricted ourselves below by studying values of $t^A(G)$ only. In what follows, $t_1^A(G)$ in expression of type $t_{0,1}^A(G)$ denotes $t^A(G)$.

There is the fourth combinatorially important cone $C_n^M = Met_n$, the metric cone. We could consider the problem of a determination of values $t_{0,1}^M(G)$. But this problem is trivially solvable. In fact, $t^M(G) = 2$ if $G$ is not a disjoint sum of complete graphs, and $t^M(G) = \infty$ otherwise.

In principle, the problem of determining of $t^N(G)$ is solved by Einhorn and Schoenberg [17], and this resolution is well known. It relates to roots of a polynomial, namely of the determinant of the Gram matrix of a representation of the distance space $d_{G,t}$. Other polynomials related to $t^N(G)$ were used by Seidel [31], Neumeier [28], Maehara [26].

For a graph $G$, the distance $d_{G,2} = d_G^*$ is a metric, and it is called the *truncated* metric of the graph $G$. If $G$ is connected and has diameter 2, then $d_G^*$ is the path metric of the graph $G$. For a graph $G$ of diameter 2, the equality $t^C(G) = 2$ ($t^H(G) = 2$) means that $G$ is an $l_1$-graph (hypermetric, respectively). For example, $t^C(K_{m\times 2}) = t^H(K_{m\times 2}) = 2$, where $K_{m\times 2}$ is the Cocktail-party graph. Conversely, if a graph $G$ of diameter 2 is not hypermetric (not $l_1$-embeddable), then $t^H(G) < 2$ ($t^C(G) < 2$, respectively).

In this paper, we apply known properties of hypermetrics and $l_1$-metrics to two-distance spaces.

In Section 2, general properties of distance spaces are considered. In particular, it is noted that the convex hull $P(d)$ of representing points of $d \in Neg_n$ is a polytope distinct from a simplex iff $d$ belongs to the boundary of the cone $Neg_n$. But if $d$ belongs to the boundary of $Hyp_n$, $P(d)$ can remain a simplex. The role of $P(d)$ plays here the Delaunay polytope $P_D(d)$. Some properties of $t^{H,N}(G)$ are described in Section 3. Taking in attention that any hypermetric space has a spherical representation, we give, in Section 4, known bounds on $t^Q(G)$ of such a representation using the smallest eigenvalue of the

Gram matrix $Q$. In Sections 5 and 6, we give lower bounds on $t^{H,N}(G)$. The bound on $t^N(G)$ was known early, but it was not related to $t^N(G)$. In Section 7, we recall a relation of two-distance spaces with equiangular lines. We find in Section 8 exact values of $t^H(G)$ for some classes of bipartite graphs, and describe the corresponding Delaunay polytopes $P_D(G)$. In Section 9, we give $t^H(G)$ and $P_D(G)$ for $G$ with small number of vertices. We consider some examples of hypermetric two-distance spaces with $t^H(G) = 2$ and $t^H(G) = \frac{3}{2}$, respectively, in Sections 10 and 11.

# 2 Distance spaces

See [10] for a good survey of finite distance spaces.

A finite distance space $(V, d)$ is a finite set $V$ and a matrix $D = \{d(ij)\}$ or a function $d$ on the set $V^2$ of all pairs of $V$, such that

(1) $D$ is nonnegative, i.e. $d(ij) \geq 0$ for all $(ij) \in V^2$,

(2) $D$ is symmetric, $d(ij) = d(ji)$,

(3) $D$ vanishes on diagonal, i.e. $d(ii) = 0$.

A finite distance space $(V, d)$ with $|V| = n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ distances $d(ij)$. Hence $d$ can be considered as a point of the nonnegative orthant $\mathbf{R}_+^{\binom{n}{2}}$ of the Euclidean space $\mathbf{R}^{\binom{n}{2}}$. Let $E_n$ be the set of all $\binom{n}{2}$ unordered pairs (edges) of distinct points of $V$. Then the coordinates of $\mathbf{R}^{\binom{n}{2}}$ are indexed by $(ij) \in E_n$, i.e. $\mathbf{R}^{\binom{n}{2}} = \mathbf{R}^{E_n}$.

There are some sub-cones in $\mathbf{R}_+^{\binom{n}{2}}$ given by linear conditions on the distance $d$ and tightly related to combinatorics. For example, if $d$ satisfies all triangle inequalities $d(ij) + d(ik) - d(jk) \geq 0$, then $d$ is a *semi-metric*. All semi-metrics form the metric cone $Met_n$. The following inequality

$$\sum_{1 \leq i < j \leq n} b_i b_j d(ij) \leq 0, \; b_i \in \mathbf{Z}, \; 1 \leq i \leq n, \tag{2}$$

where $\sum_{1 \leq i \leq n} b_i = 1$, is a generalization of the triangle inequality. It is called the *hypermetric* inequality. Since the hypermetric inequality is linear in $d$, all distances satisfying (2) for all $b_i \in \mathbf{Z}$ with $\sum b_i = 1$ fill out the hypermetric cone $Hyp_n$.

If $d$ satisfies (2) for all $b \in \mathbf{Z}^n$ with $\sum b_i = 0$, then $d$ is called a distance of *negative type*. These distances form the negative type cone $Neg_n$.

*Remark 1.* Since the condition $\sum_{i=1}^n b_i = 0$ and the inequalities (2) are homogeneous, $d \in Neg_n$ satisfies (2) for all rational $b_i$, and by continuity, for all $b \in \mathbf{R}^n$. □

The cut cone $Cut_n$ is the convex hull of *cut metrics* $c_S$ for $S \subset V$. The cut metrics span the common extreme rays of $Cut_n$, $Hyp_n$ and $Met_n$. For $S \subseteq V$, the *cut* $\delta(S) \subseteq E_n$ is the set of edges having exactly one vertex in the set $S$. Then $c_S$ is the indicator vector of the cut $\delta(S)$, i.e. the vector of the space $\mathbf{R}^{E_n}$ such that

$$c_S(ij) = \begin{cases} 1 & \text{if } |\{ij\} \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Clearly, $c_{V-S} = c_S$.

There are important problems on isometric embedding of a distance space $(V, d)$ into an Euclidean space with Euclidean and squared Euclidean distances. If $(V, d)$ is embedded into an $m$-dimensional Euclidean space, it can be considered as a set of $n = |V|$ points in $\mathbf{R}$. These points (and the vectors whose endpoints they are) are called a *representation* (of dimension $m$) of the distance space $(V, d)$.

We are interested only in representations in Euclidean spaces with Euclidean and squared Euclidean distances. The famous result of Schoenberg (1935) says that a distance space $(V, d)$ is of negative type if and only if it has a representation with $d$ equal to squared Euclidean distance.

Similarly to $t^N_{0,1}(G)$, one can define the values $t^E_{0,1}(G)$, i.e. minimal and maximal values of $t$ such that the distance space $(V, d_{G,t})$ has an Euclidean representation with distance equal to Euclidean distance. Since $(d_{G,t})^2 = d_{G,t^2}$, both these representations of a two-distance space are, in a sense, equivalent. Obviously, $(t^E_{0,1}(G))^2 = t^N_{0,1}(G)$.

An advantage of the representation of $(V, d)$ in $\mathbf{R}$ with squared Euclidean distance is that

*there is a correspondence between linear dependences on the set of distances and linear dependences on the set of representing vectors.*

Let vectors $v_i \in \mathbf{R}^m$, $i \in V$, represent a distance space $(V, d)$, $d \in Neg_n$, such that

$$d(ij) = (v_i - v_j)^2. \tag{4}$$

For $d \in Neg_n$, let

$$\mathcal{B}(d) = \{b \in \mathbf{R}^n : \sum_{1 \le i < j \le n} b_i b_j d(ij) = 0, \ \sum_{i=1}^n b_i = 0\}.$$

**Fact 1**. *$\mathcal{B}(d)$ is a subspace of $\mathbf{R}^n$.*

**Proof**. If $d(ij) = (v_i - v_j)^2$, then the inequality (2) with $\sum_{i=1}^n b_i = 0$ takes the form $(\sum_{i=1}^n b_i v_i)^2 \ge 0$. For $b \in \mathcal{B}(d)$, we have the linear equality $\sum_{i=1}^n b_i v_i = 0$. Since the condition $\sum_{i=1}^n b_i = 0$ is linear, too, we obtain that

$$\mathcal{B}(d) = \{b \in \mathbf{R}^n : \sum_{i=1}^n b_i v_i = 0, \ \sum_{i=1}^n b_i = 0\}$$

is a subspace of the space $\mathbf{R}^n$. $\square$

For $d \in Neg_n$, let $P(d)$ be the convex hull of endpoints of representing vectors $v_i$, $i \in V$. It is known that $P(d)$, up to translations and rotations, depends only on $d$.

**Proposition 1** *Dimension of $P(d)$ is equal to $n - 1 - dim\mathcal{B}(d)$. In particular, if $d$ is an inner point of $Neg_n$, then $P(d)$ is an $(n-1)$-dimensional simplex.*

**Proof**. Since $b_n = -\sum_{i=1}^{n-1} b_i$, we have

$$\sum_{i=1}^n b_i v_i = \sum_{i=1}^{n-1} b_i (v_i - v_n).$$

Without loss of generality, we can suppose that the $(n-1)$ vectors $v_i - v_n$ belongs to $\mathbf{R}^{n-1}$. If the point $d$ lies in the interior of the cone $Neg_n$, then the sum $\sum_{i=1}^{n-1} b_i(v_i - v_n)$ is not zero for all $b \in \mathbf{R}^n$. In other words, the vectors $v_i - v_n$, $1 \le i \le n-1$, are linearly independent, and $P(d)$ is an $(n-1)$-dimensional simplex.

If $d$ lies on the boundary of $Neg_n$, then the space $\mathcal{B}(d)$ is not empty, and there are linear dependencies between vectors $v_i - v_n$. Obviously, dimension of $P(d)$ is equal to $n - 1 - \dim\mathcal{B}(d)$. □

Usually one proves assertions like Proposition 1 using some characteristic determinants that we consider below.

Let $d$ be an inner point of $Neg_n$. Then $P(d)$ is an $(n-1)$-dimensional simplex. We take the vectors $v_i - v_n$, $1 \le i \le n-1$, as a basis of $\mathbf{R}^{n-1}$. Let $Gr_n(d)$ be the Gram matrix of these vectors, i.e. $(Gr_n(d))_{ij} = (v_i - v_n, v_j - v_n)$, $1 \le i, j \le n-1$. Using (4) it is easy to see that

$$(Gr_n(d))_{ij} = \frac{1}{2}(d_{in} + d_{jn} - d_{ij}).$$

Note that, using the Gram matrix, we can rewrite the inequality (2) $b^T Db \le 0$ as follows $b^T Db = -2b'^T Gr_n(d)b' \le 0$, i.e. $b'^T Gr_n(d)b' \ge 0$ for any $b' \in \mathbf{R}^{n-1}$.

The Gram matrix is positive semidefinite. It is positive definite if the vectors $v_i - v_n$ are linearly independent. In other words, its minimal eigenvalue is positive. When the point $d$ belongs to the boundary of $Neg_n$, then these vectors are linearly dependent and the Gram matrix is positive semidefinite with zero as the smallest eigenvalue. In this case dimension of $P(d)$ is equal to $n - 1 - f$, where $f$ is the multiplicity of the zero eigenvalue.

The Gram matrix is not symmetric with respect to indices, the index $n$ is special. But $Gr_n(d)$ is tightly related to the *Cayley-Menger matrix* $CM_n(d)$ of the order $n+1$, where

$$CM_n(d) = \begin{pmatrix} 0 & j_n^T \\ j_n & D \end{pmatrix},$$

and $j_n = (1, ..., 1)_T$ is the all-one column. It is known (see, for example, [6]) that

$$\det Gr_n(d) = \frac{(-1)^n}{2^{n-1}} \det CM_n(d).$$

It is well known (see, for example, [14]) that $Cut_n \subseteq Hyp_n \subseteq Neg_n$, and for $n \ge 7$ all these inclusions are strict. Obviously, $Hyp_n \subseteq Met_n$, but $Met_n$ and $Neg_n$ are not comparable. The cones $Cut_n$, $Hyp_n$ and $Met_n$ are polyhedral, while $Neg_n$ is not polyhedral. Moreover, all extreme rays of $Cut_n$ are extreme rays of $Hyp_n$ and $Met_n$, and all facets of $Met_n$ are facets of $Cut_n$ and $Hyp_n$.

Note that the cones $Hyp_n$ and $Neg_n$ both are described by the inequalities (2) with $\sum_{i=1}^n b_i$ equal to 1 and 0, respectively. But it is sufficient a finite number of these inequalities for $Hyp_n$, while the description of $Neg_n$ needs infinite number of inequalities (2).

Since a hypermetric space $(V, d)$ is of negative type, it has a representation with $d$ equal to squared Euclidean distance. Assouad (1980) proved that $(V, d)$ is hypermetric if and only if the endpoints of the vectors of this representation lie on an empty sphere of the lattice affinely generated by these vectors. An $n$-dimensional lattice is an Abelian group of vectors of $\mathbf{R}^n$ integrally generated by $n$ linearly independent vectors. An empty sphere of a lattice is a sphere such that no lattice point lies in the interior of the sphere. One considers usually an empty sphere such that the lattice points on the sphere affinely generate $\mathbf{R}^n$. In this case, the convex hull of all lattice points on an empty sphere is a *Delaunay polytope* $P_D$ of the lattice. So $P(d)$ is the convex hull of some vertices of the Delaunay polytope $P_D$. Note that in this case $P_D$ is uniquely determined by the distance $d$ (see [14]). Hence we denote it by $P_D(d)$.

Since a $l_1$-metric space $(V, d)$ is hypermetric, the set $V$ can be represented as a set of vertices of a Delaunay polytope $P_D(d)$. But the $l_1$-embeddability implies that $P_D(d)$ can be inscribed into a box. A box is a Delaunay polytope of a rectangular lattice generated by a set of mutually orthogonal vectors. In general, the dimension of the box, where $P_D(d)$ is inscribed, is greater than dimension of $P_D(d)$.

Let $(V, d)$ be hypermetric. Then we can take the center of the corresponding Delaunay polytope $P_D(d)$ as origin. Substituting the representation (4) with $v_i^2 = r^2(d)$, where $r(d)$ is the radius of $P_D(d)$, into (2) with $\sum_{i=1}^n b_i = 1$, we obtain

$$(\sum_{i=1}^n b_i v_i)^2 \geq r^2(d).$$

This means that any affine integer combination $v(b) \equiv \sum_{i=1}^n b_i v_i$ of vectors $v_i$, being a point of the lattice affinely generated by these vectors, does not lie inside the empty sphere circumscribing $P_D(d)$.

Similarly to $\mathcal{B}(d)$ we can introduce the set

$$\mathcal{B}^H(d) = \{b \in \mathbf{Z}^n : \sum_{1 \leq i < j \leq n} b_i b_j d(ij) = 0, \ \sum_{i=1}^n b_i = 1\}.$$

Obviously $\mathcal{B}^H(d) \neq \emptyset$ iff $d$ lies on the boundary of $Hyp_n$. Substituting the expression (4) for $d$, we obtain

$$\mathcal{B}^H(d) = \{b \in \mathbf{Z}^n : (\sum_{1 \leq i < j \leq n} b_i v_i)^2 = r^2(d), \ \sum_{i=1}^n b_i = 1\}.$$

In other words, if $b \in \mathcal{B}^H(d)$, then the point $v(b)$ is a vertex of $P_D(d)$. There are two possibilities: either $v(b) \in V$, or $v(b)$ is a new vertex of $P_D(d)$. The first case is possible only if there is an affine dependencies between vectors $v_i$, when these vectors span an affine space of dimension not greater than $n-2$. This means that $d$ lies on the boundary of $Neg_n$, too. Since $v_i$ generate $P_D(d)$, the dimension of $P_D(d)$ is not greater than $n-2$. Hence we have

**Proposition 2** *If $dim P_D(d) \leq n-2$, then $d$ lies on both the boundaries of $Hyp_n$ and $Neg_n$.* □

When a point $d$ comes to the boundary of $Hyp_n$, then the simplex $P_D(d)$ is glued with some other simplexes of the L-partition of the lattice determined by the representation of $d$. If $d$ comes to a facet $F$ of $Hyp_n$ determined by exactly one hypermetric equality, then the glued simplexes form a special Delaunay polytope called *repartitioning* polytope. It is studied in detail in [3].

Let the facet $F$ is determined by a hypermetric equality with $b \in \mathbf{Z}^n$, $\sum_{i=1}^n b_i = 1$. Let

$$V_+ = \{i : b_i > 0\},\ V_- = \{i : b_i < 0\},\ V_0 = \{i : b_i = 0\}. \tag{5}$$

Let vectors $v_i$ represent $d \in F$. Then the vectors $v_i$, $i \in V$, and the vector $v_0 = v(b) = \sum_{i=1}^n b_i v_i$ represent the repartitioning polytope. It is constructed as follows. Let $S_-$ be the simplex spanned by endpoints of vectors $v_i$ for $i \in V_- \cup \{0\}$. Similarly, let $S_+$ and $S_0$ be simplexes spanned by $v_i$ for $i \in V_+$ and $V_0$, respectively. The simplexes $S_+$ and $S_-$ span spaces that intersect in the point $v_c = \frac{1}{k}\sum_{i \in V_+} b_i v_i = \frac{1}{k}(v_0 + \sum_{i \in V_-} b_i v_i)$, where $k = \sum_{i \in V_+} b_i = 1 + \sum_{i \in V_-} |b_i|$. The point $v_c$ is the common barycenter of both the simplexes $S_+$ and $S_-$. Let $V^{n'}_{n_+,n_-}$ be the convex hull of dimension $n'$ of vertices of both the simplexes $S_+$ and $S_-$, where $n_+ = |V_+| - 1$ and $n_- = |V_-|$ are dimensions of $S_+$ and $S_-$. The vertices of the simplex $S_0$ do not belong to the space spanned by $V^{n'}_{n_+,n_-}$. The repartitioning polytope is the convex hull of vertices of $V^{n'}_{n_+,n_-}$ and $S_0$. It is denoted as $V^{n-2}_{n_+,n_-}$, where $n$ is the number of its vertices and $n-2$ is its dimension. Note that the notation $V^m_{p,q}$ does not describe a concrete polytope, but a class of affinely equivalent repartitioning polytopes of dimension $m$ with simplexes of dimensions $p$ and $q$ intersecting in the common barycenter.

If the points of a representation lie on a sphere, then the representation is called *spherical*. We saw that $(V, d)$ has a spherical representation if $d$ belongs to $Hyp_n$. Moreover, since every simplex can be inscribed into a sphere, the distance space has a spherical representation if $d$ is an inner point of $Neg_n$. In this case the radius $r$ of the circumscribing $P(d)$ sphere is given (see [6]) by

$$r^2(d) = -\frac{1}{2}\frac{\det D}{\det CM_n(d)}.$$

We see that, according to Proposition 1, the only case when $P(d)$ may be not inscribed into a sphere is the case when $d$ belongs to the boundary of $Neg_n$. But there are cases when $P(d)$ is inscribed into a sphere although $d$ lies on the boundary of $Neg_n$.

Call a distance space *regular* (of *strength 1*, in terms of [28]) if its distance matrix $D$ has the all-one vector $j_n$ as an eigenvector. This means that the sum of all elements of a row (or a column) of $D$ does not depend on this row (and column). It is well known that if a regular distance space has a representation, then it has necessarily a spherical representation, too. In fact, we can take the origin of the representation space in the center of mass $\frac{1}{n}\sum_{i=1}^n v_i$ of the vectors $v_i$ such that then $\sum_{i=1}^n v_i = 0$. Let $\alpha$ be the eigenvalue of $D$ corresponding to the eigenvector $j_n$. Then taking in attention (4), we have

$$(Dj_n)_i = \sum_{j=1}^n (v_i - v_j)^2 = nv_i^2 + \sum_{i=1}^n v_j^2 = \alpha,$$

i.e. $v_i^2$ does not depend on $i$, that is all $v_i$ have the same *norm* =squared length $r^2$.

Setting in the above equality $v_i^2 = v_j^2 = r^2$, we obtain the following value of the radius of the spherical representation of a regular distance space:

$$r^2(d) = \frac{1}{2n}\sum_{j=1}^{n} d(ij). \tag{6}$$

In particular, setting $d(ij) = a^2$, we obtain the following very useful formula for the squared radius of a regular $(n-1)$-dimensional simplex with length of edges $a$

$$r^2 = \frac{a^2}{2}\frac{n-1}{n}. \tag{7}$$

# 3 Two-distance spaces

For fixed $G$ and varying $t$, the two-distances $d_{G,t}$ form a line in $\mathbf{R}^{\binom{n}{2}}$ going through the point $(1,1,...,1) = d_{G,1}$. For $t_k = t_k^A$, $k = 0,1$, the points $d_{G,t_k}$ are the endpoints of a segment of this line lying in the cone $C_n^A$. The segments of the line $d_{G,t}$ lying in $Cut_n$, $Hyp_n$ and $Neg_n$ are contained each in other, respectively.

There are $2^{\binom{n}{2}} - 1$ lines $d_{G,t}$, since there are $2^{\binom{n}{2}} - 1$ graphs on $n$ vertices distinct from $K_n$. In fact, we consider subsets $E$ of the set $E_n$ of all unordered pairs (=edges) of distinct points of $V$. The (labeled) graph $G = G(E)$ induced by the set $E$ of its edges is a convenient denotation of the subset $E$. The set of all labeled $n$-point graphs is partitioned into classes of isomorphic graphs. Note that the cones $C_n^A$, $A = C, H, M, N$, are invariant under permutations of coordinates. Hence the values of $t_{0,1}^A$ are the same for all isomorphic graphs for all $A$. So it is natural to consider non-labeled graphs only.

For the complete graph $K_n$, $d_{K_n,t}$ degenerates into the point $(1,1,...,1)$. For the empty graph $\overline{K_n}$, the line $d_{\overline{K_n},t}$ coincides with the central axis $(t,t,...,t)$ of the cones $Cut_n$, $Hyp_n$, $Met_n$ and $Neg_n$. We see, that the graph $G$ determines, in a sense, the angle between the line $d_{G,t}$ and the central axis. The points, where the line $d_{G,t}$ intersects the boundary of the cones $Cut_n$ and $Hyp_n$, are given by $t = t_{0,1}^{C,H}(G)$, respectively.

Since the cut metric $c_S$ takes only two values 0 and 1, it is the two-distance semi-metric $d_{K(S,\overline{S}),0}$, where $\overline{S} = V - S$ and $K(S,\overline{S})$ is the complete bipartite graph with the partition $V = S \cup \overline{S}$.

Below we use the common notations $K_n$, $P_n$, $C_n$ for the complete graph, the path and the cycle, each on $n$ vertices. $K_{p,q}$ is the complete bipartite graph with parts of size $p$ and $q$. The *direct product* $G_1 \times G_2$ of graphs $G_1$ and $G_2$ is the graph with the set of all pairs $(v_1,v_2)$, $v_1 \in V_1$, $v_2 \in V_2$ as the set of vertices. Two vertices $(v_1,v_2)$ and $(u_1,u_2)$ are adjacent iff one of the coordinates coincide and the other coordinates are adjacent in the corresponding graph. Denote by $G_1 + G_2$ the disjoint union of graphs $G_1$ and $G_2$ such that $kG$ is the disjoint union of $k$ graphs isomorphic to $G$. The graph $\overline{K_1 + \overline{G}}$ is called *suspension* of $G$ and is denoted as $\nabla G$.

Let $Q=(q_1,...,q_k)$ be a sequence of positive integers. Denote by $K(Q)$ the disjoint sum $\sum_{i=1}^k K_{q_i}$. The graphs $K(Q)$ play a special role. The distance spaces $(V,d_{K(Q),t})$ are just all *ultrametrics* in the class of two-distance spaces. An ultrametric is a metric $d$ satisfying the inequalities $d(ij)\le\max\{d(i,k),d(j,k)\}$ for every $k\in V$, and any $i,j\in V$. It is proved in [22] (see also [26]) that all Euclidean representations of ultrametrics are $(|V|-1)$-dimensional simplexes. This implies, and it is obvious for two-distance spaces $(V,d_{K(Q),t})$, that

$$t^N(K(Q))=\infty. \tag{8}$$

The representation of $(V,d_{K(Q),t})$ for every $t$ is the simplex being the convex hull of $m$ unit simplices $P(d_{K_{q_i},t})$ spanning mutually non-intersecting $(q_i-1)$-dimensional spaces, $1\le i\le k$. Moreover, it is easy to verify that the simplex $P(d_{K(Q),t})$ is a Delaunay polytope for all $t\ge 1$. Hence

$$t^C(K(Q))=t^H(K(Q))=\infty.$$

The following graphs are special cases of $\overline{K(Q)}$ which is the complete multipartite graph $K_{q_1,...,q_k}$. In particular, if $k=2$, $K_{q_1,q_2}$ is the complete bipartite graph. $\overline{K_1+K_q}=\nabla\overline{K_q}=K_{1,q}$ is a star, a special case of $K_{p,q}$. If $q_i=q$, $1\le i\le k$, then $K_{q,...,q}=K_{k\times q}$ is the complete $k$-partite graph. In particular, if $q=2$, $K_{2,...,2}=K_{k\times 2}$ is the Cocktail-party graph.

The distance $d_{\overline{K(Q)},t}$ is a metric for all $t$ such that $0\le t\le 2$. Moreover, if $m=2$, then $\overline{K(Q)}=K_{q,p}$ and $d_{K_{q,p},0}$ is a cut metric and spans an extreme ray of $Cut_n$. Otherwise, $d_{\overline{K(Q)},0}$ is a 0-extension of $d_{K_m,1}$, and lies on a face of $Cut_n$. In all these cases, $d_{\overline{K(Q)},0}$ is $l_1$-metric. Similarly, the metric $d_{\overline{K(Q)},2}$ spans an extreme ray of $Met_n$ if $\overline{K(Q)}$ contains $K_{2,3}$ as an induced subgraph. In this case $d_{\overline{K(Q)},2}$ is not hypermetric.

If $G\ne\overline{K(Q)}$, then $d_{G,t}$ is not a metric for $t<\frac{1}{2}$.

The property $d_{\overline{G},t}=td_{G,\frac{1}{t}}$ implies that $d_{K(Q),t}$ is a metric for $t\ge\frac{1}{2}$ but not hypermetric for $t=\frac{1}{2}$.

Obviously,

$$t_0^H(G)\le t_0^C(G)\le t^C(G)\le t^H(G).$$

By the properties of $d_{G,t}$, and since $l_1$-metrics and hypermetrics are metrics, if $G\ne K(Q)$, then $t^{C,H}(G)\le 2$.

(1) and (8) imply that $t_0^N(\overline{K(Q)})=0$.

Recall that a subgraph $H$ of $G$ with the set of vertices $V'\subseteq V$ is called *induced* subgraph if edges of $H$ are all edges of $G$ with both ends in $V'$. The following lemma is useful in what follows.

**Lemma 1** *If $H$ is an induced subgraph of $G$, then $t^A(G)\le t^A(H)$ for $A=C,H,N$.*

**Proof**. Obviously, any representation of $G$ provides a representation for every induced subgraph of $G$. Similarly, if $d_{G,t}$ is an $l_1$-metric or a hypermetric, then for every induced subgraph $H$ of $G$ the distance $d_{H,t}$ is an $l_1$-metric or a hypermetric, respectively. □

# 4 Vector representations of two-distance spaces

Obviously, det$Gr_n(d_{G,t})$ is a polynomial in $t$. It was used in [17]. The authors of [21] use the polynomial $P_G(t)$ =det$CM_n(d_{G,t}) = -(-2)^{n-1}$det$Gr_n(d_{G,t})$. Maehara [26] considers the polynomial $P'_G(s)$ =det$CM_n(d_{\overline{G},1-s}) = P_{\overline{G}}(1-s)$ and proves that $s^{n-1}P'_G(\frac{1}{s}) = \phi(G;-s) - (-1)^n\phi(\overline{G};s-1)$, where $\phi(G;s) = det(sI - A(G))$.

When $t < t^N(G)$, the Gram matrix is not singular and $P_G(t) \neq 0$. If $t = t^N(G)$, there is a dependency between representing vectors and $P_G(t) = 0$. Hence $t^N(G)$ is the smallest root of $P_G(t)$ with $t > 1$.

Unfortunately, $t^H(G)$ is not always given by a root of a known polynomial. But, for a hypermetric distance, we have a spherical representation that simplifies some considerations.

Using the adjacency matrix $A(G)$ of $G$, we can express the distance matrix $D(G,t)$ of the two-distance space $(V, d_{G,t})$ as follows

$$D(G,t) = A(G) + tA(\overline{G}) = A(G) + t(J - I - A(G)) = t(J - I) + (1-t)A(G), \quad (9)$$

where $I$ is the identity matrix and $J$ is the all-one matrix.

If $t$ is sufficient near to 1, then $d_{G,t}$ is hypermetric, since $Hyp_n \subset Neg_n$. Hence there is a Delaunay polytope $P_D(G,t) = P_D(d_{G,t})$ such that the set of vertices of $G$ is a subset of vertices of $P_D(G,t)$, and the distance is equal to the squared Euclidean distance. When $d_{G,t}$ is an interior point of $Hyp_n$, then $P(d_{G,t}) = P_D(G,t)$ is a simplex.

The polytope $P_D(G,1)$ is a regular $(n-1)$-dimensional simplex $P(d_{G,1})$ with the length of edges 1. When $t$ increases (or decreases) such that $d_{G,t}$ lies in the interior of the hypermetric cone $Hyp_n$, then $P_D(G,t)$ continues to be simplex, but not regular. When $d_{G,t}$ goes out on the boundary of $Hyp_n$, $P_D(G,t)$ ceases to be simplex. But there are two possibilities depending on whether there are affine dependencies between representing points or not: either the set $V$ spans a basic $(n-1)$-dimensional simplex of $P_D(G,t)$ or $P_D(G,t)$ has dimension less than $n-1$ and $V$ spans a sub-polytope of $P_D(G,t)$. The second case occurs when $t^H(G) = t^N(G)$, i.e. the boundaries of both the cones $Neg_n$ and $Hyp_n$ coincide in the point $d_{G,t^H(G)}$. We shall see later that both these cases occur for bipartite $G$.

For $t = t^H(G)$, denote $P_D(G,t)$ as $P_D(G)$. Note that the distance space generated by all vertices of $P_D(G)$ is not, in general, a two-distance space. For example, the unit cube $\gamma_n$ realizes all distances $k$, for $1 \leq k \leq n$. For the star $K_{1,n}$, we have $t^H(K_{1,n}) = 2$ and $P_D(K_{1,n}) = \gamma_n$.

Let the origin be in the center of $P_D(G,t)$, and let $v_i$ represent the vertex $i$ of $G$. Then

$$d_{G,t}(i,j) = (v_i - v_j)^2 \quad (10)$$

Let $r$ be the radius of the sphere circumscribing $P$. Then (10) implies

$$v_iv_j = \begin{cases} r^2 - \frac{1}{2} & \text{if } (ij) \in E(G), \\ r^2 - \frac{1}{2}t & \text{if } (ij) \notin E(G), \\ r^2 & \text{if } i = j. \end{cases} \quad (11)$$

Recall that a regular distance space has a spherical representation with radius given by (6). Of course, a two-distance space $(V, d_{G,t})$ is regular if and only if the graph $G$ is regular.

Let $G$ be regular of valency $q$. Then $\sum_{j=1}^n d_{G,t}(ij) = q + t(n-1-q)$. Hence, by (6), for the squared radius of the spherical representation we have

$$r^2(G,t) = \frac{q + t(n-1-q)}{2n}. \tag{12}$$

Of course if a space $(V, d)$ has a spherical representation with a radius $r$, then it has a spherical representation with any radius greater than $r$ (may be of dimension one greater).

We note the following useful fact proved in [21].

**Fact 2**. *Let $G = G_1 + G_2$, where $G_1 + G_2$ is disjoint union of $G_1$ and $G_2$, or $G = \overline{G_1 + G_2}$ such that $V = V_1 \cup V_2$. Let the distance space $(V, d_{G,t})$ have a representation. Then this representation is as follows: for $i = 1, 2$, $(V_i, d_{G_i,t})$ has a spherical representation such that $V_1$ and $V_2$ span orthogonal spaces that intersect in at most one point.* □

Let the vectors $v_i$, $1 \le i \le n$, give a representation of the two-distance space $(V, d_{G,t})$. The matrix $Q(G,t) \equiv (v_i v_j)_{1 \le i,j \le n}$ is the Gram matrix of the vectors $v_i$, $1 \le i \le n$. If this representation is spherical, then, according to (11),

$$Q(G,t) = r^2 J - \frac{t}{2}(J - I) + \frac{t-1}{2} A(G) = r^2 J - \frac{t+1}{4}(J - I) - \frac{t-1}{4} B(G), \tag{13}$$

where $B(G) = (J - I) - 2A(G)$ is the Seidel $(\mp 1)$-adjacency matrix of $G$. Comparing with (9), we obtain

$$Q(G,t) = r^2 J - \frac{1}{2} D(G,t).$$

As a Gram matrix, $Q(G,t)$ is positive semidefinite, and all its eigenvalues are nonnegative. Let $\lambda_0(G,t)$ be the minimal eigenvalue of $Q(G,t)$. We set

$$t_0^Q(G) = \min\{t : \lambda_0(G,t) \ge 0\}, \;\; t^Q(G) = \max\{t : \lambda_0(G,t) \ge 0\}.$$

Obviously, $t_0^Q(G) \le t_0^H(G)$, and $t^Q(G) \ge t^H(G)$. These inequalities hold as equalities if the representation given by $Q(G,t)$ is a hypermetric representation, i.e. the set of endpoints of vectors $v_i$, $1 \le i \le n$, is a subset of vertices of a Delaunay polytope.

If $G$ is a regular graph of valency $q$, then one can give explicit values of $t_{0,1}^Q(G,t)$. In this case, the all one vector $j$ is an eigenvector of $A(G)$ with the eigenvalue $q$. It is easy to see that $j$ is also an eigenvector of the matrix $Q(G,t)$ with the eigenvalue $(r^2 - \frac{t}{2})n + \frac{t}{2} + \frac{t-1}{2} q$. Note that this eigenvalue is nonnegative only if $r^2 \ge r^2(G,t)$, where $r^2(G,t) \equiv \frac{q+t(n-1-q)}{2n}$ is $r^2(d_{G,t})$ of (6). It is no wonder, since $r(G,t)$, the radius of the sphere with the center in the center of mass, is the radius of the minimal sphere circumscribing $P(G,t)$.

Let $\lambda(G)$ be the second largest eigenvalue of $G$ and $-\mu(G)$ be the smallest eigenvalue of $G$. The smallest eigenvalue is negative, since all eigenvalues of $A(G)$ are real and the sum of all eigenvalues equals the trace of $A(G)$, i.e. it is zero. Then

$$\lambda_0(G,t) = \begin{cases} \frac{t}{2} - \lambda(G)\frac{1-t}{2} & \text{if } t \le 1 \\ \frac{t}{2} - \mu(G)\frac{t-1}{2} & \text{if } t \ge 1 \end{cases}$$

It is known that $\mu(G) > 1$ if $G \neq K(Q)$ (see, for example, [5], Corollary 3.5.4.). Hence we have

$$t_0^Q(G) = \frac{\lambda(G)}{1+\lambda(G)},\ t^Q(G) = \frac{\mu(G)}{\mu(G)-1} \tag{14}$$

Note that $\mu(G) = k+1$ for the bipartite graph $K_{k+1,k+1}$. Hence $t^Q(K_{k+1,k+1}) = \frac{k+1}{k}$.

The condition (14) is a sufficient condition that the vertices of $G$ have a representation by points $v_i$ of a sphere with distances (10). But (14) says nothing about what is the obtained distance space. Is it a metric, hypermetric or $l_1$-space?

Note that the smallest eigenvalue of $Q(G,t)$ is equal to 0 if and only if the vectors $v_i$ are linearly dependent.

Usually one interests in two-distance sets in an Euclidean space of given dimension $m$ of maximal possible size $n(m)$. Obviously, the two-distance set with $n(1) = 3$ collinear points is given by two points and their mid-point. A regular pentagon provides $n(2) = 5$ points of the maximal two-distance set in $\mathbf{R}^2$. It is proved in [17] that two-distance sets in $\mathbf{R}^3$ with $n(3) = 6$ points are represented by 6 polyhedrons.

Recall that the $m$-dimensional *Johnson polytope* $JP(m+1,2)$ is the convex hull of middle points of edges of a regular $m$-dimensional simplex. ($JP(4,2)$ is a regular octahedron). Its 1-skeleton is the triangular graph $T(m+1)$, i.e. the line graph of $K_{m+1}$. Besides, $JP(m+1,2)$ is a section of the $(m+1)$-dimensional unit cube by the affine plane $\sum_1^{m+1} x_i = 2$. The last description shows that $JP(m+1,2)$ is inscribed in a sphere such that it is the convex hull of the simplex $JP(m,1)$ and the Johnson polytope $JP(m,2)$ both lying in parallel hyperplanes. $JP(m+1,2)$ has $\binom{m+1}{2}$ vertices which represent the distance $d_{G,2}$ for $G = T(m+1)$. The representation of the two-distance $d_{T(m+1),2}$ provides the lower bound $\binom{m+1}{2} \leq n(m)$.

For $m = 6$ and $m = 22$, two-distance sets are known with $\frac{1}{2}m(m+3) > \frac{1}{2}m(m+1)$ points given by maximal sets of equiangular lines. The best general upper bound is due to Blokhuis [7] who has shown that $n(m) \leq \binom{m+2}{2} = \frac{1}{2}m(m+3)+1$. It was known that this bound is tight for $m = 1$ and is not tight for $m = 2,3$.

Recently, P. Lisoněk [25] proved that the Blokhuis bound is not tight for $2 \leq m \leq 7$ and found a 8-dimensional two-distance set with $n(8) = \binom{8+2}{2} = 45$ points proving that Blokhuis bound is tight for $m = 8$. He noted that, for $m \leq 9$, one can move the simplex $JP(m,1)$ of $JP(m+1,2)$ into a hyperplane parallel to the hyperplane, where $JP(m,1)$ lies, such that the convex hull $\tilde{JP}(m+1,2)$ of the moved simplex and $JP(m,2)$ represents the distance $d_{G,2}$, where $G = \tilde{T}(m+1)$ is obtained from the triangular graph $T(m+1)$ by switching a maximal clique (of size $m$). It occurs that, for $m = 9$, the hyperplane, where the moved simplex lies, coincides with the hyperplane, where $JP(9,2)$ lies. In other words, the polytope $\tilde{JP}(10,2)$ has dimension 8. Since it has $\binom{10}{2} = 45$ vertices, it realize the Blokhuis bound for $m = 8$.

We can reformulate problem of a maximal two-distance set in a space of given dimension in terms of $t^N(G)$ as follows. Let $P(G)$ be $P(d_{G,t})$ for $t = t^N(G)$, and $m(G) = \dim P(G)$. Let $m(n) = \min_G\{m(G) : |V(G)| = n\}$

*What is minimal dimension $m(n)$ of polytopes $P(G)$ with $n$ vertices?*

The function $m(n)$ is the reciprocal function to $n(m)$.

It is proved in [23] the following striking fact (Theorem 2 in [23]): if $n > 2m(G) + 3$, then $t^N(G) = \frac{k+1}{k}$, the quotient of two neighboring positive integers such that $k < \frac{1}{2} + \sqrt{\frac{1}{2}m(G)}$. Neumaier [28] improved this result showing that it is true for $n \geq \max\{5, 2m(G) + 1\}$. We reformulate this fact as follows:

**Theorem A** *If* $2 \leq m(G) \leq \frac{n-1}{2}$, *then* $t^N(G) = \frac{k+1}{k}$ *for some positive integer* $k < \frac{1}{2} + \sqrt{\frac{n-1}{4}}$.

# 5 Bounds on $t^N(G)$

Schoenberg [30] considered the following problem: find a distance space $(V, d)$ with minimal distance 1 such that $(V, d)$ has an $(n-2)$-dimensional representation, $n = |V|$, and the *diameter* of $d$

$$t_n = \max_{(ij)} d(ij)$$

is minimal. He called these spaces *quasi-regular*. Schoenberg gave bounds for diameter $t_n$ and conjectured that quasi-regular distance spaces are two-distance spaces.

Seidel [31] proved that, for any $n$, there exists a unique (up to labeling of points) quasi-regular distance space. This implies the validity of Schoenberg's conjecture and that the bound given by Schoenberg is the exact value of $t_n$.

According to Proposition 1, $d$ lies on the boundary of the cone $Neg_n$. The following nice interpretation of this problem given in [30] shows that $t_n$ is, in fact, a lower bound on $t^N(G)$.

Consider a subset $B(t) \subset \mathbf{R}^{\binom{n}{2}}$ of all $d$ such that $1 \leq d(ij) \leq t$ for all $(ij)$. Obviously $B(t)$ is the convex hull of all points $d_{G,t}$, where $G$ takes all $2^{\binom{n}{2}}$ values. $B(t)$ is $\binom{n}{2}$-dimensional cube whose edges are parallel to the coordinate axes of $\mathbf{R}^{\binom{n}{2}}$, and the vertex $d_{K_n,t}$ is the point (1,1,...,1). The vertices $d_{G,t}$ and $d_{\overline{G},t}$ are opposite vertices of the cube $B(t)$. Schoenberg showed that $t_n$ is the value of $t > 1$, when a vertex of $B(t)$ touch for the first time the boundary of the cone $Neg_n$.

Let $G_0$ be the graph corresponding to the vertex of $B(t_n)$ touching the boundary of $Neg_n$. Obviously, $t_n = t^N(G_0) \leq t^N(G)$. In fact, if $G$ is not isomorphic to $G_0$, we have here a strict inequality, and this fact Schoenberg proved also. The graph $G_0$ is isomorphic to one of the bipartite graphs $K_{k,k}$ or $K_{k,k+1}$ depending on $n = 2k$ or $n = 2k + 1$.

The same bound was found also in [15], where *metric transforms* of a distance space were considered. In particular, a value of $c_n$ was given, where $c_n$ is the largest $c$ such that the metric transform $d^c_{G,2} = d_{G,2^c}$ of the truncated metric $d_{G,2}$ has an Euclidean representation for all $n$-points graphs $G$. It was shown in [15] that, for the bipartite graphs $K_{k,k}$ and $K_{k,k+1}$ the value of $c_n$ is exact. In our terms, we have $d_{G,2^{2c_n}} \in Neg_n$ for all $G$, and $d_{G_0,2^{2c_n}}$ belongs to the boundary of $Neg_n$, i.e. $t^N(G_0) = 2^{2c_n}$, or $c_n = \frac{1}{2}log_2 t^N(G_0)$.

We give once more proof of this fact using the inequalities (2) of negative type.

Denote by $h_G(t, b)$ the left hand side of the inequality (2) with $d = d_{G,t}$. According to Remark 1, we consider rational $b_i$, and suppose that $\sum_{i=1}^n |b_i| = 2$. Let $V_+(b) =$

$\{i \in V : b_i > 0\}$, $V_-(b) = \{i \in V : b_i < 0\}$, and $V(b) = V_+(b) \cup V_-(b) \subseteq V$. Note that $\sum_{i \in V_+(b)} b_i = \sum_{i \in V_-(b)} |b_i| = 1$. Let $G(b)$ be the subgraph of $G$ induced on the set $V(b)$. Let $K(b)$ be the complete bipartite graph $K_{p,q}$ on the set $V(b)$ with the partition $(V_+(b), V_-(b))$, i.e. $p = |V_+(b)|$, $q = |V_-(b)|$. Let $E_b(G) = E(G(b)) \Delta E(K(b))$ be the symmetric difference between the sets of edges of the graphs $G(b)$ and $K(b)$.

Similarly as it was shown in [3], one can show that

$$h_G(b,t) = (t-1) - \frac{t}{2}\sum_{i=1}^{n} b_i^2 - (t-1)\sum_{(ij)\in E_b(G)} |b_i||b_j|. \tag{15}$$

If we set

$$h_G(t) = \max_b h_G(b,t),$$

then

$$t^N(G) = \max\{t : h_G(t) \leq 0\}.$$

**Proposition 3** *We set*

$$f_n(k) = \begin{cases} \frac{k}{k-1} & \text{if } n = 2k, \\ \frac{2k(k+1)}{2k^2-1} & \text{if } n = 2k+1. \end{cases} \tag{16}$$

*Then*

$$f_n(k) \leq t^N(G) \leq 2 + \sqrt{3}$$

*with an equality in the left hand side if* $G = K_{k,k}$ *or* $G = K_{k,k+1}$.

**Proof**. We show that $h_G(b,t) \leq 0$ for $t$ equal to values given in the right hand side of (16) for all $b$ with $\sum b_i = 0$. Obviously, maximum of $h_G(b,t)$ is achieved for $b \in \mathbf{R}^n$ such that the second and the third terms of (15) are minimal in absolute value. The sum $\sum_{i=1}^n b_i^2$ with $\sum_{i \in V_+(b)} b_i = \sum_{i \in V_-(b)} |b_i| = 1$ takes its minimal value when $b_i$'s are almost equal for all $i$.

If $n = 2k$, then the sum $\sum_{i=1}^n b_i^2$ takes the minimal value $\frac{2}{k}$ for $|b_i| = \frac{1}{k}$. Hence we have

$$h_G(b,t) \leq (t-1) - \frac{t}{2}\frac{2}{k} = \frac{k-1}{k}(t - \frac{k}{k-1}).$$

This implies that $h_G(b, \frac{k}{k-1}) \leq 0$ for all $b$ if $n = 2k$.

Now, let $n = 2k+1$. Then the sum $\sum_{i=1}^n b_i^2$ takes the minimal value $\frac{2k+1}{k(k+1)}$ when $b_i = |V_+(b)|^{-1}$ for $i \in V_+(b)$, $b_i = -|V_-(b)|^{-1}$ for $i \in V_-(b)$, and either $|V_+(b)| = |V_-(b)| - 1 = k$ or $|V_+(b)| - 1 = |V_-(b)| = k$. In these cases

$$h_G(b,t) \leq (t-1) - \frac{t}{2}\frac{(2k+1)}{k(k+1)} = \frac{2k^2-1}{2k(k+1)}(t - \frac{2k(k+1)}{2k^2-1}).$$

This implies that $h_G(b, \frac{2k(k+1)}{2k^2-1}) \leq 0$ for all $b$ if $n = 2k+1$.

Let $G = K_{k,k}$ with the partition $V = V_1 \cup V_2$ such that $|V_1| = |V_2| = k$. Then for $b$ such that $b_i = \frac{1}{k}$ for $i \in V_1$, and $b_i = -\frac{1}{k}$ for $i \in V_2$, we have $h_G(b, \frac{k}{k-1}) = 0$.

If $G = K_{k,k+1}$ with the partition $V = V_1 \cup V_2$ such that $|V_1| = k$ and $|V_2| = k+1$, we set $b_i = \frac{1}{k}$ for $i \in V_1$ and $b_i = -\frac{1}{k+1}$ for $i \in V_2$, we have $h_G(b, \frac{2k(k+1)}{2k^2-1}) = 0$.

Hence the above bound is tight for $G = K_{k,k}$ and $G = K_{k,k+1}$.

One can reformulate Theorem 1 of [20] as the inequality

$$t^N(G) \le 2 + \sqrt{3}$$

valid for $n \ge 4$, i.e. as an upper bound on $t^N(G)$. This bound is implied by the following. If $n \ge 4$, then any representation of $(V, d_{G,t})$ contains a representation of a 4-point distance space $(V_4, d_{G,t})$. But examples of all 4-points distance spaces (see below) show that $t^N(G) \le t^N(P_3 + K_1) = 2 + \sqrt{3}$ if $t^N(G) \ne \infty$. □

Note that the upper bound of Proposition 3 does not depend on the dimension $m(G)$ of the corresponding representation of $G$. One can find in [21] more exact upper bounds depending on $m(G)$. We reformulate Theorems 1, 2 and 3 of [21] as follows.

**Theorem B** *Let $m(G)$ be dimension of a representation of $d_{G,t}$ for $t = t^N(G)$. Then*

*1. If $m(G) = n-2$, then $t^N(G) \le t^N(P_3 + K_{n-3}) = \frac{9(n-3)-1+\sqrt{33(n-3)^2+14(n-3)+1}}{4(n-3)}$;*

*2. If $\frac{2}{3}n \le m(G) \le n-3$, then $t^N(G) \le 2 + \sqrt{2}$;*

*3. If $\frac{n}{2} \le m(G) < \frac{2}{3}n$, then $t^N(G) \le \tau^2$.* □

We can reformulate Theorem A as the item 4 of Theorem B:

*4. If $m(G) \le \frac{n-1}{2}$, then $t^N(G) = \frac{k+1}{k}$ for $k < \frac{1}{2} + \sqrt{\frac{n-1}{2}}$.*

The polytope $P(G,t)$ for $G = K_{k,k}$ or $G = K_{k,k+1}$ and $t = t^N(G)$ has dimension $n-2$. For $G = K_{k,k}$, it is a special case of the *repartitioning polytope*, and is a Delaunay polytope.

Recall that $t^M(K_{k,k}) = t^M(K_{k,k+1}) = 2$, what is greater than $t^N(K_{k,k})$ and $t^N(K_{k,k+1})$. Since there are graphs with $t^N(G) > 2$, we have examples exhibiting incomparability of the cones $Met_n$ and $Neg_n$.

# 6 Bounds on $t^H(G)$

Let $b$ determine a $(2q+1)$-gonal hypermetric inequality, i.e. $\sum_{i=1}^n |b_i| = 2q+1$. Note that since $\sum_{i=1}^n |b_i| \equiv \sum_{i=1}^n b_i = 1 \pmod 2$, the first sum here is odd. Note that $\sum_{i\in V_+(b)} b_i = \sum_{i\in V_-(b)} |b_i| + 1 = q+1$.

It is shown in [3] that

$$h_G(b,t) = q^2(t - \frac{q+1}{q}) - \frac{t}{2}\sum_{i=1}^n |b_i|(|b_i|-1) - (t-1)\sum_{(ij)\in E_b(G)} |b_i||b_j|, \qquad (17)$$

where $E_b(G)$ is defined in the previous section.

**Proposition 4** *Let $n = 2k+1$ or $n = 2k+2$ and $G$ be not a sum of complete graphs. Then*

$$\frac{k+1}{k} \le t^H(G) \le 2,$$

*with equality in the left hand side if $G = K_{k,k+1}$ or $G = K_{k+1,k+1}$, and equality in the right hand side iff the truncated distance $d^*_G$ is hypermetric.*

**Proof**. Let

$$h_G(q,t) = \max\{h_G(b,t) : \sum_{i=1}^{n} |b_i| = 2q+1\}.$$

We show that $h_G(q,t) \le 0$ for $t = \frac{k+1}{k}$ and all $q \ge 1$. Obviously, $\sum_{(ij)\in E_b(G)} |b_i||b_j| \ge 0$. Similarly $\sum_{i=1}^n |b_i|(|b_i|-1) \ge 0$, since $x(x-1) \ge 0$ for all integer $x$. Hence $\max_b h_G(b,t)$ is achieved for $b \in \mathbf{Z}^n$ such that the second and the third terms of (17) are minimal in absolute value.

Let $a = \lfloor \frac{2q+1}{n} \rfloor$. The sum $\sum_{i=1}^n |b_i|(|b_i|-1)$ with $\sum_{i=1}^n |b_i| = 2q+1$ takes its minimal value when $|b_i|$'s are almost equal for all $i$. This means that either $|b_i| = a$ or $|b_i| = a+1$. If $|b_i| = a$ for $x$ values of $i$, then $|b_i| = a+1$ for $n-x$ values of $i$. We have $xa+(n-x)(a+1) = 2q+1$, i.e. $x = n(a+1)-(2q+1)$. In this case $\sum_{i=1}^n |b_i|(|b_i|-1) = a(2q+1-\frac{n}{2}(a+1))$, and therefore

$$h_G(q,t) \le q^2(t - \frac{q+1}{q}) - ta(2q+1-\frac{n}{2}(a+1)).$$

We have $2q+1 = an+s$, where $0 \le s \le n-1$. If $a = 0$, then $2q+1 \le n-1 \le 2k+1$, i.e. $q \le k$. Since $\frac{k+1}{k} \le \frac{q+1}{q}$ for $q \le k$, it follows that

$$h_G(q, \frac{k+1}{k}) \le 0 \text{ for } 2q+1 < n.$$

Hence we suppose below that $a \ge 1$.

We have $q = \frac{1}{2}(an+s-1)$. Substituting $q$ in the above inequality with $a$ by this value, we obtain

$$h_G(q,t) \le t[(\frac{1}{4}(an+s-1)^2 - a(an+s-\frac{n}{2}(a+1)] - \frac{1}{4}((an+s)^2-1).$$

Setting $f(a,n,s) = a(n-2)(an+2s)+(s-1)^2$, one can rewrite the above inequality as follows

$$h_G(q,t) \le \frac{1}{4}f(a,n,s)\left[t - \frac{n}{n-2} - \frac{2(s-1)(n-1-s)}{(n-2)f(a,n,s)}\right].$$

Since $f(a,n,s) \ge 0$, the inequality $h_G(q,t) \le 0$ is valid if the expression in the square parentheses is not greater than zero. If $s > 0$, then $(s-1)(n-1-s) \ge 0$. This, in particular, holds when $n$ is even. Hence, for $n = 2k+2$ even, the expression in the square parentheses takes the maximal value $t - \frac{n}{n-2} = t - \frac{k+1}{k}$ for $s = 1$ or for $s = n-1$. For $n = 2k+1$ odd, the expression in the square parentheses takes the maximal value

$$t - \frac{n}{n-2} + \frac{2}{(n-2)(n-1)} = t - \frac{n+1}{n-1} = t - \frac{k+1}{k}$$

for $s = 0$ and $a = 1$ when $f(1,n,0) = (n-1)^2$. In both these cases we obtain that $h_G(q,t) \le 0$ for $t = \frac{k+1}{k}$ and all $q$.

It is not difficult to verify that for $G = K_{k,k+1}$ and $G = K_{k+1,k+1}$ this bound is tight.

Since a hypermetric is a metric, there is an obvious upper bound $t^H(G) \leq 2$ if $G \neq K(Q)$. Obviously this bound is archived if the truncated distance $d^*_G$ is hypermetric. □

We reformulate here Proposition 2 for a two-distance space. Recall that $P_D(G)$ is the Delaunay polytope of the distance space $(V, d_{G,t})$ for $t = t^H(G)$.

**Proposition 5** *If $dimP_D(G) \leq n-2$, then $t^N(G) = t^H(G)$.* □

Note that $P_D(K_{k+1,k+1})$ is a special case of the class of repartitioning polytopes $V^{2k}_{k,k}$ of dimension $2k = 2(k+1) - 2 = n - 2$. Hence Proposition 5 explains why we have $t^N(K_{k,k}) = t^H(K_{k,k})$ in both Propositions 3 and 4.

The repartitioning polytope $P_D(K_{k+1,k+1})$ is important for what follows. It is the convex hull of two regular $k$-dimensional simplexes $S_1$ and $S_2$, spanning orthogonal spaces and intersecting in the common center. The edges of both these simplexes $S_i$ have norm $t^N(K_{k+1,k+1}) = t^H(K_{k+1,k+1}) = \frac{k+1}{k}$. We denote this polytope shortly as $B_k$. So, the $2k$-dimensional polytope $B_k$ has $2k+2$ vertices and $(k+1)^2$ facets. Each facet is a simplex and is obtained by deleting by a vertex from both the simplexes $S_1$ and $S_2$. By construction the set of vertices of $B_k$ form a two-distance space with distances 1 and $\frac{k+1}{k}$.

Note that the right hand side of Proposition 4 also has the form $\frac{k+1}{k}$ for $k = 1$. It is worth to recall Theorem A. Taking in attention Theorem A and Proposition 2, we give the following conjecture.

**Conjecture 1**. *In the case when the conditions of Theorem A hold, the distance $d_{G,t^N}$ lies on the boundary of $Hyp_n$.*

# 7 Two-distance spaces and equiangular lines

Suppose that $4r^2 = 1 + t$ in the spherical representation (11) of the two-distance $d_{G,t}$. Then $v_i v_j = \pm\frac{t-1}{2}$ if $i \neq j$, i.e. the vectors $v_i$ span equiangular lines. If $G$ is regular of valency $q$, this is possible only if $\frac{1+t}{4} \geq r^2(G,t)$, i.e. if

$$t \begin{cases} \leq 1 + \frac{2}{n-2q-2} & \text{for } n > 2q+2, \\ \geq 1 - \frac{2}{2q+2-n} & \text{for } n < 2q+2. \end{cases} \tag{18}$$

The last inequality shows that if $q > \frac{n-2}{2}$, then regular $G$ always has a spherical representation spanning equiangular lines.

If we replace a vector $v_i$ by $-v_i$, we obtain again a two-distance $d_{G^{sw},t}$. Here the graph $G^{sw}$ is obtained from $G$ by *switching* of the vertex $i$. The edges and non-edges of $G$ incident to the vertex $i$ are interchanged in $G^{sw}$. The switching of $G$ by a set of vertices is clear. Two graphs are called *switching equivalent* if one is a switching of other. Clearly, switching equivalent graphs determine isomorphic sets of eqiangular lines.

According to (13), if $r^2 = \frac{1+t}{4}$, we have

$$Q(G,t) = \frac{t+1}{4}I - \frac{t-1}{4}B(G).$$

This expression shows explicitly that $Q(G,t)$ is related to equiangular lines, since the Seidel matrix $B(G)$ takes $(\mp 1)$-values.

We can easy find the acute angle $\alpha$ between the corresponding equiangular lines. If $r^2 = \frac{1+t}{4}$ and the maximal inner product $v_i v_j = r^2 \cos\alpha$ is equal, according to $Q(G,t)$, to $\frac{t-1}{4}$, we have

$$\cos\alpha = \frac{t-1}{t+1}. \tag{19}$$

Note that, considering equiangular lines, one corresponds usually adjacency to $v_i$ and $v_j$ with negative inner product $v_i v_j$, i.e. one uses the complemented graph $\overline{G}$. Following to Neumaier [28], and using (13), we rewrite $Q(G,t)$ for $r^2 = \frac{1+t}{4}$ with $A(\overline{G})$:

$$Q(G,t) = \frac{t-1}{2}\left(\frac{1}{t-1}I - A(\overline{G}) + \frac{1}{2}J\right). \tag{20}$$

Hence if the largest eigenvalue $\lambda_{max}(\overline{G}) \leq \frac{1}{t-1}$, then $Q(G,t)$ is positive semidefinite, and we have

**Proposition 6** *[28]. Let $\overline{G}$ be a graph which is switching equivalent to a graph $H$ with $\lambda_{max}(H) \leq \frac{1}{t-1}$. Then $d_{\overline{G},t}$ is represented by equiangular lines with the angle $arcos\frac{t-1}{t+1}$.*

If $G$ is regular, then eigenvalues of $G$ and $\overline{G}$ are related as $\lambda(\overline{G}) = -(1+\lambda(G))$, $\lambda(G) \neq q(G)$, and $q(\overline{G}) = n-1-q(G)$. Hence the minimal eigenvalue $\lambda_{min}(G) = -\mu(G)$ corresponds to the second largest eigenvalue $\lambda_2(\overline{G})$ of $\overline{G}$.

There are two upper bounds on the number of equiangular lines in an $m$-dimensional space. The *absolute* bound does not depend on the angle between lines:

$$n \leq n_a(m) = \frac{m(m+1)}{2}.$$

The *special* bound is valid for $m < \frac{1}{\cos^2\alpha}$:

$$n \leq n_s(\alpha, m) = \frac{m(\cos^{-2}\alpha - 1)}{\cos^{-2}\alpha - m}.$$

In [24], the following analogue of Theorem A is proved.

**Theorem C** *If the number of equiangular lines in $\mathbf{R}^m$ is greater than $2m$, then $\frac{1}{\cos\alpha}$ is an odd integer.* $\square$

Denoting this integer by $2k+1$ we have $\cos\alpha = \frac{1}{2k+1}$. According to (19), we have in this case that

$$t = t(k) \equiv \frac{k+1}{k}.$$

For this value of $t = t(k)$, $r^2 = \frac{1+t}{4} = \frac{2k+1}{4k}$.

Take along each line two opposite vectors $\pm v$ of norm $v^2 = 2k+1$. Then $v_i v_j = \pm 1$ for $i \neq j$. It is convenient to denote the pair of opposite vectors $(v, -v)$ as $(v, v^*)$. The set $\mathcal{V}$ of all these vectors is a special case of an odd system (see [13]). An *odd system* is a set $\mathcal{V}$ of vectors with odd inner products, and, in particular, having odd norms. Let

$$L(\mathcal{V}) = \{\sum_{v\in\mathcal{V}} b_v v : \sum_{v\in\mathcal{V}} b_v = 1 \pmod 2\}.$$

It is proved in [13] that $L(\mathcal{V})$ is a lattice. Let $\mathcal{U}_k$ be an $m$-dimensional odd system corresponding to a set of equiangular lines, i.e. $\mathcal{U}_k$ is a set of vectors of norm $2k+1$ with inner products $\pm 1$. Let $P(\mathcal{U}_k)$ be the convex hull of all vectors of norm $2k+1$ of the lattice $L(\mathcal{U}_k)$.

$P(\mathcal{U}_k)$ is a symmetric $m$-dimensional polytope and turns out very often to be a symmetric Delaunay polytope of the lattice $L(\mathcal{U}_k)$. This implies that the corresponding two-distance space is hypermetric.

Let $Q(G,t)$ be the Gram matrix of $\mathcal{U}_k$. If $G$ is not regular, then according to (20) $\lambda_{max}(\overline{G}) \leq k$. If $G$ is regular, then, by (14), $t^Q(G) = \frac{k+1}{k}$ if $\mu(G) = k+1$. This implies that $\lambda_2(\overline{G}) = k$. The cases $k=1$ and $k=2$ are especially interesting, since, for these $k$, conditions are known when $P(\mathcal{U}_k)$ is a Delaunay polytope. One knows many regular graphs with the second largest eigenvalue 2. We consider some of them in the last section.

For $\cos\alpha = \frac{1}{2k+1}$, the special bound takes the form

$$n_s(k,m) = \frac{4k(k+1)m}{(2k+1)^2 - m}. \tag{21}$$

Recall that this formula is valid only for $m < \frac{1}{\cos^2\alpha} = (2k+1)^2$.

Sets of equiangular lines, where the special bound is attained, are of a special interest. They are related to so-called *regular two-graphs*. Obviously, if the special bound is tight, then $n = n_s(k,m)$ is an integer. For a given $k$, there is a number of values of dimension $m$ such that $n_s(k,m)$ is an integer.

There is a minimal value of $m$ such that the integer $n_s(k,m) > m$. It is easy to find that the minimal value is equal to $m = 2k+1$ when

$$n_s(k, 2k+1) = 2(k+1).$$

Let $\mathcal{U}_k^0$ be the odd system corresponding to the set of $2(k+1)$ eqiangular lines. It consists of $2k+2$ vectors $u_i$, $1 \leq i \leq 2k+2$, of norm $2k+1$ with pairwise inner products $-1$ and of $2k+2$ its opposite $u_i^*$. It is easy to verify that $\sum_{i=1}^{2k+2} u_i = 0$. Note that, in any odd system of norm $2k+1$, every set of vectors with mutual inner products $-1$ has at most $2k+2$ vectors.

Let $\mathcal{V}_k^0 = \frac{1}{\sqrt{4k}}\mathcal{U}_k^0$, and $v_i = \frac{1}{\sqrt{4k}} u_i$. Let $V = \{1, 2, ..., 2k+1, 2k+2\}$, and $V = V_1 \cup V_2$, $|V_1| = p$, $|V_2| = q$, $p+q = 2k+2$, be a partition of $V$. Then the set of vectors $\{v_i : i \in V_1, v_j^* : j \in V_2\}$ represents the two-distance $d_{K_{p,q},t(k)}$, where $t(k) = \frac{k+1}{k}$. In the next section, we show that this representation is exact (i.e. with $t = t^H(G)$) if $p = k$, $q = k+2$ with $t^H(K_{k,k+2}) = t(k)$.

Baranovski [4] proves (in other terms) that $P(\mathcal{V}_k^0)$ coincides with the convex hull of $\mathcal{V}_k^0$, and $P(\mathcal{V}_k^0)$ is a symmetric Delaunay polytope of the lattice $L(\mathcal{V}_k^0)$ which is the Coxeter lattice $A_{2k+1}^{k+1}$. He denotes $P(\mathcal{V}_k^0)$ as $\mathcal{A}^{2k+1}$.

For a partition $V = V_1 \cup V_2$, $|V_1| = p$, $|V_2| = q$, let $S^{p-1}$ and $S^{q-1}$ be regular simplexes with edges of norm $t(k)$ such that these simplexes are the convex hulls of $v_i$, $i \in V_1$, and $v_i^*$, $i \in V_2$, respectively. Let $S_{p,q}(t)$ be the convex hull of $S^{p-1}$ and $S^{q-1}$. The distances between vertices of $S_{p,q}(t)$ from distinct simplexes are equal to 1. Then $\mathcal{A}^{2k+1}$ is the convex hull of both $S_{p,q}(t)$ and its opposite with $p+q = 2k+2$. If $q = 0$, and $p = 2k+2$, we obtain that $\mathcal{A}^{2k+1}$ is the convex hull of the regular simplex $S^{2k+1}$ and its opposite. So, all edges of $\mathcal{A}^{2k+1}$ have norm 1 or $t(k)$, and its diagonals have norm $1 + t(k)$. Each facet of $\mathcal{A}^{2k+1}$ is the $2k$-dimensional repartitioning polytope $P_D(K_{k+1,k+1}) = S_{k+1,k+1}(t(k))$, which we denoted in previous section as $B_k$. The polytope $B_k$ is the convex hull of two regular $k$-dimensional simplexes spanning orthogonal spaces and intersecting in the common center. Note that $\mathcal{A}^3 = \gamma_3$, $B_1 = \gamma_2$, where $\gamma_n$ is the unit $n$-dimensional cube.

# 8 Complete bipartite graphs

We saw that the complete bipartite graphs $K_{k,k+1}$ and $K_{k+1,k+1}$ provides the lower bound on $t^H(G)$ for $G$ with the same number of vertices. We obtained that $t^H(K_{k,k+1}) = t^H(K_{k+1,k+1}) = t(k)$, and $P_D(K_{k,k+1}) = P_D(K_{k+1,k+1}) = B_k$. Now we consider the complete bipartite graphs $K_{p,q}$ for other values of $p$ and $q$.

The case $p = 1$ is special. Recall that $\gamma_q$ is the unit $q$-dimensional cube, which is the unique Delaunay polytope of the lattice $\mathbf{Z}^q$.

**Proposition 7** *For $q \geq 2$, $t^H(K_{1,q}) = 2$ and $P_D(K_{1,q}) = \gamma_q$.*

**Proof**. The graph $K_{1,1}$ is the complete graph $K_2$ with $t^H(K_2) = \infty$. Hence we have to consider $q \geq 2$. By Proposition 4, $t^H(K_{1,q}) \leq 2$, for $q \geq 2$. The $q$ mutually orthogonal edges of $\gamma_q$ give a representation of $d_{K_{1,q},2}$. Obviously this representation generates the lattice $\mathbf{Z}^q$. $\square$

Let $V = V_1 \cup V_2$, where $|V_1| = p$ and $|V_2| = q$, are the set of vertices of $K_{p,q}$, and let $p \leq q$.

Recall that $h_G(k,t) = \max_b\{h_G(b,t) : \sum_{i=1}^n |b_i| = 2k+1\}$.

**Lemma 2** *For $G = K_{p,q}$, $h_G(k,t) = k(k+1)((1 - f_{p,q}(k))t - 1)$, where*

$$f_{p,q}(k) = \frac{b_1(k - \frac{1}{2}(b_1+1)p) + b_2(k+1-\frac{1}{2}(b_2+1)q) + k}{k(k+1)},$$

*and $b_1 = \lfloor \frac{k}{p} \rfloor$, $b_2 = \lfloor \frac{k+1}{q} \rfloor$.*

**Proof**. Let $b$ define a hypermetric inequality. Recall that $V_+(b) = \{i \in V : b_i > 0\}$, $V_-(b) = \{i \in V : b_i < 0\}$. Let $b'$ be such that $b'_j = b_j$, $j \in V_2$, $b'_i = b_i$, $i \in V_1 - \{i_1, i_2\}$, $b'_{i_1} = b_{i_1} - \varepsilon$, $b'_{i_2} = b_{i_2} + \varepsilon$, $\varepsilon$ is a positive integer. Let $\delta h_G(t) = h_G(b,t) - h_G(b',t)$. Then $\delta h_G(t) = t(\varepsilon(b_{i_2} - b_{i_1}) + \varepsilon^2)$. Obviously if $b_{i_2} - b_{i_1} < -\varepsilon$, then $h_G(b',t) > h_G(b,t)$. We see that if there are two indexes $i, i'$ both in $V_1$ or $V_2$ such that $|b_i - b_{i'}| > 1$, then there is a perturbation $b \to b'$ such that $h_G(b',t) > h_G(b,t)$. Hence, for $b$ with maximal $h_G(b,t)$,

$b_i$'s of each part differs at most on 1. In particular, the coefficients of each part have the same sign.

If $b$ defines a $(2k+1)$-gonal inequality, then $|\sum_{i\in V_2} b_j|$ is equal either to $k$ or $k+1$. But comparing with (17), we see that if $|\sum_{i\in V_2} b_j| = k+1$, i.e. if $V_+(b) \subseteq V_2$, then $h_G(b,t)$ is not less than $h_G(b,t)$ for $V_+(b) \subseteq V_1$, since $q \geq p$.

Let $b$ define a $(2k+1)$-gonal inequality. Then we saw that $\max_b h_G(b,t)$ for $G = K_{p,q}$ is achieved for $b_i$ such that $\sum_{i\in V_1} b_i = -k$, $\sum_{i\in V_2} b_i = k+1$, and $|b_i - b_{i'}| \leq 1$ for $i, i'$ from the same part. Let $b_1 = \lfloor \frac{k}{p} \rfloor$, $b_2 = \lfloor \frac{k+1}{q} \rfloor$. Then $b_i = -b_1$ for $(b_1+1)p - k$ values of $i \in V_1$, and $b_i = -(b_1+1)$ for other $k - b_1 p$ values of $i \in V_1$. Similarly, $b_j = b_2$ for $(b_2+1)q - (k+1)$ values of $j \in V_2$, and $b_j = b_2 + 1$ for other $k+1-b_2 q$ values of $j \in V_2$. In other words, the function $h_G(k,t)$ takes the form

$$h_G(k,t) = (k^2 - kb_1 - (k+1)b_2 + \frac{1}{2}pb_1(b_1+1) + \frac{1}{2}qb_2(b_2+1))t - k(k+1).$$

Introducing the function $f_{p,q}(k)$, we obtain the expression of $h_G(k,t)$ given in the formulation of this lemma. □

**Proposition 8** *Let $G = K_{p,q}$ with $p \leq q$, and $q = ps + r$, where $1 \leq r \leq p$. Then*

$$t^H(K_{p,q}) \leq t(p,q) = \frac{1}{1-\varphi(p,q)}, \text{ where } \varphi(p,q) \equiv \frac{(s+1)(q+r-2)}{2q(q-1)} = f_{p,q}(q-1),$$

*with equality for $p = 1, q-2, q-1, q$, when*

$$t^H(K_{p,q}) = t(p,q) = \begin{cases} 2 & \text{for } p = 1 \\ \frac{q}{q-1} & \text{for } p = q-1, q \\ \frac{q-1}{q-2} & \text{for } p = q-2 \end{cases}$$

**Proof**. Consider a $(2k+1)$-gonal inequality with $k = q-1$ and $b$ such that $h_G(b,t) = h_G(k,t)$. Then $b_2 = 1$, and $b_1 = \lfloor \frac{k}{p} \rfloor = \lfloor \frac{q-1}{p} \rfloor = \lfloor \frac{ps+r-1}{p} \rfloor = s$, since $r \geq 1$. For these values of $b_1$ and $b_2$, $f_{p,q}(q-1) = \varphi(p,q)$ and $h_G(q-1,t) = q(q-1)(\frac{t}{t(p,q)} - 1)$ should be $\leq 0$. Hence $t \leq t(p,q)$ and $h_{K_{p,q}}(q-1,t) = 0$.

Obviously, $h_{K_{p,q}}(q-1,t) = 0$ for $t = t(p,q)$, i.e. the distance $d_{G,t}$, for $G = K_{p,q}$ and $t = t(p,q)$ satisfies the above $(2q-1)$-gonal hypermetric equality. This implies that $t^H(K_{p,q}) \leq t(p,q)$.

Note that if $p = 1$, then $s = q-1$ and $r = 1$, since $r \geq 1$. Hence $\varphi(1,q) = \frac{1}{2}$ and $t(1,q) = 2$, the result of Proposition 7. Similarly if $p = q-1$, then $s = r = 1$, and if $p = q$, then $s = 0$, $r = q$. Hence $\varphi(q-1,q) = \varphi(q,q) = \frac{1}{q}$, and $t(p,q) = \frac{q}{q-1} = t^H(K_{q-1,q}) = t^H(K_{q,q})$, by Proposition 4.

Now we consider $p = q-2$, when $s = 1$, $r = 2$, and $\varphi(q-2,q) = \frac{1}{q-1} = \frac{1}{p+1}$. By previous proposition we may assume that $p \geq 2$.Hence $t(p,p+2) = \frac{p+1}{p} = t(p)$. We know from Proposition 4 that $t^H(K_{p,p+1}) = t(p)$. Since $K_{p,p+1}$ is an induced subgraph of $K_{p,p+2}$, by Proposition 4 and Lemma 1, $t(p) = t^H(K_{p+1,p+1}) \leq t^H(K_{p,p+2}) \leq t^H(K_{p,p+1}) = t(p)$, i.e. $t^H(K_{p,p+2}) = t(p)$. □

Note, that if $b_j = 0$ for some $j \in V_2$, then the hypermetric inequality with this $b$ is applied, in fact, to a graph $K_{p,q'}$ with $q' < q$. Since $K_{p,q'} \subseteq K_{p,q}$, $t^H(K_{p,q}) \le t^H(K_{p,q'})$, by Lemma 1. Hence, using induction on $q$, and that $\varphi(p,q) \le \varphi(p,q')$, we can suppose that $h_G(k,t) \le 0$ for $t = t(p,q)$, $G = K_{p,q}$, and for $k \le q-2$.

**Conjecture 2**. *$t^H(K_{p,q}) = t(p,q)$ where $t(p,q)$ is given in Proposition 8.*

For to prove this conjecture, we have to prove that $h_G(k,t) \le 0$ for $G = K_{p,q}$, $t = t(p,q)$ and all integers $k \ge q$. In other words, we have to prove that $\varphi(p,q) \le f_{p,q}(k)$ for all $k$. It is not difficult to verify that $\varphi(p,q) \le f_{p,q}(q)$. Unfortunately, the function $f_{p,q}(k)$ is not monotone on $k$ and behaves very irregular when $k$ increases. We computed $f_{p,q}(k)$ for many values of $p,q$ and $k$. For all these values the inequality $\varphi(p,q) \le f_{p,q}(k)$ holds.

The difficulty is such that, for given $p$ and $q$, there is no unique expression of $f_{p,q}(k)$ for all $k$ without the operation of taking integer part. But, for $q = ps$, we have

**Proposition 9** $t^H(K_{p,ps}) = t(p,ps) = \frac{2p(ps-1)}{ps(2p-1)-(3p-2)}$.

**Proof**. If $q$ is divided by $p$, then $r = p$. We set $k = bq+cp+a$, where $b \ge 0$, $0 \le cp+a < q$, i.e. $0 \le c \le s-1$ and $0 \le a \le p-1$. Now we have only two cases:

1) either $c \le s-1$, $a \le p-2$, or $c \le s-2$, $a = p-1$, when $b_1 = bs+c$, $b_2 = b$, and
2) $c = s-1$, $a = p-1$, when $b_1 = bs+c$, $b_2 = b+1$.

It is easy to verify that, in the second case, when $k = (b+1)q-1$,

$$f_{p,ps}((b+1)ps-1) - \varphi(p,ps) = \frac{b(s-1)}{2(ps-1)((b+1)ps-1)} \ge 0.$$

Tedious computations show that the inequality $f_{p,ps}(k) - \varphi(p,ps) \ge 0$ holds in the first case, too. □

We describe the Delaunay polytopes $P_D(K_{p,ps})$ in Proposition 10 below.

For sufficient small $t$, the distance $d_{K_{p,q},t}$ is hypermetric, and $P_D(K_{p,q},t)$ is a simplex of the following form (cf. Fact 2 from Section 4). Let $S^{p-1}$, $S^{q-1}$ be regular $(p-1)$- and $(q-1)$-dimensional simplexes with edges of norm $t$. Let $S^{p-1}$ and $S^{q-1}$ are embedded in a $(p+q-1)$-dimensional space as follows. $S^{p-1}$ and $S^{q-1}$ span non-intersecting orthogonal spaces, the segment connecting their centers is orthogonal to both these spaces, and the distance between vertices of distinct simplexes is equal to 1. Then $P_D(K_{p,q},t)$ is the convex hull of $S^{p-1}$ and $S^{q-1}$. We denote the $(p+q-1)$-dimensional simplex $P_D(K_{p,q},t)$ as $S_{p,q}(t)$. Obviously $S^{p-1}$ and $S^{q-1}$ are faces of $S_{p,q}(t)$. Moreover, any face of $S_{p,q}(t)$ is either $S^{p'-1}$, $S^{q'-1}$ or $S_{p',q'}(t)$ for some $p' \le p$, $q' \le q$. Besides $S_{p',q'}(t)$ is the Delaunay polytope $P_D(K_{p',q'},t)$.

Using (7), it can be shown that the squared radius of $S_{p,q}(t)$ is equal to

$$R^2_{p,q}(t) = \frac{pq - (p-1)(q-1)t^2}{4pq - 2(2pq-p-q)t}.$$

Note that if $R^2_{p,q}(t) = \frac{1}{2}$, then the centers of $S^{p-1}$, $S^{q-1}$ and $S_{p,q}(t)$ coincide, and if $R^2_{p,q}(t) > \frac{1}{2}$, then the center of $S_{p,q}(t)$ lies beyond its boundary. For $t = 2$ and $q \ge 2$, $R^2_{1,q}(2) = \frac{q}{4}$ is the squared radius of the unit $q$-dimensional cube $\gamma_q$.

Take the center of the sphere circumscribing $S_{p,q}(t)$ as origin. Let $\mathcal{W}_{p,q}(t)$ be $p+q$ pairs of opposite vectors $(w_i, w_i^*)$, $i \in V$, of norm $R^2_{p,q}(t)$ such that $w_i$, $i \in V$, represent vertices of the simplex $S_{p,q}(t)$. Denote by $\mathcal{D}_{p,q}(t)$ the convex hull of vectors of $\mathcal{W}_{p,q}(t)$. In other words, similar to $\mathcal{A}^{2k+1}$, $\mathcal{D}_{p,q}(t)$ is the convex hull of $S_{p,q}(t)$ and its opposite.

Conjecture 1 and the cases of $K_{2,2s}$ and $K_{k,k+2}$, considered below, imply

**Conjecture 3** *The polytope $\mathcal{D}_{p,ps+2}(t)$ for $t = t(p, ps+2)$ is a Delaunay polytope.*

Recall that the Delaunay polytope $B_k$ is defined in the previous section. For $q = ps$, and $p \geq 3$, let $D_{p,s}$ be the following polytope. We set a copy of the simplex $S^{p-1}$ in the sphere circumscribing $S_{p,q}(t)$ such that its vertices touch the sphere and the space spanned by this copy is parallel to the space spanned by the original simplex $S^{p-1}$.

Set $\mathcal{D}_{2,2s}(t(2,2s)) = \mathcal{D}^{2s+1}_{2,s}$.

**Proposition 10** *For $p \geq 1$, $s \geq 1$,*

$$P_D(K_{p,ps}) = \begin{cases} \gamma_s & \text{if } p = 1,\ s \geq 2, \\ B_{p-1} & \text{if } p \geq 2,\ s = 1, \\ \mathcal{D}^{2s+1}_{2,s} & \text{if } p = 2,\ s \geq 2, \\ D_{p,s} & \text{if } p \geq 3,\ s \geq 2. \end{cases}$$

**Proof**. The case $p = 1$ is considered in Proposition 7. It is shown in Proposition 8 that $t^H(K_{p,p}) = t(p,p)$ (the case $s = 1$). In previous section we show that $P_D(K_{p,p}) = B_{p-1}$.

Recall that, for $G = K_{p,ps}$, $p \geq 2$, the distance $d_{G,t}$ satisfies the $p$ equalities with $b_j = 1$ for all $j \in V_2$, $b_i = -s$, $i \in V_1 - \{i_0\}$, $b_{i_0} = -(s-1)$. This equalities, for $i \in V_1$, determine $p$ vectors $v(b) = w'_i \equiv w_0 + w_i$, where $w_0 \equiv \sum_{j \in V_2} w_j - s \sum_{i \in V_1} w_i$, and $w_i \in \mathcal{W}_{p,ps}(t)$.

Using (11) for $G = K_{p,ps}$, $r^2 = R^2_{p,ps}(t)$ and $t = t(p, ps)$, one can show that $w_0 = 0$ only if $s = 1$ and $w_0 = -\sum_{i \in V_1} w_i$ if $p = 2$.

Hence if $s = 1$, then $w'_i = w_i$, and the convex hull of $2p$ vectors $w_i$, $i \in V$, is $B_{p-1}$.

If $p = 2$, $s \geq 2$, then $w'_i = w^*_i$ for $i \in V_1$. For $t = t(2, 2s)$, the distance $d_{K_{2,2s},t}$ satisfies additionally $2s$ $(4s-1)$-gonal equalities with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V_2 - \{j_0\}$, $b_{j_0} = 0$. These equalities provide additionally $2s$ vectors $v(b) = w_0 - w_i = w^*_i$, $i \in V_2$, of the system $\mathcal{W}_{2,2s}(t(2,2s))$. This implies that $\mathcal{D}^{2s+1}$, the convex hull of $\mathcal{W}_{2,2s}(t(2,2s))$, is the Delaunay polytope $P_D(K_{2,2s})$.

If $p > 2$ and $s > 1$, then $w'_i \neq w_i, w^*_i$, but $w'_i - w'_{i'} = w_i - w_{i'}$ for $i, i' \in V_1$. Hence the convex hull of endpoints of the $p$ vectors $w'_i$ is a copy of $S^{p-1}$ parallel to $S^{p-1}$, and the convex hull of $2p + ps$ vectors $w_j$, $j \in V_2$ and $w_i$, $w'_i$, $i \in V_1$, is the polytope $D_{p,s}$. □

We give, in Proposition 11 below, an infinite sequence of pairs of bipartite graphs $(K_{p,p+1}, K_{p,p+2})$ such that $t^H$'s of both the graphs of the sequence coincide and the Delaunay polytope of the first graph is a facet of the Delaunay polytope of the second graph.

Let $t_{p,s} = t(ps, ps+2) = \frac{2(ps+1)}{2ps-s+1}$. Then $t_{p,1} = \frac{p+1}{p} = t(p)$. For $q = ps+2$, we set $\mathcal{D}_{p,q}(t_{p,s}) = \mathcal{D}^{ps+p+1}_{p,s}$, $\mathcal{W}_{p,sp+2}(t_{p,s}) = \mathcal{W}_{p,s}$. Note that $\mathcal{D}^{2p+1}_{p,1} = \mathcal{A}^{2p+1}$,

**Proposition 11** *If Conjecture 2 is true for $q = ps+2$, then it is true for $q = ps+1$, too, and for $p \geq 2$, $s \geq 1$, $P_D(K_{p,ps+1})$ belongs to the class $V^{ps+p}_{p,ps}$ and is a facet of $P_D(K_{p,ps+2}) = \mathcal{D}^{ps+p-1}$.*

*Remark* . Note that Conjecture 2 is true for $q = ps + 2$ and either $p = 1$ or $s = 1$. In fact, the cases $p = 1$ and $s = 1$ are noted in Proposition 8. We saw that $P_D(K_{p,p+1})$ is a facet of $\mathcal{A}^{2p+1}$.

**Proof**. It is easy to verify that $t(p, ps + 1) = t(p, ps + 2) = t_{p,s}$. Suppose that $t^H(K_{p,ps+2}) = t_{p,s}$. Since $K_{p,ps+1}$ is an induced subgraph of $K_{p,ps+2}$, by Lemma 1, we have $t^H(K_{p,ps+1}) \geq t^H(K_{p,ps+2}) = t_{p,s}$. But, by Proposition 8, $t^H(K_{p,ps+1}) \leq t(p, ps + 1) = t_{p,s}$. Hence $t^H(K_{p,ps+1}) = t(p, ps + 1)$ if Conjecture 2 is true for $K_{p,ps+2}$.

Let $V = V_1 \cup V_2$ be the set of vertices of $K_{p,ps+2}$. Suppose that the set of vertices of $K_{p,ps+1}$ is the set $V' = V_1 \cup V_2'$, where $V_2'$ is $V_2$ without a vertex. Note that the distance $d_{K_{p,ps+1},t}$, for $t = t_{p,s}$, satisfies the $(2ps + 1)$-gonal equality with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V_2'$.

Similarly, for $t = t_{p,s}$, the distance $d_{K_{p,ps+2},t}$ satisfies the following $p + ps + 2$ equalities:

1) $(2ps + 1)$-gonal equalities with $b_i = -s$, $i \in V_1$, $b_j = 1$, $j \in V_2 - \{j_0\}$, $b_{j_0} = 0$; we have $ps + 2$ such equalities for $j_0 \in V_2$;

2) $(2ps + 3)$-gonal equalities with $b_i = -s$, $i \in V_1 - \{i_0\}$, $b_{i_0} = -(s + 1)$ and $b_j = 1$, $j \in V_2$. We have $p$ such equalities for $i_0 \in V_1$.

Using (11) for $G = K_{p,ps+2}$, $t = t_{p,s}$ and $r^2 = R^2_{p,ps+2}$, one can show that $w_0 = \sum_{j\in V_2} w_j - s\sum_{i\in V_1} w_i = 0$ for $w_i \in \mathcal{W}_{p,s}$. Then the equalities 1) and 2) provide $sp + p + 2$ vectors $v(b) = w_0 - w_i = w_i^*$, $i \in V$, of the system $\mathcal{W}_{p,s}$. This implies that $\mathcal{D}^{ps+p+1}_{p,s}$, the convex hull of $\mathcal{W}_{p,s}$, is the Delaunay polytope $P_D(K_{p,ps+2})$. The convex hull of vectors $w_i$, $i \in V' = V - \{j_0\}$ and $w^*_{j_0}$ for $j_0 \in V_2$ and $t = t_{p,s}$ is $P_D(K_{p,ps+1})$. It is a facet of $\mathcal{D}^{ps+p+1}$ orthogonal to the vector $\beta\sum_{i\in V_1} w_i - \sum_{j\in V_2-\{j_0\}} w_j$, where $\beta = \frac{(s-1)(ps+1)}{(s-1)p+2}$. □

**Corollary** $t^H(K_{2,2s+1}) = t^H(K_{2,2s+2}) = t(2, 2(s + 1)) = \frac{2(2s+1)}{3s+1}$. *$P_D(K_{2,2s+1})$ belongs to the class $V^{2(s+1)}_{2,2s}$ and is a facet of $P_D(K_{2,2s+2}) = \mathcal{D}^{2s+1}$.*

**Proof**. We have $K_{2,2s+2} = K_{2,2(s+1)}$. By Proposition 10, Conjecture 2 holds for $K_{2,2s+2}$. Now we can apply Proposition 11. □

We denote the facet $P_D(K_{p,ps+1})$ of $\mathcal{D}^{p(s+1)+1}_{p,s}$ by $F^{p(s+1)}_{p,s}$. Note that $F^{2p}_{p,1} = B_p$.

The cases of Proposition 7 and the above Corollary cover all complete bipartite graphs $K_{p.q}$ with $p + q \leq 9$. In Table below, we write out the values of $t^H(K_{p,q})$ and $P(K_{p,q})$ for $3 \leq p + q \leq 9$, $1 \leq p \leq q$.

| $p+q$ | $K_{p,q}$ | $t^H(K_{p,q})$ | $P_D(K_{p,q})$ |
|---|---|---|---|
| 3 | $K_{1,2}=P_3$ | 2 | $\gamma_2=B_1$ |
| 4 | $K_{1,3}$ | 2 | $\gamma_3=\mathcal{A}^3$ |
| 4 | $K_{2,2}=C_4$ | 2 | $\gamma_2=B_1$ |
| 5 | $K_{1,4}$ | 2 | $\gamma_4$ |
| 5 | $K_{2,3}$ | $\frac{3}{2}$ | $B_2$ |
| 6 | $K_{1,5}$ | 2 | $\gamma_5$ |
| 6 | $K_{2,4}$ | $\frac{3}{2}$ | $\mathcal{A}^5=\mathcal{D}^5$ |
| 6 | $K_{3,3}$ | $\frac{3}{2}$ | $B_2$ |
| 7 | $K_{1,6}$ | 2 | $\gamma_6$ |
| 7 | $K_{2,5}$ | $\frac{10}{7}$ | $F^6_{2,2}$ |
| 7 | $K_{3,4}$ | $\frac{4}{3}$ | $B_3$ |
| 8 | $K_{1,7}$ | 2 | $\gamma_7$ |
| 8 | $K_{2,6}$ | $\frac{10}{7}$ | $\mathcal{D}^7_{2,2}$ |
| 8 | $K_{3,5}$ | $\frac{4}{3}$ | $\mathcal{A}^7$ |
| 8 | $K_{4,4}$ | $\frac{4}{3}$ | $B_3$ |
| 9 | $K_{1,8}$ | 2 | $\gamma_8$ |
| 9 | $K_{2,7}$ | $\frac{7}{5}$ | $F^8_{2,3}$ |
| 9 | $K_{3,6}$ | $\frac{30}{23}$ | $D_{3,2}$ |
| 9 | $K_{4,5}$ | $\frac{5}{4}$ | $B_4$ |

# 9 Small two-distance sets with $n \le 7$

In [17], the numbers $g(n)$ of $n$-point graphs $G \neq K(Q)$, i.e. with $t^N(G) < \infty$, are given for $n = 4, 5, 6, 7$. In Table 1 below, we give these numbers together with numbers $h(n)$ of graphs having $t^H(G) = 2$, i.e. with the hypermetric distance $d^*_G$, and numbers $h^N(n)$ of graphs with $t^H(G) = t^N(G)$.

**Table 1**.

| $n$ | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| $g(n)$ | 6 | 27 | 145 | 1029 |
| $h(n)$ | 6 | 23 | 95 | ? |
| $h^N(n)$ | 1 | 3 | ? | ? |

We use the Coxeter's notations of some polytopes. Let
$\alpha_n$ be an $n$-dimensional regular simplex, with 1-skeleton $K_{n+1}$,
$\beta_n$ be an $n$-dimensional cross-polytope with length of edges 1, with 1-skeleton $K_{n\times 2}$,
$\gamma_n$ be an $n$-dimensional unite cube, $\gamma_n = \gamma_1^n$, with 1-skeleton $K_2^n$.

We denote by $PyrB$ the pyramid with the base $B$ whose lateral edges have length greater than 1. Then dim$PyrB$ = 1+dim$B$. The 1-skeleton of $PyrB$ is $K_1 + G$, where $G$ is the 1-skeleton of $B$. Besides, $Pyr^*(PyrB)$ denotes a polytope such that apex of the second pyramid is at distance 1 from the apex of the first pyramid.

It is known all combinatorial types of Delaunay polytopes in dimensions 2, 3 and 4. There are 2, 5 and 19 types in these dimensions, respectively (see [16]).

The 2-dimensional Delaunay polytope distinct from a simplex is a rectangle, of the combinatorial type $\gamma_2$. The combinatorial types of 3-dimensional Delaunay polytopes are the simplex $\alpha_3$, the cross-polytope $\beta_3$, the prism $\alpha_2\times\gamma_1$, the pyramid $Pyr\gamma_2$ and the cube $\gamma_3$.

Note that $\gamma_2 = P_D(K_{2,2})$ is the special case $B_1$ of the class $V^2_{1,1}$ of repartitioning polytopes. Similarly, $Pyr\gamma_2 = P_D(K_1 + K_{2,2})$ is the special case $PyrB_1$ of the class $V^3_{1,1}$.

In tables below, we give $t^H(G)$ and $P_D(G)$ of graphs $G$ with $t^N(G) < \infty$. We use symbols of graphs from [17]. The symbol of a graph $G$ is the triple $(n.r.s)$, where $n$ and $r$ are the numbers of vertices of $G$ and of edges of $\overline{G}$, respectively, and $s$ differs distinct graphs with the same $n$ and $r$. The symbol $(n.r.s)'$ corresponds to the complement $\overline{G}$. (Graphs given in [17] have edges of length $t$, i.e. they are the complements of our graphs.) The $D$-symbol is either the *Delaunay symbol* denoting a Delaunay polytope of dimensions 3 and 4 in Tables IV and V or the symbol of Table VI taken from [16].

**n=3**. There is only one graph $P_3 = K_{1,2}$ on 3 vertices distinct from $K(Q)$, i.e. with $t^N(G) < \infty$. We have $t^N(K_{1,2}) = 4$ and $t^H(K_{1,2}) = 2$ (see Propositions 3 and 4), $P_D(K_{1,2}) = \gamma_2 = B_1$.

**n=4**. The values of $t^N(G)$ are taken from [17]. They can be found also in [21].

Since $Cut_4 = Hyp_4 = Met_4$, we have $t^C(G) = t^H(G) = 2$ for all these six graphs $G$ on 4 vertices with $t^N(G) < \infty$. Moreover, since $\lambda_{min}(C_4) = -2$, we have $t^H(C_4) = t^N(C_4) = 2$ and $P_D(C_4) = P(d_{C_4,2}) = \gamma_2$.

Besides these 6 four-point two-distance sets in $\mathbf{R}^2$ there is only one another two-distance set, the pentagon.

**Table 2. 4-point graphs**

| symbol | $G$ | $t^N(G)$ | $t^H(G)$ | $P_D(G)$ |
|---|---|---|---|---|
| (4.1.1) | $K_4 - e = \nabla P_3$ | 3 | 2 | $\beta_3$ |
| (4.2.1)$'$ | $\overline{P_3 + K_1}$ | $2+\sqrt{3}$ | 2 | $\alpha_2\times\gamma_1$ |
| (4.2.1) | $P_3 + K_1$ | $2+\sqrt{3}$ | 2 | $Pyr\gamma_2 = PyrB_1$ |
| (4.2.2) | $C_4$ | 2 | 2 | $\gamma_2 = B_1$ |
| (4.3.1) | $K_{1,3}$ | 3 | 2 | $\gamma_3$ |
| (4.3.2) | $P_4$ | $\tau^2$ | 2 | $\alpha_2\times\gamma_1$ |

**n=5**. We have $Cut_5 = Hyp_5 \subset Met_5$. Hence $t^C(G) = t^H(G)$ for all 27 five-point graphs with $t^N(G) < \infty$. The values $t^N(G)$ can be found in [17]. There are 3 five-point graphs having 3-dimensional Delaunay polytopes $P_D(G)$, i.e. satisfying the conditions of Proposition 5. Hence for these graphs we have $t^H(G) = t^N(G) = 2$.

Among other 27-3=24 graphs there are additionally 20 graphs with $t^H(G) = 2 < t^N(G)$. The 4 graphs with $t^H(G) < 2$ are the 3 non-hypermetric five-point graphs (of diameter 2) and one graph with a non-hypermetric distance $d^*_G$. These graphs were found for the first time in [1]. One of these graphs is the complete bipartite graph $K_{2,3}$, the unique graph of Propositions 3 and 4, giving the minimal values of $t^N(G)$ and $t^H(G)$ for 5-point graphs. The edges $e'$ and $e$ in $K_{2,3} + e'$ and $K_{2,3} + e$ are added to distinct parts of the bipartite graph $K_{2,3}$. The Delaunay polytopes $P_D(G)$ of all these 4 graphs are 4-dimensional repartitioning polytopes. But only $P_D(K_{2,3}) = B_2$. $K_{2,3} - V$ is the graph $K_{2,3}$ without two edges incident to a vertex of the part of size 2.

Besides the graph $K_1 + C_4$, there are 5 graphs of the same type $G = K_1 + H$.

Note the graph $(5.5.4) = (5.5.4)' = C_5$, having $t^N(C_5) = \tau^2$ with a two-dimensional Euclidean representation. We have $t^H(C_5) = 2$ and $P_D(C_5)$ is the Johnson polytope $PJ(5,2)$ of dimension 4, whose 1-skeleton is the triangular graph $T(5)$.

The Johnson polytope $PJ(n,k)$ is the section of the cube $\gamma_n$ by the hyperplane $\{x : xj_n = k\}$ orthogonal to the diagonal of $\gamma_n$ spanned by the all-one $n$-dimensional vector $j_n$. We have $PJ(n, n-k) = PJ(n,k)$, $PJ(n,1) = \alpha_{n-1}$, $PJ(4,2) = \beta_3$.

**Table 3. 5-point graphs**

| $N$ | symbol | $G$ | $t^H(G)$ | $P_D(G)$ | $D$-symbol |
|---|---|---|---|---|---|
| | | $G$ with $\dim P_D(G) = 3$ and $t^H(G) = t^N(G)$ | | | |
| 1 | (5.2.2) | $\nabla C_4$ | 2 | $\beta_3$ | $F_2^3$ |
| 2 | (5.4.6) | $\overline{P_5}$ | 2 | $\alpha_2 \times \gamma_1$ | $F_1^3$ |
| 3 | $(5.4.5)'$ | $K_1 + C_4$ | 2 | $Pyr\gamma_2 = PyrB_1$ | $F^4$ |
| | | $G$ with $P_D(G)$ of type $V_{2,2}^4$ | | | |
| 4 | (5.4.4) | $K_{2,3}$ | $\frac{3}{2}$ | $B_2$ | $A$ |
| 5 | (5.3.2) | $\nabla^2(3K_1) =$ $= K_{2,3} + e'$ | $\frac{5}{3}$ | $\simeq V_{2,2}^4$ | $A$ |
| 6 | (5.3.4) | $K_{2,3} + e$ | $\frac{5}{3}$ | $\simeq V_{2,2}^4$ | $A$ |
| 7 | (5.5.3) | $K_{2,3} - e$ | $\frac{5}{3}$ | $\simeq V_{2,2}^4$ | $A$ |
| | | $G$ with $P_D(G) = PJ(5,2)$ | | | |
| 8 | (5.2.1) | $\nabla^2(K_2 + K_1)$ | 2 | $PJ(5,2)$ | $F_4^4$ |
| 9 | (5.3.1) | $\nabla(K_3 + K_1)$ | 2 | $PJ(5,2)$ | $F_4^4$ |
| 10 | (5.3.3) | $\nabla P_4$ | 2 | $PJ(5,2)$ | $F_4^4$ |
| 11 | (5.4.3) | $\overline{K_{2,3} - V}$ | 2 | $PJ(5,2)$ | $F_4^4$ |
| 12 | (5.5.4) | $C_5$ | 2 | $PJ(5,2)$ | $F_4^4$ |
| | | $G$ with $P_D(G)$ of type $PyrP$ | | | |
| 13 | $(5.2.1)'$ | $2K_1 + P_3$ | 2 | $Pyr^2\gamma_2 = Pyr^2B_1$ | $F^8$ |
| 14 | $(5.3.1)'$ | $K_1 + K_{1,3}$ | 2 | $Pyr\gamma_3$ | $F_1^6$ |
| 15 | $(5.3.3)'$ | $K_1 + P_4$ | 2 | $Pyr(\alpha_2 \times \gamma_1)$ | $F_1^7$ |
| 16 | $(5.4.2)'$ | $K_1 + \overline{K_1 + P_3}$ | 2 | $Pyr(\alpha_2 \times \gamma_1)$ | $F_1^7$ |
| 17 | (5.5.1) | $K_1 + (K_4 - e)$ | 2 | $Pyr\beta_3$ | $B$ |
| 18 | $(5.3.4)'$ | $K_2 + P_3$ | 2 | $Pyr^*(Pyr\gamma_2$ | $F^8$ |
| | | Other graphs | | | |
| 19 | (5.1.1) | $\nabla^3(2K_1) =$ $= \nabla(K_4 - e)$ | 2 | $\beta_4$ | $C$ |
| 20 | $(5.4.1)'$ | $K_{1,4}$ | 2 | $\gamma_4$ | $F_6^3$ |
| 21 | (5.4.2) | $\nabla(K_1 + P_3)$ | 2 | $\beta_3 \times \gamma_1$ | $F_5^4$ |
| 22 | $(5.4.3)'$ | $K_{2,3} - V$ | 2 | $\gamma_4$ | $F_6^3$ |
| 23 | (5.4.5) | $\nabla(2K_2)$ | 2 | $\alpha_2 \times \alpha_2$ | $F_5^5$ |
| 24 | (5.4.6) | $P_5$ | 2 | $\alpha_2 \times \alpha_2$ | $F_5^5$ |
| 25 | (5.5.2) = $= (5.5.2)'$ | | 2 | $\alpha_2 \times \alpha_2$ | $F_5^5$ |
| 26 | $(5.5.1)'$ | $\nabla(2K_1 + K_2)$ | 2 | $\alpha_2 \times \gamma_2$ | $F_1^4$ |
| 27 | $(5.5.3)'$ | $\overline{K_{2,3} - e}$ | 2 | $\alpha_3 \times \gamma_1$ | $F_4^6$ |

**n=6**. In this case $Cut_6 = Hyp_6$. Hence $t^C(G) = t^H(G)$. There are 145 six-point graphs $G$ of not the form $K(Q)$, i.e. with $t^N(G) < \infty$. It is noted in [1] that $d_G^*$ is hypermetric, i.e. $t^H(G) = 2$, if either $G$ does not contain one of the four 5-point graphs with $t^H(G) < 2$ (and there are 48 such 6-point graphs), or $G$ is not one of the two 6-point graphs $G_1$ and $G_2$. Here $G_1 = K_{2,4} - 3e$, where all the 3 deleted edges are incident in $K_{2,4}$ to the same vertex of the part of size 2. The graph $G_2 = K_{2,4} - 2e + e'$, where the two deleted edges are incident in $K_{2,4}$ to the same vertex of the part of size 2, and the edge $e'$ connects vertices of degree 2 of the part of size 4 in $K_{2,4} - 2e$.

The distance $d^*_G$ for $G = G_1$ and $G = G_2$ does not satisfy the 7-gonal inequality with $b = (1, 1, 1, 1, 1, -1, -2)$. For this inequality and $G = G_1, G_2$, $h_G(t, b) = 5t - 9$. Hence $h_G(t, b) \le 0$ if $t \le \frac{9}{5}$. This implies $t^H(G) = \frac{9}{5} < 2$ for these two graphs.

Note that among 5 three-dimensional Delaunay polytopes there are two polytopes, $\beta_3$ and $\alpha_2 \times \gamma_1$, having 6 vertices with two distances between them. Obviously, for the corresponding graphs $(6.3.1)' = K_{3\times 2}$ and $(6.6.1)' = K_2 \times K_3$, we have $t^{H,N}(K_{3\times 2}) = t^{H,N}(K_2 \times K_3) = 2$. As it shown in [17], there are 4 another 6-point two-distance spaces $d_{G,t}$ with a 3-dimensional representations and with the same $t^N(G) = \tau^2$.

Among 19 four-dimensional Delaunay polytopes, there are two 6-vertex polytopes, $Pyr^2\gamma_2$ and $B_2$, giving two-distance sets $d^*_G$ with $G = 2K_1 + C_4$ and $K_{3,3} = (6.6.3)'$, respectively. The 4 Delaunay polytopes $\beta_3$, $\alpha_2 \times \gamma_1$, $Pyr^2\gamma_2$ and $B_2$ are examples, where the conditions of Proposition 5 hold. Hence $t^H(2K_1 + C_4) = t^N(2K_1 + C_4) = 2$, and $t^H(K_{3,3}) = t^N(K_{3,3}) = \frac{3}{2}$.

**n=7**. We have $Cut_7 \ne Hyp_7$, and there are 7-point graphs $G$ with $t^C(G) < t^H(G)$. The first such graph was found by Avis [2]. Now, one knows 26 7-point graphs with this property. These graphs can be found in [12] and [14]. All these graphs lie on extreme rays of $Hyp_7$, and they are subgraphs of the Schläfli graph. The corresponding hypermetric distances are two-distances $d^*_G$. They have the common Delaunay polytope $P_D(G) = P_{Schl}$. The polytope $P_{Schl}$ is a 6-dimensional asymmetric Delaunay polytope which is the convex hull of the representation of $Schl$.

For to find $t^C(G)$, one needs to find the facet of $Cut_7$, where $d_{G,t}$ lies. Obviously, this facet is not hypermetric.

# 10 Graphs with $t^H(G) = 2$

Recall that $d_{G,2} = d^*_G$ is the truncated distance of the graph $G$. The equality $t^H(G) = 2$ means that the distance $d^*_G$ is hypermetric. The graphs $G$ having hypermetric $d^*_G$ are studied in [1] and [12]. In particular, it is proved in [12] the following assertion.

**Proposition 12** *If $G$ is a connected regular graph, then $d^*_G$ is hypermetric iff $\lambda_{min}(G) \ge -2$, where $\lambda_{min}(G)$ is the smallest eigenvalue of $G$.*

Recall that $\mu(G) = -\lambda_{min}(G)$ in (14). If $\mu(G) \le 2$, then (14) shows that $t^Q(G) \ge 2$ with equality only if $\mu(G) = 2$. Hence if $\lambda_{min}(G) > -2$, when $t^Q(G) > 2$, the Gram matrix $Q(G, 2)$ is not singular, and $P(d_{G,2})$ is a simplex. But if $\lambda_{min}(G) = -2$, then there are dependencies between representing vectors, and the conditions of Proposition 2 hold. Therefore if $\mu(G) = 2$, we have $t^H(G) = t^N(G) = 2$.

Note the class of strongly regular graphs with $\lambda_{min}(G) = -2$. These graphs were classified by Seidel (see [5], Theorem 3.12.4(i)). These graphs are

the triangular graph $T(n)$, $n \geq 5$,
the square $n \times n$ grid $K_n \times K_n$ (also called a lattice graph $L_2(n)$), $n \geq 3$,
the Cocktail party graph $K_{n\times 2}$, $n \geq 2$,
the Petersen $Pe$, the Clebsch $Cle$, the Schläfli $Schl$, the Shrikhande $Shr$, and three Chang $Ch_i$ graphs.

If $G$ is a connected regular graph with $\lambda_{min}(G) > -2$, then $G$ is a complete graph or an odd cycle.

Since all these graphs (except $Schl$ and $Ch_i$) are $l_1$-graphs, we have that $t^C(G) = 2$ if $G$ is a strongly regular graph with $t^H(G) = 2$, $G \neq Schl, Ch_i$, $1 \leq i \leq 3$.

# 11 Graphs with $t^H(G) = \frac{3}{2}$

In this section we give examples of regular graphs $G$ with $t^H(G) = \frac{3}{2}$ represented by odd systems of norm $2k+1$ related to equianglar lines. According to (19), the angle between these lines is equal to $arccos\frac{1}{5}$, i.e. $k = 2$.

Let $\mathcal{U}_2 = \{u_i, u_i^* : i \in V\}$ be the odd system of vectors of norm 5 related to the representation. It is proved in [13] that $P(\mathcal{U}_2)$ is a Delaunay polytope if the odd system $\mathcal{U}_2$ of norm 5 is not pillar. Recall that the cardinality of a maximal set of vectors of $\mathcal{U}_2$ with mutual inner products $-1$ is not greater than 6. Let $\{u_i : i \in C\}$ be such a set with $|C| \leq 6$, and let $v \in \mathcal{U}_2$. Then the vector $v$ partitions $C$ into subsets $C_+ = \{i \in C : u_iv = 1\}$ and $C_- = \{i \in C : u_iv = -1\}$.

The odd system $\mathcal{U}_2$ is called *pillar* if this partition does not depend on the vector $v$. Let $\mathcal{U}_2^+$ be a subset of $\mathcal{U}_2$ containing from each pair $(u_i, u_i^*)$ of opposite vectors exactly one vector. W.l.o.g., we denote the vector as $u_i$. Let $G^+$ be the graph with $V$ as the set of vertices. Two vertices $i, j$ of $G^+$ are adjacent iff $u_iu_j = -1$. If $\mathcal{U}_2$ is not pillar, then there are two vertices of $G^+$ having distinct neighborhoods in $C$, i.e. having distinct partitions of $C$.

Note that distance between endpoints of two non-opposite vectors (of norm 5) $u, u' \in \mathcal{U}_2$ is equal to 12 if $uu' = -1$ and to 8 if $uu' = 1$. Hence $\mathcal{U}_2^+$ represents the distance space $(8d_{G,t}, V)$, where $G = \overline{G^+}$ and $t = \frac{3}{2}$.

Let $G'$ be a regular subgraph of $G^+$ with $n$ vertices and valency $q$. It is represented by a subset of $\mathcal{U}_2^+$. Using (6), we find that the squared radius of the convex hull of endpoints of vectors of this representation is equal to $r^2 = 4 + \frac{2q-4}{n}$. Since $r^2 \leq 5$, we have to have $n \geq 2q - 4$.

We saw in Section 7 that a regular graph $G$ with $\lambda_2(\overline{G}) = 2$ has such a representation if its valency satisfies inequalities (18). For $t = \frac{3}{2}$, these inequalities take the form $n \leq 2q+6$. For the graph $\overline{G}$ of valency $\overline{q} = n - q - 1$, the last inequality takes the form $\overline{q} \leq \frac{n+4}{2}$.

Call a graph $G$ with $\lambda_2(G) = 2$ *non-pillar* if it has the following property. There is a maximal clique $C$ and two vertices $i, j \in V - C$ such that the neighborhoods of $i$ and $j$ in $C$ are distinct, i.e. the partitions of $C$ determined by $i$ and $j$ are distinct.

**Proposition 13** *Let $G$ be a regular non-pillar graph of valency $q \leq \frac{n+4}{2}$ with $\lambda_2(G) = 2$ of multiplicity $f \geq 2$. Then $t^H(\overline{G}) = \frac{3}{2}$.*

**Proof**. We saw that $\overline{G}$ has a representation by an odd system $\mathcal{U}_2$ related to equiangular lines at angle $arccos\frac{1}{5}$ if valency $q$ of $G$ satisfies $q \le \frac{n+4}{2}$. The condition that $G$ is non-pillar implies that the odd system $\mathcal{U}_2$ is not pillar. Hence $P(\mathcal{U}_2)$ is a Delaunay polytope. The dimension of this polytope is equal to $n - f \le n - 2$. By Proposition 2, $d_{\overline{G},\frac{3}{2}}$ lies on the boundary of $Hyp_n$. Hence this representation is exact, and $t^H(\overline{G}) = \frac{3}{2}$. □

The most important case relates to equiangular lines corresponding to a regular two-graph. In table below, we give dimensions $m$ for which one knows sets of $n_s(2,m) = \frac{24m}{25-m}$ (see (21)) equiangular lines corresponding to regular two-graphs. As usual, $N$ is the number of known non-isomorphic two-graphs, and $\overline{N}$ denotes that this number is exact (cf. [13]).

| $m$ | 5 | 10 | 13 | 15 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|
| $n_s(2,m)$ | 6 | 16 | 26 | 36 | 126 | 176 | 276 |
| $N$ | $\overline{1}$ | $\overline{1}$ | $\overline{4}$ | 227 | 1 | 1 | $\overline{1}$ |

The minimal set of 6 lines is represented by the 5-dimensional odd system $\mathcal{V}_2^0 = \frac{1}{\sqrt{8}}\mathcal{U}_2^0$. The Delaunay polytope $P(\mathcal{V}_2^0) = \mathcal{A}^5$ is considered in the end of Section 7.

Now we consider in detail the case $m = 10$. Let $\mathcal{U}_2^1$ be the corresponding odd system. It is not pillar. The polytope $P(\mathcal{U}_2^1)$ coincides with the convex hull of $\mathcal{U}_2^1$ and is a Delaunay polytope. $\frac{1}{\sqrt{2}}P(\mathcal{U}_2^1)$ is the symmetrization of the cut polytope $PCut_5$ which is the convex hull of indicator vectors $c_S$ of all cuts $\delta(S)$ of $K_5$ defined in Section 2.

The symmetrization of the cut polytope $PCut_5$ is the convex hull of all indicator vectors of cuts and their complements in the complete graph $K_5$. Let $V_5 = \{1, 2, ..., 5\}$ and $E_5 = \{ij : 1 \le i < j \le 5\}$ be the sets of vertices and edges of $K_5$, respectively.

Let $j_{10}$ be the all-one vector. Denote by $c^*(S) = j_{10} - c(S)$ the indicator vector of the complement $\delta^*(S) = E_5 - \delta(S)$ of the cut $\delta(S)$. Since $c(V_5 - S) = c(S)$, we can use only $S \subseteq V_5$ with $|S| \le 2$. We set $\mathcal{S} = \{S : S \subset V_5, |S| \le 2\}$.

Denote by $P_5$ the convex hull of all vectors $c(S)$, $c^*(S)$, $S \in \mathcal{S}$. It is shown in [19] that $P_5$ is, up to a multiple, a Delaunay polytope of the 10-dimensional isodual lattice $Q_{10}$ mentioned in [9].

All the vectors $c(S)$ and $c^*(S)$ are vertices of the 10-dimensional unit cube $\mathbf{B}^{10}$. In fact the set $C_{10}$ of all 32 these (0,1)-vectors is the set of all codewords of a linear binary code with parameters $[n, k, d] = [10, 5, 4]$.

The set $C_{10}$ with the Hamming distance $d$ between its points is an $l_1$ metric space $(C_{10}, d)$. The distance between points $a, b \in C_{10}$ is equal to the *norm* $(a-b)^2$ of the vector $a - b$. Since $|\delta(S)|$ take only two values 4 and 6 for $S \subseteq E_5$, $S \ne \emptyset$, $E_5$, we obtain that any subset of $(C_{10}, d)$ without pairs of complemented vectors is a two-distance $l_1$-space with distances 4 and 6, i.e. it is $4d_{G,\frac{3}{2}}$ for some $G$.

The center of $P_5$ is $\frac{1}{2}j_{10}$, and the squared radius of $P_5$ is $\frac{5}{2}$. Hence the set of vectors $\sqrt{2}c(S)$, $\sqrt{2}c^*(S)$, $S \in \mathcal{S}$, form the odd system $\mathcal{U}_2^1$.

We consider graphs on subsets of $\mathcal{U}_2^1$ without opposite vectors such that two vertices are adjacent iff they are at distance 12. It is easy to verify that the corresponding graph on 16 vertices $\sqrt{2}c(S)$, $S \in \mathcal{S}$, is the strongly regular Clebsch graph $Cle$ with parameters (16,10,6,6) and with $\lambda_2(Cle) = 2$. By construction, $Cle$ represents the distance space

$12d_{Cle,\frac{2}{3}} = 8d_{\overline{Cle},\frac{3}{2}}$. Using (12) with $t = \frac{2}{3}$, $q = 10$, $n = 16$, we find that $12r^2(Cle, \frac{2}{3}) = 5$, the norm of $\mathcal{U}_2^1$. Here $Cle$ spans $P_5$.

The switching class of $Cle$ contains another two strongly regular graphs, namely, the grid $L_2(4)$ and the Shrikhande graph $Shr$ [5]. It is shown in [19] that $L_2(4)$ and $Shr$ define facets of $P_5$ which are 9-dimensional Delaunay polytopes. Since $L_2(4)$ and $Shr$ have $n = 16$ vertices, by Proposition 5, we have $t^H(\overline{L_2(4)}) = t^H(\overline{Shr}) = \frac{3}{2}$. The polytope $P_5$ has another facet defined by the Petersen graph $Pe$. Since dimension of the facet is 9 and $\overline{Pe} = T(5)$ has 10 vertices, we cannot apply Proposition 5. We have only $t^H(T(5)) \geq \frac{3}{2}$. In fact, we saw in the previous section, that $t^H(T(5)) = 2$.

In [5], 3 regular proper subgraphs of $Shr$ are described. These graphs have 6, 10 and 12 vertices. The graphs on 10 and 12 vertices are denoted as $G_{10}$ and $G_{12}^1$ in [19]. They define 8-dimensional faces of $P_5$. There is once more regular 12-vertex subgraph $G_{12}^0$ of $Shr$ determining a 8-dimensional face of $P_5$ which is a line graph. Again, using Proposition 5, we obtain that $t^H(\overline{G}) = \frac{3}{2}$ for $G = G_{10}, G_{12}^0, G_{12}^1$.

There are exactly 15 vertices of $P_5$ lying at the same distance from a vertex of $P_5$. These 15 vertices affinely generate a 9-dimensional hyperplane $H$ and induce the triangular graph $T(6)$. The graph $T(6)$ is 1-skeleton of a Delaunay polytope which is an intersection of $P_5$ by the hyperplane $H$. By Proposition 5 we obtain that $t^H(\overline{T(6)}) = \frac{3}{2}$.

Recall that $P_5$ is inscribed in the unit cube $\gamma_{10}$. Since $t^C(G) \leq t^H(G)$ for any graph $G$, we have that $t^C(G) = t^H(G)$ for all above graphs, excluding $T(5)$.

Concluding this section we give some popular non-pillar strongly regular graphs $G$ with $\lambda_2(G) = 2$. A survey of strongly regular graphs with $\lambda_2(G) = 2$ is given in [18].

1) The graphs $GQ(3,t)$ corresponding to generalized quadrangles with lines of size $s + 1 = 4$, $t = 1, 3, 5, 9$. $GQ(3,1) = L_2(4)$.

2)The negative Latin square graphs $NL_2(m)$, $4 \leq m \leq 10$, $m \neq 7$. $NL_2(4) = Cle$, $NL_2(8) = GQ(3,5)$, $NL_2(10)$ is the Higman-Sims graph.

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